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The Impact of Educational Campaigns on the Global Dynamics of Cholera Epidemics*

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Abstract In this study, we examine a cholera epidemic model that incorporates educational transmission and delve into the long-term dynamics of the epidemic equilibrium point, particularly in relation to the spread of information about disease transmission. We find that if the basic reproduction number R_0 is less than or equal to 1, the disease is destined to be eradicated, and the unique disease-free equilibrium point becomes globally stable. It is noteworthy that the stability of this disease-free equilibrium is not influenced by the extent of disease education, the variety of infections, or the rate of education. On the other hand, if the basic reproduction number is greater than 1, we study the existence of the endemic equilibrium, and demonstrate that the disease will persist if $\beta_0 > D_0$. To assess the impact of educational campaigns on disease control, we further establish the asymptotic behavior of both infected and susceptible populations in response to educational interventions. Our findings indicate that well-executed educational campaigns can substantially contribute to the management and mitigation of the disease's effects.

Keywords Cholera epidemic model, education communication, asymptotic behavior

MSC(2010) 34D05, 34D25, 34E10, 92B20.

1. Introduction

Cholera is an ancient disease that has cased at least seven global pandemic. According to statistics, 3 to 5 million people worldwide are infected with cholera each year, resulting in over 100000 deaths. This acute intestinal infectious disease is caused by Vibrio cholera and is characterized by rapid onset, fast transmission, and widespread occurrence. It can be transmitted to humans through direct contact with infected individuals and indirect contact with contaminated water environments [15, 19]. After ingesting water or food contaminated with Vibrio cholera, individuals may experience symptoms within 12 hours to 5 days [10], which can include vomiting and diarrhea. If infected individuals are not treated promptly, cholera can lead to severe dehydration and even death [24]. Cholera remains a sig-

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nificant public health challenge in many parts of the world [20], particularly in areas lacking clean drinking water and underdeveloped sanitation facilities [21]. Various measures have been implemented across different regions to control the spread of cholera infection. Education and media campaigns play a crucial role in preventing the spread of cholera, especially in developing countries that may not have effective healthcare to prevent the spread of cholera. These campaigns can provide essential information to raise individuals awareness about cholera and help individuals to change harmful habits, such as drinking from unclean water sources or consuming contaminated water products; see [3, 5, 17, 22]. Recently, Denu et al. [11] studied a time delayed HIV/AIDS epidemic model with education dissemination and they considered the asymptotic behavior of the endemic equilibrium with respect to the education dissemination. In [12], they further proved the existence of traveling wave solution and spreading speed of a diffusive time-delayed HIV/AIDS epidemic model with information and education campaigns.

In order to understand the impact of control strategies on the spread of cholera infection, various mathematical models have been used to simulate the spread of cholera. Results have indicated that intervention measures are crucial for controlling disease spread [1, 2, 4, 6–8, 13, 16, 19, 23, 28–30]. In this paper, we incorporate information and educational initiatives into the SIR epidemic model to analyze their effects on controlling the transmission of cholera. We assume that the dynamics of cholera involve the interactions among human hosts, bacteria, and the environment, including human-to-human transmission, indirect environmental transmission, and bacterial shedding into the water environment. In our model, we use the compartment $B(t)$ to represent the bacteria concentration in the environment.

Thus, we divide that population with a total size of $N(t)$ at time t into four classes, which are susceptible ($S_0(t)$, $S_1(t)$), infected individuals $I(t)$, and the removed category $R(t)$. The $S_0(t)$ class, also known as the general susceptible group, consists of uninfected individuals who are vulnerable to infection after contacting with infected individuals. In addition, group $S_0(t)$ can directly access information and educational activities related to the disease. We denote by $Z(t)$ the amount of the educational information from the information and education campaigns. The main educational strategies are enhancing people's awareness of cholera and encouraging them to drink purified water to reduce exposure to the cholera virus. As a result of participating in educational campaigns, some individuals in $S_0(t)$ will change their behavior and enter other compartments based on the information they have received. The group $S_1(t)$ represents individuals whose behavior has been changed due to information and educational campaigns. We assume that only through natural death and infection can individuals leave group $S_1(t)$. Furthermore, we assume that the infection between $S_0(t)$ and $Z(t)$ will transfer to $S_1(t)$ at a rate of γ. The compartment $R(t)$ represents the recovery group of infected individuals.

Based on the above discussion, we consider the following system of nonlinear differential equations:

$$\begin{cases} \frac{d}{dt}S_0(t) = \mu_H\left(N_0 - S_0(t)\right) - \gamma S_0(t)Z(t) - \beta_0 S_0(t)I(t) - \beta_2 S_0(t)B(t), \\ \frac{d}{dt}S_1(t) = \gamma S_0(t)Z(t) - \beta_1 S_1(t)I(t) - \beta_3 S_1(t)B(t) - \mu_H S_1(t), \\ \frac{d}{dt}Z(t) = qI(t) - mZ(t), \\ \frac{d}{dt}I(t) = \beta_0 S_0(t)I(t) + \beta_1 S_1(t)I(t) + \beta_2 S_0(t)B(t) + \beta_3 S_1(t)B(t) - (\sigma + \mu_H)I(t), \\ \frac{d}{dt}B(t) = \eta I(t) - \mu_B B(t), \\ \frac{d}{dt}R(t) = \sigma I(t) - \mu_H R(t). \end{cases} \tag{1.1}$$

We list the interpretation of the parameters present in (1.1) as the following Table 1.

Table 1. A list of model parameters and their interpretations

Symbol	Description
μ_H	Natural birth/death ratio of humans
σ	Recovery rate of populations
μ_B	Net degradation rate of the bacteria
q	Rate of increase of information w.r.t. $I(t)$
m	Death rate of information
γ	Effective response rates of $S_1(t)$
N_0	Carrying capacity of constant size
β_0, β_1	Interpersonal infection transmission rate
β_2, β_3	Indirect environmental infection transmission rate

Here, we give a brief description of model (1.1). The total rate of individuals entering the susceptible group $S_0(t)$ is $\mu_H N_0$. After interacting with the educational information $Z(t)$, a portion of the $S_0(t)$ group will transition to the $S_1(t)$ group at a rate of γ. In addition, interactions between the $S_0(t)$ and $S_1(t)$ groups and the infected group $I(t)$ lead to individuals becoming infected and moving to the infection compartment $I(t)$ at transmission rates β_0 and β_1, respectively. The parameter β_0 represents the infection transmission rate of group $S_0(t)$, where $\beta_0 S_0(t)$ represents the total number of contacts per unit time between a single infected individual and susceptible number of $S_0(t)$. Hence $\beta_0 S_0(t)I(t)$ represents the total number of contacts between susceptible individuals and the entire infected population per unit time. Similarly, $\beta_1 S_1(t)I(t)$ has the same explanation. Moreover, interactions between groups $S_0(t)$ and $S_1(t)$ and the bacterial contaminated water environment $B(t)$ lead to infection at transmission rates of β_2 and β_3, respectively. The nonlinear term $\beta_2 S_0(t)B(t)$ represents the total number of contacts per unit time that susceptible individuals in $S_0(t)$ have with the contaminated water environment. Finally, for the $Z(t)$ compartment equation, we assume that the change in information within the group is proportional to the number of organizations providing information and educational activities. The negative term $-m$ indicates a lack of information due to resource constraints. Areas with higher infection rates should implement more extensive information and education campaigns, while those with lower infection rates should conduct corresponding levels of outreach. According to [9, 14], waterborne diseases are mainly transmitted through indirect contact between humans and the environment, and providing clean drinking water can significantly reduce or even prevent diseases spread. In this paper, we assume that the direct contact rate $\beta_0 < \frac{\sigma+\mu_H}{N_0}$ is relatively small. Therefore, diseases may not occur without indirect water transmission. A flow diagram is provided in Fig.1 to illustrate the interactions in the cholera epidemic as represented in our model (1.1).

