Generalized Variational Inclusion Problem Involving Averaged Operator and Cayley Operator

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Abstract In this paper, we consider and study a variational inclusion problem involving averaged and Cayley operators in the context of a Hilbert space. It is shown that our problem is equivalent to a fixed-point equation. We define an iterative algorithm based on a fixed-point formulation to obtain the solution. Our result is supported by a numerical example, computation tables, and a convergence graph are also provided.

Keywords Cayley operator, averaged operator, variational inclusion, resolved operator

MSC(2010) 40D25, 47A05, 47A10, 47A30.

1. Introduction

In recent years, the study of variational inclusion problems has received significant attention due to its many uses in a variety of disciplines, including engineering, economics, and optimization; for more details, see [6, 11, 13, 14]. Variational inclusions are generalizations of variational inequalities [4, 8]. These problems encompass a variety of mathematical models and provide a unified framework to address various equilibrium problems. By encompassing single-valued and multivalued mappings, variational inclusion extends the classical variational inequality framework, allowing greater flexibility and broader applicability.

In practical terms, variational inclusion has significant implications for the development of algorithms and computational methods. Researchers and practitioners leverage their principles to design efficient algorithms that handle large-scale and high-dimensional problems. As we delve deeper into variational inclusion, we will explore its theoretical properties, practical applications, and the various algorithms developed for its solution. Understanding variational inclusion improves our ability to tackle diverse problems and enriches the mathematical toolbox available for future innovations. For further generalization of variational inclusion, we refer to [1, 2, 9, 12].

Baillon et al. [5] introduced the concept of the averaged operator while dealing with the asymptotic behavior of nonexpansive mappings and semigroups. Average

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operators play a significant role in optimization, variational inequalities, and study-ing dynamical systems. Average operators are fundamental in studying iterative algorithms for fixed-points, optimization, and variational inequalities. Their prop-erties ensure convergence and stability in various applications [7, 10], making them a crucial tool in applied mathematics and computational sciences.

The resolvent operator technique, introduced by Hassouni and Moudafi [1] in 1994, is a powerful tool for analyzing variational inequalities involving maximal monotone mappings and single-valued mappings, termed variational inclusions. The Cayley operator [3], which incorporates a resolvent operator as introduced by Fang and Huang [15] in 2004, provides further insight into this area. Notably, the Cayley transform is renowned for mapping skew-symmetric matrices to special orthogonal matrices, a fundamental concept in linear algebra. In the context of Hilbert spaces, this transform maps linear operators in a manner that preserves essential structural properties. The Cayley transform is a homographic transformation that offers sig-nificant applications in various fields, including real analysis, complex analysis, and quaternionic analysis.

This paper explores the theoretical aspects of a generalized variational inclusion problem by establishing the existence and convergence of solutions. We construct a new iterative algorithm involving the averaged operator and the Cayley operator and provide a detailed analysis of its convergence properties. The proposed algorithm is tested through numerical examples to demonstrate its efficacy and practical utility using MATLAB R2024a.

2. Preparatory results

Throughout this paper, $\mathcal{H}$ is a real Hilbert space with its usual norm $\|.\|$ and the inner product $\langle \cdot, \cdot \rangle$. We denote by $2^{\mathcal{H}}$ the set of all non-empty subsets of $\mathcal{H}$.

Definition 2.1. A mapping $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is,

(i) Lipschitz continuous, if there exists a constant $\lambda > 0$ such that,

$$\|\mathcal{A}(\tilde{a}) - \mathcal{A}(\tilde{b})\| \leq \lambda \|\tilde{a} - \tilde{b}\|, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(ii) non-expansive,

$$\|\mathcal{A}(\tilde{a}) - \mathcal{A}(\tilde{b})\| \leq \|\tilde{a} - \tilde{b}\|, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(iii) monotone if,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \geq 0, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(iv) strictly monotone if $\mathcal{A}$ is monotone and,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle = 0 \text{ iff } \tilde{a} = \tilde{b};$$

(v) strongly monotone if there exists a constant $t > 0$ such that,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \geq t\|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(vi) relaxed Lipschitz continuous if there exists a constant $c > 0$ such that,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \leq -c\|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

Definition 2.2. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is called β-averaged if for $\beta \in (0,1)$

$$\mathcal{T} = (1 - \beta)I + \beta N,$$

where $N : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator and I is an identity operator.

Definition 2.3. A multi-valued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is,

(i) monotone if,

$$\langle \tilde{a} - \tilde{b}, \tilde{\tilde{a}} - \tilde{\tilde{b}} \rangle \geq 0 \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H} \ \text{ and } \ \tilde{\tilde{a}} \in M(\tilde{a}), \tilde{\tilde{b}} \in M(\tilde{b}),$$

(ii) maximal monotone if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$.

Definition 2.4. The resolvent operator $\mathcal{R}_{I,\lambda}^{M} : \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) = [I + \lambda M]^{-1}(\tilde{a}), \ \forall \ \tilde{a} \in \mathcal{H} \text{ and } \lambda > 0, \tag{2.1}$$

where I is an identity operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone multi-valued mapping.