The remainder of the paper is organized as follows. In Section 2, we state our main results. The global existence, uniqueness, and positivity of solutions of model (2.1) is considered in Section 3. In Section 4, we investigate the stability of disease-free equilibrium. In Section 5, we study the existence and uniqueness of endemic equilibrium and analyze the asymptotic behavior of the infection and susceptible populations with respect to education information. We discuss the permanence of disease in Section 6.

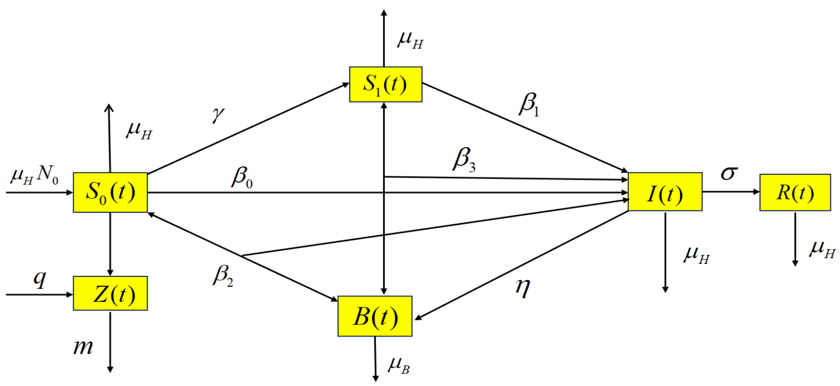

Figure 1. A flow diagram describing the dynamics of cholear epidemic model

2. Statement of main results

Consider (1.1). Note that the first five equations of (1.1) decouple from the last equation and the solution to the last equation is completely determined by the solution of the decouple sub-system formed by the first five equations. Hence, in the following, we focus on studying of the dynamics the sub-system

$$\begin{cases} \frac{d}{dt}S_0(t) = \mu_H\left(N_0 - S_0(t)\right) - \gamma S_0(t)Z(t) - \beta_0 S_0(t)I(t) - \beta_2 S_0(t)B(t), \\ \frac{d}{dt}S_1(t) = \gamma S_0(t)Z(t) - \beta_1 S_1(t)I(t) - \beta_3 S_1(t)B(t) - \mu_H S_1(t), \\ \frac{d}{dt}Z(t) = qI(t) - mZ(t), \\ \frac{d}{dt}I(t) = \beta_0 S_0(t)I(t) + \beta_1 S_1(t)I(t) + \beta_2 S_0(t)B(t) + \beta_3 S_1(t)B(t) - (\sigma+\mu_H)I(t), \\ \frac{d}{dt}B(t) = \eta I(t) - \mu_B B(t), \end{cases} \tag{2.1}$$

where $\mu_H, q, m, \eta, \sigma, \mu_B > 0, \beta_j > 0$ for every $j \in \{0,1,2,3\}$ and $\gamma > 0$, and subject to the initial conditions

$$(S_0(0), S_1(0), Z(0), I(0), B(0)) \in \mathbb{R}_+^5,\ S_0(0) > 0,\ B(0) > 0. \tag{2.2}$$

By the standard theory on differential equations, model (2.1) with initial conditions satisfying (2.2) has a unique non-negative solution, which is defined for all $t \geq 0$. In addition, we define $N(t) = S_0(t) + S_1(t) + I(t)$. It holds that

$$\frac{d}{dt}N(t) = \mu_H N_0 - \mu_H N(t) - \sigma I(t) \leq \mu_H N_0 - \mu_H N(t),\ t > 0.$$

By comparison principle for Ordinary Differential Equations,

$$\lim_{t\to\infty} \sup N(t) \leq N_0 \ \ \textit{and} \ \ N(t) \leq \max\{N(0), N_0\}, \ \ \forall t \geq 0. \tag{2.3}$$

Therefore, we have

$$\lim_{t\to\infty} \sup Z(t) \leq \frac{q}{m} \lim_{t\to\infty} \sup I(t) \leq \frac{qN_0}{m} \ \ , Z(t) \leq \max\left\{Z(0), \frac{q}{m}\max\left\{N_0, N(0)\right\}\right\}, \tag{2.4}$$

and

$$\lim_{t\to\infty} \sup B(t) \leq \frac{\eta}{\mu_B} \lim_{t\to\infty} \sup I(t) \leq \frac{\eta N_0}{\mu_B}, B(t) \leq \max\left\{Z(0), \frac{\eta}{\mu_B}\max\left\{N_0, N(0)\right\}\right\}, \tag{2.5}$$

which implies that the set

$$\chi = \left\{((S_0, S_1, Z, I, B)(t) \in \mathbb{R}_+^5 : \sum_{j=0}^{1} S_j(t) + I(t) \leq N_0,\ Z(t) \leq \frac{qN_0}{m},\ B(t) \leq \frac{\eta N_0}{\mu_B}\right\} \tag{2.6}$$

is forward invariant for the flow generate by the solution of (2.1). In (2.1), when $I(t) \equiv 0$, $Z(t) = Z(0)e^{-mt} \to 0$ as $t \to \infty$. In this case, $(S_0(t), S_1(t), B(t)) \to (N_0, 0, 0)$ as $t \to \infty$. Then set $\mathbf{E^0} := (N_0, 0, 0, 0, 0)^T$ is the corresponding disease-free equilibrium of (2.1). We also find that any equilibrium solution $\bar{\mathbf{E}} = \left(\bar{S}_0, \bar{S}_1, \bar{Z}, \bar{I}, \bar{B}\right)^T$ of (2.1) is uniquely determined by

$$\bar{S}_0 = \frac{\mu_H N_0}{\mu_H + \left(\Delta + \frac{q}{m}\gamma\right)\bar{I}}, \ \ \bar{Z} = \frac{q\bar{I}}{m} \ \ \textit{and} \ \ \bar{S}_1 = \frac{q\mu_H N_0 \gamma \bar{I}}{m\left(\mu_H + \left(\Delta + \frac{q}{m}\gamma\right)\bar{I}\right)\left(\mu_H + p\bar{I}\right)}, \tag{2.7}$$

where $\bar{I}$ as the remaining components, $\Delta = \beta_0 + \beta_2 \frac{\eta}{\mu_B}$ and $p = \beta_1 + \beta_3 \frac{\eta}{\mu_B}$. We call $\bar{\mathbf{E}} = \left(\bar{S}_0, \bar{S}_1, \bar{Z}, \bar{I}, \bar{B}\right)^T$ of (2.1) with $\bar{I} > 0$ as endemic equilibrium. Using the linearization of (2.1) at disease-free equilibrium $(N_0, 0, 0, 0, 0)^T$ and the next-generation matrix theory given in [26,27], we can verify that the basic reproduction number R_0 of (2.1) is given by

$$R_0 := R_{0I} + R_{0B} = \frac{\beta_0 N_0}{\sigma + \mu_H} + \frac{\beta_2 N_0 \eta}{\mu_B(\sigma + \mu_H)}, \tag{2.8}$$

where

$$R_{0I} = \frac{\beta_0 N_0}{\sigma + \mu_H}$$

is the basic reproduction number induced by the direct human-to-human transmission, and

$$R_{0B} = \frac{\beta_2 N_0 \eta}{\mu_B(\sigma + \mu_H)}$$

is the basic reproduction number induced by the indirect environment-to-human transmission. Set

$$D_0 = \frac{\sigma + \mu_H}{N_0}, \tag{2.9}$$

which is a critical important rate of the model (1.1). Note that $\sigma + \mu_H$ represents the rate at which an individual leaves the infective class, and the total size of the population is N_0 at the disease-free equilibrium. Then D_0 represents the per capita mortality rate of infected individuals. Then together with (2.8) and (2.9), we have

$$R_0 = \frac{1}{D_0}(\beta_0 + \frac{\beta_2 \eta}{\mu_B}) = \frac{\Delta}{D_0}.$$

It is easy to see that $R_0 \le 1$ if and only if $\Delta \le D_0$ and $R_0 > 1$ if and only if $\Delta > D_0$. In the following, we give the results on stability of the disease-free equilibrium $\mathbf{E^0}$ and existence of endemic equilibrium point.

Theorem 2.1. (i) *The disease-free equilibrium $\mathbf{E^0}$ is linearly stable if and only if $\Delta < D_0$. Furthermore, if $\Delta + p < D_0$, then the disease-free equilibrium is globally stable.*

(ii) *If $\Delta > D_0$, system* (2.1) *has a unique endemic equilibrium solution $\mathbf{E}^* = ({S_0}^*, {S_1}^*, Z^*, I^*, B^*)^T$, where ${S_0}^*, {S_1}^*, Z^*$ and B^* are given by* (2.7) *with $\bar{I} = I^*$.*

From Theorem 2.1(i), we know that no endemic equilibrium exists when $\Delta + p < D_0$, and the stability of the disease-free equilibrium is independent of the amount of disease education, different infections, and education rates. When $\Delta > D_0$, $\mathbf{E}^* = ({S_0}^*, {S_1}^*, Z^*, I^*, B^*)^T$ is the unique endemic equilibrium solution of (2.1). We point out that I^* is uniquely determined by the unique positive solution of the algebraic equation

$$G(I) = \sigma + \mu_H = D_0 N_0, \tag{2.10}$$

where

$$G(I) := \frac{\Delta \mu_H N_0}{\mu_H + (\Delta + \tau\gamma)\, I} + \frac{p \mu_H N_0 \tau \gamma I}{(\mu_H + pI)\,(\mu_H + (\Delta + \tau\gamma)\, I)}. \tag{2.11}$$

It is very difficult to derive an explicit expression of I^*, which makes the analysis of endemic equilibrium solutions more complex. To analyse the effect of the education term on disease control, we consider the asymptotic behavior of I^* and examine how the parameters on education dissemination affect I^*. Thus we set

$$\tau = \frac{q}{m}.$$

Then $\tau > 1$ means that there is enough information about diseases and $\tau < 1$ means the amount of education is relatively small. We give the asymptotic behavior of I^* and (S_0^*, S_1^*) with respect to $\tau = \frac{q}{m}$ as follows.

Theorem 2.2. *Suppose that $\Delta = \max\{\Delta, p\} > D_0$, $\Delta > \sqrt{p\tau\gamma}$, and let $\mathbf{E}^* = ({S_0}^*, {S_1}^*, Z^*, I^*, B^*)^T$ denote the unique endemic equilibrium solution of* (2.1). *One has*

(i) *I^* is non-increasing with respect to $\tau = \frac{q}{m}$ and*

$$\frac{\mu_H N_0 \left(\Delta - D_0\right)}{\left(\Delta + \tau\gamma\right)\left(\sigma + \mu_H\right)} \leq I^* \leq \frac{\mu_H}{D_0}. \tag{2.12}$$

Furthermore, if $\Delta > p$ then I^* is strictly monotone decreasing with respect to $\tau = \frac{q}{m}$.

(ii) *If $D_0 \geq p$, then*

$$\lim_{\tau\to\infty} I^* = 0. \tag{2.13}$$

Moreover, if $D_0 > p$, then there is a constant $C > 0$ such that

$$\frac{\mu_H N_0 \left(\Delta - D_0\right)}{\left(\Delta + \tau\gamma\right)\left(\sigma + \mu_H\right)} \leq I^* \leq \frac{C}{\tau}, \quad \tau > 0. \tag{2.14}$$

(iii) *If $D_0 < p$, then*

$$\lim_{\tau\to\infty} I^* = I^*_\infty > 0, \tag{2.15}$$

where I^*_∞ is the unique positive solution of the equation $D_0 = \frac{p\mu_H}{\mu_H + pI^*_\infty}$.

Theorem 2.2 states that effective educational campaigns can significantly aid in controlling and reducing the impact of the disease. In fact, Theorem 2.2 (ii) shows $\lim_{\tau\to\infty} I^* = 0$ when $D_0 > p$. The biological assumption implies that when p is less than the critical death rate D_0, the size of the infected population at the endemic diseases is significantly controlled. On the other hand, according to Theorem 2.2 (iii), if p exceeds the critical death rate D_0, the size of the infected population at the endemic equilibrium will remain uniformly bounded away from zero as a function of τ.

In the following theorem, we also study the asymptotic behavior of the susceptible populations (S_0^*, S_1^*) with respect to τ.

Theorem 2.3. *Suppose that $\Delta = \max\{\Delta, p\} > D_0$, $\Delta > \sqrt{p\tau\gamma}$, and let $\mathbf{E}^* = \left({S_0}^*, {S_1}^*, Z^*, I^*, B^*\right)^T$ denote the unique endemic equilibrium solution of (2.1). Then one has*

(i) *If $D_0 = p$, then*

$$\lim_{\tau\to\infty} \left(S_0^*, S_1^*\right) = \left(0, pN_0\right). \tag{2.16}$$

(ii) *If $D_0 > p$, then*

$$\lim_{\tau\to\infty} \left(S_0^*, S_1^*\right) = \left(\frac{\mu_H N_0}{\mu_H + \gamma Z^*_\infty}, \frac{pN_0\gamma Z^*_\infty}{\mu_H + \gamma Z^*_\infty}\right), \tag{2.17}$$

where Z^*_∞ is the unique solution of the algebraic equation $D_0 = \frac{\Delta\mu_H + p\gamma Z^*_\infty}{\mu_H + \gamma Z^*_\infty}$.

(iii) *If $D_0 < p$, by setting $\lim_{\tau\to\infty} I^* = I^*_\infty > 0$, it holds that*

$$\lim_{\tau\to\infty} \left(S_0^*, S_1^*\right) = \left(0, \frac{p\mu_H N_0}{\mu_H + pI^*_\infty}\right). \tag{2.18}$$

Next, we will investigate that whether the disease will permanently exist in the population when $\beta_0 > D_0$, $\Delta = \max\{\Delta, p\}$.

Theorem 2.4. *Suppose that $\beta_0 > D_0$, $\Delta = \max\{\Delta, p\}$. There exists a positive constant $\bar{m} > 0$ such that any solution $(S_0(t), S_1(t), Z(t), I(t), B(t))$ of (2.1) with initial value in χ satisfies*

$$\frac{\mu_H\left(\beta_0 - D_0\right)}{D_0\left(\Delta + \tau\gamma\right)} \le \limsup_{t\to\infty} I(t) \le \frac{\sigma + \mu_H}{D_0} \tag{2.19}$$

and

$$\liminf_{t\to\infty} I(t) > \bar{m}. \tag{2.20}$$

3. Basic properties of solution

In this section, we will study the global existence, uniqueness, and positivity of solutions of model (2.1). Set

$$H = \begin{pmatrix} \mu_H\left(N_0 - S_0\right) - \gamma S_0 I \\ \gamma S_0 Z - \mu_H S_1 \\ qI - mZ \\ -(\sigma + \mu_H)I(t) \\ \eta I - \mu_B B \end{pmatrix} \text{ and } Q = \begin{pmatrix} -\beta_0 S_0 I - \beta_2 S_0 B \\ -\beta_1 S_1 I - \beta_3 S_1 B \\ 0 \\ \beta_0 SI + \beta_1 S_1 I + \beta_2 S_0 B + \beta_3 S_1 B \\ 0 \end{pmatrix}.$$

Then model (2.1) can be written as

$$\frac{d}{dt}(S_0, S_1, Z, I, B)^T = H(S_0, S_1, Z, I, B)^T + Q(S_0, S_1, Z, I, B)^T, \quad t > 0. \tag{3.1}$$

By the standard theory of ordinary differential equations, (3.1) has a unique local solution (S_0, S_1, Z, I, B) defined on a maximal interval of existence $(0, T_{\max})$ for some $T_{\max} \in (0, \infty]$. When $T_{\max} < \infty$,

$$\lim_{t\to T_{\max}} \sup\left[\sum_{i=0}^{1} |S_i(t)| + |Z(t)| + |I(t)| + |B(t)|\right] = \infty. \tag{3.2}$$

Theorem 3.1. *Suppose that* $\min\{S_0(0), S_1(0), Z(0), I(0), B(0)\} > 0$. *The solution $S_0(t)$, $S_1(t)$, $Z(t)$, $I(t)$, $B(t)$ of (2.1) with initial value* $(S_0(0), S_1(0), Z(0), I(0), B(0))$ *is positive and uniformly bounded in $t \in (0, \infty)$.*

Proof. By continuity of the map

$$t \in (0, T_{\max}] \mapsto \min\left\{S_0(0), S_1(0), Z(0), I(0), B(0)\right\}, \tag{3.3}$$

there is $0 < T' < T_{\max}$ such that

$$\min\left\{S_0(t), S_1(t), Z(t), I(t), B(t)\right\} > 0 \ \ \forall t \in [0, T']\,. \tag{3.4}$$

Let $T'_{\max}$ denote the supremum of positive number in $(0, T_{\max})$ satisfying (3.4). We then show that $T'_{\max} = T_{\max}$. Let

$$M = \max_{t\in[0,T'_{\max}]}\{Z(t), B(t), I(t)\} > 0.$$

Hence, by definition of $T'_{\max}$, we have

$$\frac{d}{dt}S_0(t) \geq \mu_H N_0 - (M+1)\left(\mu_H + \gamma + \beta_0 + \beta_2\right) S_0, \;\; \forall 0 < t \leq T'_{\max},$$

and by comparison principle for differential equation it holds that

$$S_0(t) \geq h_0 := \min\left\{S_0(0), \frac{\mu_H N_0}{\left(\mu_H + \gamma + \beta_0 + \beta_2\right)(M+1)}\right\} \;\; \forall t \in [0, T'_{\max}].$$

In addition, we have

$$\frac{d}{dt}S_1(t) \geq -\left(\beta_1 M + \mu_H + \beta_3 M\right) S_1(t), \;\; \forall 0 < t \leq T'_{\max}.$$

By comparison principle for differential equation it holds that

$$S_1(t) \geq h_1 := e^{-(\beta_1 M + \mu_H + \beta_3 M)T'_{\max}} S_1(0) \;\; \forall t \in [0, T'_{\max}].$$

Form the equation of $Z(t)$, we have

$$\frac{d}{dt}Z(t) \geq -mZ(t), \;\; \forall 0 < t \leq T'_{\max},$$

which implies that

$$Z(t) \geq h_z := e^{-mT'_{\max}} Z(0), \;\; \forall t \in [0, T'_{\max}].$$

Similarly, it can be inferred that

$$B(t) \geq h_b := e^{-\mu_B T'_{\max}} B(0), \;\; \forall t \in [0, T'_{\max}].$$

Observe also that

$$\frac{d}{dt}I(t) \geq -(\sigma + \mu_H)I(t), \;\; \forall 0 < t \leq T'_{\max},$$

which implies that

$$I(t) \geq e^{-(\sigma+\mu_H)T'_{\max}} I(0), \;\; \forall t \in [0, T'_{\max}].$$

Observe form the last inequality and the definition of $T'_{\max}$ that $I(t) > 0$ for every $t \in [0, T'_{\max}]$. Thus, by continuity of the map $t \in [0, T'_{\max}] \mapsto I(t)$ and compactness of the set $[0, T'_{\max}]$, we have that

$$h_i := \min_{t \in [0, T'_{\max}]} I(t) > 0.$$

Therefore, we obtain that

$$\min\{(S_0(T'_{\max}), S_1(T'_{\max}), Z(T'_{\max}), I(T'_{\max}), B(T'_{\max})\} \geq \min\{h_0, h_1, h_z, h_i, h_b\} > 0.$$

This together with the continuity of the mapping defined in (3.3) implies that there is some $T'' \in (T'_{\max}, T_{\max})$ such that (3.4) holds for every $t \in (0, T'')$, which contradicts the definition of $T'_{\max}$. Therefore, we have that $T'_{\max} = T_{\max}$. Next, since the solution is positive, it follows from the arguments used to establish (2.3),(2.4) and (2.5) that

$$N(t) = S_0(t) + S_1(t) + I(t) \le \max\{N(0), N_0\}, \quad \forall t \in [0, T'_{\max}],$$

$$B(t) \le \max\left\{B(0), \frac{\eta}{\mu_B}\max\{N_0, N(0)\}\right\}, \quad \forall t \in [0, T'_{\max}],$$

and

$$Z(t) \le \max\left\{Z(0), \frac{q}{m}\max\{N_0, N(0)\}\right\}, \quad \forall t \in [0, T'_{\max}].$$

Hence, in view of (3.2), we conclude that $T_{\max} = \infty$. Let

$$(S_0(0), S_1(0), Z(0), I(0), B(0))$$

be a nonnegative initial data and $T_{\max} \in (0, \infty)$ be the maximal time of existence of the associated unique classical solution $(S_t(0), S_1(t), Z(t), I(t), B(t))$. Consider

$$\left(S_0(0) + \frac{1}{n}, S_1(0) + \frac{1}{n}, Z(0) + \frac{1}{n}, I(0) + \frac{1}{n}, B(0) + \frac{1}{n}\right)$$

and let $(S_0^n(t), S_1^n(t), Z^n(t), I^n(t), B^n(t))$ denote the associated unique classical solution. Since

$$\left(S_0(0) + \frac{1}{n}, S_1(0) + \frac{1}{n}, Z(0) + \frac{1}{n}, I(0) + \frac{1}{n}, B(0) + \frac{1}{n}\right)$$

satisfies the theorem conditions for each n, then $(S_0^n(t), S_1^n(t), Z^n(t), I^n(t), B^n(t))$ is defined for all time $t > 0$ with

$$\min\{(S_0^n(t), S_1^n(t), Z^n(t), I^n(t), B^n(t))\} > 0 \ \ \forall t > 0.$$

Since

$$\lim_{n\to\infty}\left(S_0(0) + \frac{1}{n}, S_1(0) + \frac{1}{n}, Z(0) + \frac{1}{n}, I(0) + \frac{1}{n}, B(0) + \frac{1}{n}\right)$$
$$= (S_0(0), S_1(0), Z(0), I(0), B(0)),$$

then

$$\lim_{n\to\infty}(S_0^n(t), S_1^n(t), Z^n(t), I^n(t), B^n(t)) = (S_0(t), S_1(t), Z(t), I(t), B(t)),$$
$$\forall t \in (0, T_{\max}),$$

which implies that

$$\min\{S_0(t), S_1(t), Z(t), I(t), B(t)\} \ge 0, \quad \forall t \in (0, T_{\max}).$$

Therefore, the last inequality implies that $T_{\max} = \infty$. □

4. The disease-free equilibrium

In this section, we will investigate the stability of disease-free equilibrium point. We linearize system (2.1) at an equilibrium point $\tilde{E} = \left(\tilde{S}_0, \tilde{S}_1, \tilde{Z}, \tilde{I}, \tilde{B}\right)^T$ and obtain the linearized system

$$\begin{cases} \frac{d}{dt}S_0(t) = -\left(\mu_H + \gamma\tilde{Z} + \beta_0\tilde{I} + \beta_2\tilde{B}\right)S_0(t) - \gamma\tilde{S}_0 Z(t) - \beta_0\tilde{S}_0 I(t) - \beta_2\tilde{S}_0 B(t), \\ \frac{d}{dt}S_1(t) = \gamma S_0(t)\tilde{Z} - \left(\beta_1\tilde{I} + \beta_3\tilde{B} + \mu_H\right)S_1(t) + \gamma\tilde{S}_0 Z(t) - \beta_1\tilde{S}_1 I(t) - \beta_3\tilde{S}_1 B(t), \\ \frac{d}{dt}Z(t) = qI(t) - mZ(t), \\ \frac{d}{dt}I(t) = \left(\beta_0\tilde{I} + \beta_2\tilde{B}\right)S_0(t) + \left(\beta_1\tilde{I} + \beta_3\tilde{B}\right)S_1(t) \\ \qquad + \left(\beta_0\tilde{S}_0 + \beta_1\tilde{S}_1 - (\sigma + \mu_H)\right)I(t) + \left(\beta_2\tilde{S}_0 + \beta_3\tilde{S}_1\right)B(t), \\ \frac{d}{dt}B(t) = \eta I(t) - \mu_B B(t). \end{cases} \tag{4.1}$$

Then the characteristic equation of (4.1) is

$$0 = p(\lambda, \tilde{E}) := \det\left(\lambda I - M\left(\lambda, \tilde{E}\right)\right), \tag{4.2}$$

where I is identity matrix and

$$\mathrm{M}(\lambda; \widetilde{E}) :=$$

$$\begin{pmatrix} -\left(\mu_H + \gamma\tilde{Z} + \beta_0\tilde{I} + \beta_2\tilde{B}\right) & 0 & -\gamma\tilde{S}_0 & -\beta\tilde{S}_0 & -\beta_2\tilde{S}_0 \\ \gamma\tilde{Z} & -\left(\beta_1\tilde{I} + \beta_3\tilde{B} + \mu_H\right) & \gamma\tilde{S}_0 & -\beta_1\tilde{S}_1 & -\beta_3\tilde{S}_1 \\ 0 & 0 & -m & q & 0 \\ \beta_0\tilde{I} + \beta_2\tilde{B} & \beta_1\tilde{I} + \beta_3\tilde{B} & 0 & G & \beta_2\tilde{S}_0 + \beta_3\tilde{S}_1 \\ 0 & 0 & 0 & \eta & -\mu_B \end{pmatrix},$$

where $G = \beta_0\tilde{S}_0 + \beta_1\tilde{S}_1 - (\sigma + \mu_H)$.

For convenience, we set

$$b = \mu_B - \beta_0 N_0 + \sigma + \mu_H \quad \text{and} \quad c = (\sigma + \mu_H)\mu_B - (\beta_0 N_0 \mu_B + \beta_2 N_0 \eta).$$

It then follows that, at the disease-free equilibrium $\mathbf{E^0} := (N_0, 0, 0, 0, 0)^T$,

$$p(\lambda, \mathbf{E^0}) = (\lambda + \mu_H)^2 (\lambda + m)[\lambda^2 + b\lambda + c]. \tag{4.3}$$

Lemma 4.1. *The disease-free equilibrium* $\mathbf{E^0}$ *is linearly stable if and only if* $\Delta < D_0$.

Proof. It is clear that $\lambda = -\mu_H$ and $\lambda = -m$ are always roots of the characteristic equation $0 = p(\lambda, \mathbf{E^0})$. Thus the linear stability of $\mathbf{E^0}$ is determined by the signs of the real parts of the roots of the equation

$$\lambda^2 + (\mu_B - \beta_0 N_0 + \sigma + \mu_H)\lambda + (\sigma + \mu_H)\mu_B - (\beta_0 N_0 \mu_B + \beta_2 N_0 \eta) = 0. \tag{4.4}$$

According to the literature, it is known that $b > 0$ always holds true. Denote

$$\Delta_1 = \begin{vmatrix} b & 0 \\ 1 & c \end{vmatrix}. \tag{4.5}$$

It can be seen that when $c > 0$, we have $|\Delta_1| > 0$ and when $c < 0$, we have $|\Delta_1| < 0$.

By the Routh-Hurwitz Criterion [18], the eigenvalues λ of (4.4) have negative real parts if and only if the $c > 0$, that is $(\sigma + \mu_H)\,\mu_B - (\beta_0 N_0 \mu_B + \beta_2 N_0 \eta) > 0$. Then $R_0 < 1$ and we have $\Delta < D_0$. Therefore, the conclusion is confirmed. □

Now we prove the Theorem 2.1 (1).

Proof. [Theorem 2.1(1)] Lemma (4.1) has shown that the disease-free equilibrium $\mathbf{E^0}$ is linearly stable if and only if $\Delta < D_0$. Recall that

$$S_0(t) + S_1(t) + I(t) \le N_0 \quad \forall\, t \ge 0.$$

Consider the system of equations

$$\begin{cases} \frac{d}{dt} I(t) = \beta_0 S_0(t) I(t) + \beta_1 S_1(t) I(t) + \beta_2 S_0(t) B(t) + \beta_3 S_1(t) B(t) - (\sigma + \mu_H) I(t), \\ \frac{d}{dt} B(t) = \eta I(t) - \mu_B B(t). \end{cases}$$

Furthermore, we have

$$\begin{cases} \frac{d}{dt} I(t) \le (\beta_0 N_0 + \beta_1 N_0 - (\sigma + \mu_H))\, I(t) + (\beta_2 N_0 + \beta_3 N_0)\, B(t), \\ \frac{d}{dt} B(t) = \eta I(t) - \mu_B B(t). \end{cases}$$

Analyze the following system of differential equations

$$\begin{cases} \frac{d}{dt} u_1(t) = (\beta_0 N_0 + \beta_1 N_0 - (\sigma + \mu_H))\, u_1(t) + (\beta_2 N_0 + \beta_3 N_0)\, u_2(t), \\ \frac{d}{dt} u_2(t) = \eta u_1(t) - \mu_B u_2(t). \end{cases} \tag{4.6}$$

Set

$$U = (u_1, u_2)^T \quad \text{and} \quad A = \begin{pmatrix} (\beta_0 N_0 + \beta_1 N_0 - (\sigma + \mu_H)) & \beta_2 N_0 + \beta_3 N_0 \\ \eta & -\mu_B \end{pmatrix}.$$

Then we denote equation system (4.6) as $\frac{d}{dt} U = AU$. When $\Delta + p < D_0$, the eigenvalues of matrix A are less than 0. Furthermore, by comparing the principles of differential equations, it can be concluded that $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} B(t) = 0$. This in turn implies that

$$\lim_{t\to\infty} (S_0(t), S_0(t), Z(t)) = (N_0, 0, 0).$$

It follows that $\lim_{\mathrm{t}\to\infty} (S_0(t), S_1(t), Z(t), I(t), B(t)) = \mathbf{E^0}$. Therefore, the conclusion is valid. □

5. The endemic equilibrium

In this section, we will study the existence and uniqueness of endemic equilibrium, showing that if $\Delta > D_0$ then (2.1) exists the endemic equilibrium point.

For $I \geq 0, \tau > 0, \Delta > 0$, define the auxiliary function $G(I, \Delta, \tau)$ as

$$G(I, \Delta, \tau) = \frac{\Delta \mu_H N_0}{\mu_H + (\Delta + \tau\gamma) I} + \frac{p\mu_H N_0 \tau\gamma I}{(\mu_H + pI)(\mu_H + (\Delta + \tau\gamma) I)}, \tag{5.1}$$

where $\min\{\Delta, p\} > 0$. Note that $\bar{E} = \left(\bar{S}_0, \bar{S}_1, \bar{Z}, \bar{I}, \bar{B}\right)^T$ is an equilibrium of (2.1) with $\bar{I} \neq 0$ if and only if $G\left(\bar{I}, \Delta, \tau\right) = \sigma + \mu_H = D_0 N_0$.

Lemma 5.1. *For every $\tau > 0$ and $\Delta \geq 0$, $\sup_{I \geq 0} G(I, \Delta, \tau)$ is finite and achieved. Moreover, the function $\Delta \in [0, +\infty) \mapsto \max_{I \geq 0} G(I, \Delta, \tau)$ is strictly increasing and $\max_{I \geq 0} G(I, \Delta, \tau) > D_0 N_0$ for every $\Delta > D_0$. Therefore, the following quantity*

$$D_{0,\tau} := \min\left\{\Delta \geq 0 : \max_{I \geq 0} G(I, \Delta, \tau) \geq D_0 N_0\right\} \tag{5.2}$$

is well defined.

Proof. It is clear that

$$\lim_{I \to \infty} G(I, \Delta, \tau) = 0 \textit{ and } G(I, \Delta, \tau) \geq 0, \ \forall \tau, \Delta, I \geq 0.$$

Since $I \mapsto G(I, \Delta, \tau)$ is continuous, it follows that $\sup_{I \geq 0} G(I, \Delta, \tau)$ is finite and achieved. Observe that

$$\begin{aligned} \frac{\partial G}{\partial \Delta} &= \frac{\mu_H N_0 \left(\mu_H + \tau\gamma I - \frac{p\tau\gamma I^2}{\mu_H + pI}\right)}{(\mu_H + (\Delta + \tau\gamma) I)^2} \geq \frac{\mu_H N_0 (\mu_H + \tau\gamma I - \tau\gamma I)}{(\mu_H + (\Delta + \tau\gamma) I)^2} \\ &= \frac{{\mu_H}^2 N_0}{(\mu_H + (\Delta + \tau\gamma) I)^2} > 0, \end{aligned}$$

for every $\Delta \geq 0, \tau > 0$ *and* $I \geq 0$. Thus, we can conclude that

$$\Delta \in [0, \infty) \mapsto \max_{I \geq 0} G(I, \Delta, \tau)$$

is strictly increasing. Note that $G(0, \Delta, \tau) = \Delta N_0 \geq D_0 N_0$ for every $\Delta \geq D_0$. The result follows. □

Theorem 5.1. *For every $\tau > 0$ and let $D_{0,\tau} \leq D_0$ be given by (5.2). One has*

(i) *If $0 < \Delta < D_{0,\tau}$, the algebraic equation $G(I, \Delta, \tau) = D_0 N_0$ has no nonnegative roots.*

(ii) *If $D_{0,\tau} < \Delta < D_0$, the algebraic equation $G(I, \Delta, \tau) = D_0 N_0$ has two positive roots $I_-(\Delta, \tau)$ and $I_+(\Delta, \tau)$ with $I_-(\Delta, \tau) < I_+(\Delta, \tau)$. Moreover,*

$$\partial_I G(I_-(\Delta, \tau), \Delta, \tau) > 0 \quad \textit{and} \quad \partial_I G(I_+(\Delta, \tau), \Delta, \tau) < 0,$$

the function $(\Delta, \tau) \mapsto I_\pm(\Delta, \tau)$ are of class C^1.

(iii) *If $\Delta > D_0$, there is a unique positive root $I(\Delta,\tau)$ of the algebraic equation $G(I,\Delta,\tau) = D_0 N_0$. The function $\Delta \mapsto I(\Delta,\tau)$ and $\tau \mapsto I(\Delta,\tau)$ are smooth and $\partial_I G(I(\Delta,\tau),\Delta,\tau) < 0$.*

Proof.

(i) Following from Lemma 5.1, $\max_{I\geq 0} G(I,\Delta,\tau) < D_0 N_0$ for every $0 < \Delta < D_{0,\tau}$, which implies the conclusion holds true.

(ii) Let $D_{0,\tau} < \Delta < D_0$. Then $G(0,\Delta,\tau) = \Delta N_0 < D_0 N_0 < \max_{I\geq 0} G(I,\Delta,\tau)$. This shows that $\max_{I\geq 0} G(I,\Delta,\tau)$ is achieved at an interior point $I_{\max}$. Observe that

$$\partial_I G = \frac{\left[-\Delta\mu_H N_0(\Delta+\tau\gamma) + p\mu_H N_0 \tau\gamma \frac{{\mu_H}^2 - (\Delta+\tau\gamma)pI^2}{(\mu+pI)^2}\right]}{(\mu_H + (\Delta+\tau\gamma)I)^2}. \tag{5.3}$$

Hence

$$p\mu_H N_0\tau\gamma - \Delta\mu_H N_0(\Delta+\tau\gamma) = \frac{p\mu_H N_0\tau\gamma\left[(pI)^2 + 2\mu_H pI + (\Delta+\tau\gamma)pI^2\right]}{(\mu_H+pI)^2}.$$

Computation shows that

$$I \in [0,\infty) \mapsto \frac{p\mu_H N_0\tau\gamma\left[(pI)^2 + 2\mu_H pI + (\Delta+\tau\gamma)pI^2\right]}{(\mu_H+pI)^2}$$

is strictly increasing. Therefore, $I_{\max}$ is the unique positive root of $\partial_I G = 0$. Moreover, the functions

$$I \in (0, I_{\max}) \mapsto G(I,\Delta,\tau) \quad \text{and} \quad I \in (I_{\max},\infty) \mapsto G(I,\Delta,\tau)$$

are strictly increasing and decreasing, respectively. The intermediate value theorem implies that there exist $I_-(\Delta,\tau) \in (0, I_{\max})$ and $I_+(\Delta,\tau) \in (I_{\max},\infty)$ such that $\partial_I G(I_\pm(\Delta,\tau),\Delta,\tau) = D_0 N_0$. Furthermore,

$$\partial_I G(I_-(\Delta,\tau),\Delta,\tau) > 0 \quad \text{and} \quad \partial_I G(I_+(\Delta,\tau),\Delta,\tau) < 0.$$

Then the implicit function theorem ensures that functions $(\Delta,\tau) \mapsto I_\pm(\Delta,\tau)$ are of class C^1.

(iii) Suppose $\Delta > D_0$. Since $D_0 N_0 < G(0,\Delta,\tau) = \min_{0\leq I\leq I_{\max}} G(I,\Delta,\tau)$. Therefore, there is exactly a unique solution $I(\Delta,\tau) > 0$ of the solution $G(0,\Delta,\tau) = \Delta D_0$, with $I(\Delta,\tau) \in (I_{\max},\infty)$.

□

Then for $\Delta > D_0$ and $\tau = \frac{q}{m}$, Theorem 2.1(2) follows from Theorem 5.1 (iii). Now we prove the Theorem 2.2.

Proof. [Theorem 2.2]

(i) Let $G(I, \Delta, \tau)$ be given in (5.1). Then

$$\frac{\partial}{\partial \tau} G = \frac{\mu_H N_0 \gamma I \frac{(p-\Delta)\mu_H}{\mu_H + pI}}{\left(\mu_H + (\Delta + \tau\gamma) I\right)^2}, \quad \forall \tau > 0, I \geq 0.$$

Since I^* is uniquely determined by the equation $G(I, \Delta, \tau) = D_0 N_0$ and $\Delta \geq \max\{\Delta, p\}$, the implicit theorem implies that

$$\frac{\partial}{\partial \tau} I^* = -\frac{\partial_\tau G}{\partial_I G} = \begin{cases} 0, & if \Delta = p, \\ < 0, & if \Delta > p. \end{cases}$$

By Theorem (5.1) (iii), $\partial_I G(I^*, \Delta, \tau) < 0$, which implies the monotonicity of I^* with respect to τ. A simple computation shows that

$$G\left(\frac{\mu_H (\Delta - D_0)}{D_0 (\Delta + \tau\gamma)}, \Delta, \tau\right) > \frac{\Delta \mu_H N_0}{\mu_H + \frac{\mu_H}{D_0}(\Delta - D_0)} = D_0 N_0, \quad \forall \Delta, \tau > 0.$$

By (5.1), we have

$$\begin{aligned} \partial_I G &= \frac{\left[-\Delta \mu_H N_0 (\Delta + \tau\gamma) + p\mu_H N_0 \tau\gamma \frac{\mu_H{}^2 - (\Delta + \tau\gamma) p I^2}{(\mu + pI)^2}\right]}{\left(\mu_H + (\Delta + \tau\gamma) I\right)^2} \\ &< \frac{1}{\left(\mu_H + (\Delta + \tau\gamma) I\right)^2} \left(-\Delta^2 \mu_H N_0 + p\mu_H N_0 \tau\gamma \frac{\mu_H - pI}{\mu_H + pI}\right) \\ &< \frac{\mu_H N_0}{\left(\mu_H + (\Delta + \tau\gamma) I\right)^2} \left(p\tau\gamma - \Delta^2\right) \\ &< 0, \quad \forall I, \tau > 0. \end{aligned}$$

Since $\Delta = \max\{\Delta, p\}$, the function $I \in (0, \infty) \mapsto G(I, \Delta, \tau)$ is strictly decreasing. Then $\frac{\mu_H N_0 (\Delta - D_0)}{(\Delta + \tau\gamma)(\sigma + \mu_H)} \leq I^*$. Observe that

$$\mu_H N_0 = D_0 N_0 I^* + \mu_H (S_0^* + S_1^*) + \mu_B B^*.$$

Hence $I^* \leq \frac{\mu_H}{D_0}$. Then we obtain (2.12) from the above two inequalities.

(ii) Next we proceed by contradiction to show that (2.13) holds under hypothesis $D_0 \geq$ p. Due to the monotonicity of $I^*(\Delta, \tau)$ with respect to τ, there is $I^*_\infty(\Delta) \in [0, N_0)$ such that

$$\lim_{\tau \to \infty} I^*_\infty(\Delta, \tau) = I^*_\infty(\Delta).$$

Now, we claim that $I^*_\infty = I^*_\infty(\Delta) = 0$. If not, we have $I^*_\infty > 0$. Letting $\tau \to \infty$ in the equation $G(I^*, \Delta, \tau) = D_0 N_0$, then

$$D_0 N_0 = \frac{\frac{p\mu_H N_0 \gamma I^*_\infty}{\mu_H + pI^*_\infty}}{\gamma I^*_\infty} = \frac{p\mu_H N_0}{\mu_H + pI^*_\infty} < pN_0,$$

which is in contradiction with hypothesis $D_0 \geq$ p. Thus (2.13) holds when $D_0 \geq$ p. Next, suppose that $D_0 >$ p and choose a positive constant $C_1 \gg 1$ such that

$$\frac{\mu_H N_0}{\mu_H + C_1 \gamma}\left(\Delta + \frac{p\gamma C_1}{\mu_H}\right) < D_0 N_0.$$

Then we have

$$\lim_{\tau\to\infty} G\left(C_1\tau^{-1},\Delta,\tau\right)=\frac{\mu_H N_0}{\mu_H+C_1\gamma}\left(\Delta+\frac{p\gamma C_1}{\mu_H}\right)<D_0N_0.$$

Hence, there is $\tau_*\gg 1$ such that

$$I^*\left(\Delta,\tau\right)\le\frac{C_1}{\tau}\ \ \forall\tau\ge\tau_*.$$

Therefore, since that $I^*\left(\Delta,\tau\right)$ is uniformly bounded above by N_0, we conclude that the right inequality of (2.14) holds with $C=\max\{C_1,N_0\tau_*\}$. Clearly the left inequality of (2.14) follows from(2.12).

(iii) Suppose that $D_0<$ p. Let $I^*_\infty\left(\Delta\right)$ denote the unique positive solution of the algebraic equation $D_0N_0(\mu_H+pI^*_\infty)=p\mu_H N_0$. For every $0<\varepsilon\ll 1$, we have

$$\lim_{\tau\to\infty} G\left(\left(1\pm\varepsilon\right)I^*_\infty,\Delta,I\right)=\frac{p\mu_H N_0}{\mu_H+\left(1\pm\varepsilon\right)pI^*_\infty}.$$

Since $G\left(I^*,\Delta,\tau\right)=D_0N_0$ and $G\left(I,\Delta,\tau\right)$ is strictly decreasing with respect to I, then there is $\tau_\varepsilon\gg 0$ such that

$$\left(1-\varepsilon\right)I^*_\infty<I^*\left(\Delta,\tau\right)<\left(1+\varepsilon\right)I^*_\infty,\qquad\forall\tau>\tau_\varepsilon.$$

Thus we conclude that $\lim_{\tau\to\infty} I^*_\infty\left(\Delta,\tau\right)=I^*_\infty\left(\Delta\right)$. This completes the proof.

□

Now we prove Theorem 2.3.

Proof. [Theorem 2.3]

(i) If $D_0=p$, according to Theorem 2.2, the conclusion clearly holds.

(ii) If $D_0>p$, according to Theorem 2.2, it can be concluded that $\lim_{\tau\to\infty} I^*=0$. Furthermore, according to (2.7), we have

$$\bar{S}_0=\frac{\mu_H N_0}{\mu_H+\left(\Delta+\frac{q}{m}\gamma\right)\tau^{-1}\bar{Z}},\ \bar{S}_1=\frac{q\mu_H N_0\gamma\tau^{-1}\bar{Z}}{m\left(\mu_H+\left(\Delta+\frac{q}{m}\gamma\right)\tau^{-1}\bar{Z}\right)\left(\mu_H+p\tau^{-1}\bar{Z}\right)}.$$

Then letting $\tau\to\infty$, we have $\lim_{\tau\to\infty}\left(S_0^*,S_1^*\right)=\left(\frac{\mu_H N_0}{\mu_H+\gamma Z^*_\infty},\frac{pN_0\gamma Z^*_\infty}{\mu_H+\gamma Z^*_\infty}\right)$, where Z^*_∞ is the unique solution of the algebraic equation $D_0=\frac{\Delta\mu_H+p\gamma Z^*_\infty}{\mu_H+\gamma Z^*_\infty}$.

(iii) If $D_0<p$, according to Theorem 2.2, it can be concluded that $\lim_{\tau\to\infty} I^*=I^*_\infty>0$. Furthermore, according to (2.7), we have

$$\bar{S}_0=\frac{\mu_H N_0}{\mu_H+\left(\Delta+\frac{q}{m}\gamma\right)\bar{I}},\quad\bar{S}_1=\frac{q\mu_H N_0\gamma\bar{I}}{m\left(\mu_H+\left(\Delta+\frac{q}{m}\gamma\right)\bar{I}\right)\left(\mu_H+p\bar{I}\right)}$$

and letting $\tau\to\infty$, we have $\lim_{\tau\to\infty}\left(S_0^*,S_1^*\right)=\left(0,\frac{p\mu_H N_0}{\mu_H+pI^*_\infty}\right)$. This completes the proof.

□

6. Permanence of disease

In this section, we are in a position to study the permanence of the disease when $\Delta = \max\{\Delta, p\}$ and $\beta_0 > D_0$ and prove Theorem 2.4.

Lemma 6.1. *Suppose that $\beta_0 > D_0$. Then for any solution $(S_0, S_1, Z, I, B)(t)$ of (2.1) with initial value in χ, there is*

$$\limsup_{t\to\infty} I(t) \geq \underline{m} > 0. \tag{6.1}$$

Proof. For convenience, set

$$\bar{I} := \limsup_{t\to\infty} I(t) \quad \text{and} \quad \underline{m} := \frac{\mu_H(\beta_0 - D_0)}{D_0(\Delta + \tau\gamma)}.$$

Suppose that there is a positive solution $(S_0(t), S_1(t), Z(t)I(t), B(t))$ of (2.1) with initial value in χ satisfying for any $\lambda_1 \in (0,1)$ such that

$$\bar{I} := \limsup_{t\to\infty} I(t) < \lambda_1 \underline{m}. \tag{6.2}$$

Fix $\lambda_2 \in (0, 1-\lambda_1)$. Then there is $t_1 \gg 1$ such that

$$\sup_{t \geq t_1} I(t) \leq (\lambda_1 + \lambda_2)\,\underline{m}. \tag{6.3}$$

From the equations of $\frac{d}{dt}Z(t)$ and $\frac{d}{dt}B(t)$, we obtain that

$$\frac{d}{dt}Z(t) \leq (\lambda_1 + \lambda_2)\, q\underline{m} - mZ(t), \ \ \forall t \geq t_1$$

and

$$\frac{d}{dt}B(t) \leq (\lambda_1 + \lambda_2)\, \eta\underline{m} - \mu_B B(t), \ \ \forall t \geq t_1.$$

Thus, if $\lambda_3 \in (0, 1-\lambda_1+\lambda_2)$, there is some $t_2 > t_1$ such that

$$Z(t) \leq \frac{(\lambda_1+\lambda_2+\lambda_3)\, q\underline{m}}{m}, \ \ \forall t \geq t_2 \tag{6.4}$$

and

$$B(t) \leq \frac{(\lambda_1+\lambda_2+\lambda_3)\, \eta\underline{m}}{\mu_B}, \ \ \forall t \geq t_2. \tag{6.5}$$

It then follows from (6.3), (6.4)and (6.5) that

$$\frac{d}{dt}S_0(t) \geq \mu_H N_0 - \left(\mu_H + (\lambda_1+\lambda_2+\lambda_3)\left(\tau\gamma + \frac{\eta\beta_2}{\mu_B}\right)\underline{m} + (\lambda_1+\lambda_2)\,\underline{m}\beta_0\right) S_0(t)$$

for $t \geq t_2$.

Then there is $t_3 > t_2$ such that

$$S_0(t) \geq \frac{\mu_H N_0}{\mu_H + (\lambda_1+\lambda_2+\lambda_3)(\Delta+\tau\gamma)\,\underline{m}}, \ \ \forall t \geq t_3,$$

which implies that

$$\frac{d}{dt}I(t) \geq \frac{\beta_0\mu_H N_0}{\mu_H + (\lambda_1+\lambda_2+\lambda_3)(\Delta+\tau\gamma)\,\underline{m}} I(t) - D_0 N_0 I(t), \ \ \forall t \geq t_3. \tag{6.6}$$

Observe from the formula of $(\Delta+\tau\gamma)\,\underline{m}=\frac{\mu_H(\beta_0-D_0)}{D_0}$ and $\lambda_1+\lambda_2+\lambda_3<1$, we have that

$$\begin{aligned}\frac{\beta_0\mu_H N_0}{\mu_H+(\lambda_1+\lambda_2+\lambda_3)(\Delta+\tau\gamma)\,\underline{m}}-D_0N_0 &> \frac{\beta_0\mu_H N_0}{\mu_H+(\Delta+\tau\gamma)\,\underline{m}}-D_0N_0\\ &= \frac{\beta_0\mu_H N_0}{\mu_H+\frac{\mu_H(\beta_0-D_0)}{D_0}}-D_0N_0=0.\end{aligned}$$

Therefore, the equation $\lambda-\frac{\beta_0\mu_H N_0}{\mu_H+(\lambda_1+\lambda_2+\lambda_3)(\Delta+\tau\gamma)\underline{m}}+D_0N_0=0$ has a positive root $\lambda_0>0$. Hence, the solution $\underline{I}(t)$ to the linear differential equation

$$\begin{cases}\frac{d}{dt}\underline{I}(t)=\frac{\beta_0\mu_H N_0}{\mu_H+(\lambda_1+\lambda_2+\lambda_3)(\Delta+\tau\gamma)\underline{m}}\underline{I}(t)-D_0N_0\underline{I}(t) & t>t_3,\\ \\ \underline{I}(s)=I(s)>0 & s=t_3,\end{cases}\tag{6.7}$$

blows up exponentially as t $\to\infty$. By (6.6) and (6.7), $I(t)\geq\underline{I}(t)\ \ \forall t\geq t_3$. Then

$$\limsup_{t\to\infty} I(t)\geq\limsup_{t\to\infty}\underline{I}(t)=\infty,$$

which contradicts with (6.2). Therefore, the statement of the Lemma must hold. □

Proof. [Theorem 2.4] Note that (2.1) generates a continuous semiflow on the set

$$\bar{\chi}=\{(S_0,S_1,Z_0,I_0,B_0)\in\chi:S_0>0\}$$

and we denote the semiflow by

$$\Phi_t\,(S_0,S_1,Z_0,I_0,B_0)=(S_0(t),S_1(t),Z_0(t),I_0(t),B_0(t))$$

for every $t\geq0$. By Arzela-Ascoli's Theorem, Φ_t is compact for every $t>0$.

Recall that

$$\sum_{i=0}^{1}|S_i(t)|+|I(t)|\leq 2N_0,\ |Z(t)|\leq\frac{qN_0}{m}\quad\text{and}\quad |B(t)|\leq\frac{\eta N_0}{\mu_B}.$$

Then Φ is point-wise dissipative and uniformly bounded for $t\geq0$. Therefore, it follows from [25, Theorem 2.30] that (2.1) has a compact attractor A of χ. For every $(S_0,S_1,Z_0,I_0,B_0)\in\bar{\chi}$, define the persistence function

$$\rho\,(S_0,S_1,Z_0,I_0,B_0)=I(0)\ \ \forall\,(S_0,S_1,Z_0,I_0,B_0)\in\bar{\chi}.$$

Then

$$\rho\,(\Phi_t\,(S_0,S_1,Z_0,I_0,B_0))=I(t),\ \ \forall t\geq0\quad\text{and}\quad(S_0,S_1,Z_0,I_0,B_0)\in\bar{\chi}.$$

When $I(0)>0$, $I(t)>0$ for every $t>0$. Hence

$$\rho\,(\Phi_t\,(S_0,S_1,Z_0,I_0,B_0))>0\quad\text{whenever}\quad\rho\,(\Phi_0\,(S_0,S_1,Z_0,I_0,B_0))>0.$$

Observe that the map $\rho\circ\Phi$ is continuous, and Lemma 6.1 guarantees that Φ is uniformly weakly ρ-persistent. Therefore, it follows from [25, Theorem 5.2] that there exists $\bar{m}>0$ such that

$$\lim_{t\to\infty}\inf\rho\,(\Phi_t\,(S_0,S_1,Z_0,I_0,B_0))>\bar{m}\ \ \forall\,(S_0,S_1,Z_0,I_0,B_0)\in\bar{\chi}.$$

Since $\rho\,(\Phi_t\,(S_0,S_1,Z_0,I_0,B_0))=I(t)$ for $t>0$ and $(S_0,S_1,Z_0,I_0,B_0)\in\bar{\chi}$, then $\liminf_{t\to\infty} I(t)>\bar{m}$. This completes the proof. □

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