Definition 2.5. The Cayley operator $\mathcal{C}_{I,\lambda}^{M} : \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{C}_{I,\lambda}^{M}(\tilde{a}) = [2\mathcal{R}_{I,\lambda}^{M} - I](\tilde{a}), \ \forall \ \tilde{a} \in \mathcal{H} \text{ and } \lambda > 0, \tag{2.2}$$

where $\mathcal{R}_{I,\lambda}^{M}$ is the resolvent operator defined by (2.1) and I is an identity operator.

Proposition 2.1. *[11] Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then the resolvent operator $\mathcal{R}_{I,\lambda}^{M} : \mathcal{H} \to \mathcal{H}$ defined by (2.1) is Lipschitz continuous with constant $\lambda_R > 0$ where $\lambda_R = \frac{1}{t}, t > 0$, that is,*

$$\left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| \leq \lambda_R \|\tilde{a} - \tilde{b}\|, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H}.$$

Proof. Let $\tilde{a}, \tilde{b} \in H$. Then, from (2.1), we have

$$\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) = [I + \lambda M]^{-1}(\tilde{a}),$$
$$\mathcal{R}_{I,\lambda}^{M}(\tilde{b}) = [I + \lambda M]^{-1}(\tilde{b}).$$

This implies that

$$\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) = [I + \lambda M]^{-1}(\tilde{a}),$$
$$\tilde{a} = \mathcal{R}_{I,\lambda}^{M}(\tilde{a})(I + \lambda M)$$
$$= I\left(\mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right) + \lambda M\left(\mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right),$$
$$\frac{1}{\lambda}\left(\tilde{a} - \mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right) = M\left(\mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right),$$
$$\frac{1}{\lambda}\left(\tilde{a} - \mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right) \in M\left(\mathcal{R}_{I,\lambda}^{M}(\tilde{a})\right).$$

Similarly we have,

$$\frac{1}{\lambda}\left(\tilde{b} - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})\right) \in M\left(\mathcal{R}_{I,\lambda}^{M}(\tilde{b})\right).$$

Since M is monotone,

$$\frac{1}{\lambda}\left\langle (\tilde{a} - \mathcal{R}_{I,\lambda}^{M}(\tilde{a})) - (\tilde{b} - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}), R_{I,\lambda}^{M}(\tilde{a})) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) \right\rangle$$
$$=\frac{1}{\lambda}\left\langle (\tilde{a} - \tilde{b}) - (\mathcal{R}_{I,\lambda}^{M}(\tilde{a})) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})), R_{I,\lambda}^{M}(\tilde{a})) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) \right\rangle \geq 0.$$

Now,

$$\|\tilde{a} - \tilde{b}\| \left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| \geq \left\langle \tilde{a} - \tilde{b}, (\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) \right\rangle$$
$$\geq \left\langle (\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})), (\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) \right\rangle$$
$$\geq t \left\| (\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) \right\|^{2},$$
$$\left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| \leq \frac{1}{t}\|\tilde{a} - \tilde{b}\|,$$
$$\left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| \leq \lambda_{R}\|\tilde{a} - \tilde{b}\|.$$

$\square$

Proposition 2.2. [11] *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then the Cayley operator $C_{I,\lambda}^{M} : \mathcal{H} \to \mathcal{H}$ defined by (2.2) is Lipschitz continuous with constant $\theta_1 > 0$, where $\theta_1 = (2\lambda_R + 1)$, $\lambda_R > 0$, that is,*

$$\left\| \mathcal{C}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{C}_{I,\lambda}^{M}(\tilde{b}) \right\| \leq \theta_1 \|\tilde{a} - \tilde{b}\| \ \forall \tilde{a}, \tilde{b} \in \mathcal{H}.$$

Proof.   Let $\tilde{a}, \tilde{b} \in H$. Then, from (2.2), we have ,

$$\left\| \mathcal{C}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{C}_{I,\lambda}^{M}(\tilde{b}) \right\| = \left\| (2\mathcal{R}_{I,\lambda}^{M} - I)(\tilde{a}) - (2\mathcal{R}_{I,\lambda}^{M} - I)(\tilde{b}) \right\|$$
$$= \left\| 2(\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b})) - I(\tilde{a}) + I(\tilde{b}) \right\|$$
$$\leq 2 \left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| + \left\| I(\tilde{a}) - I(\tilde{b}) \right\|$$
$$\leq 2 \left\| \mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{R}_{I,\lambda}^{M}(\tilde{b}) \right\| + \left\| \tilde{a} - \tilde{b} \right\|.$$

Since the resolvent operator is Lipschitz continous, we have

$$\left\| \mathcal{C}_{I,\lambda}^{M}(\tilde{a}) - \mathcal{C}_{I,\lambda}^{M}(\tilde{b}) \right\| \leq 2\lambda_R\|\tilde{a} - \tilde{b}\| + \|\tilde{a} - \tilde{b}\|$$
$$\leq (2\lambda_R + 1)\|\tilde{a} - \tilde{a}\|$$
$$\leq \theta_1\|\tilde{a} - \tilde{b}\|.$$

$\square$

Lemma 2.1. [10] *Let $\mathcal{T}$ be an averaged operator. Then for all $\tilde{a}, \tilde{b} \in \mathcal{H}$ and $\beta \in (0,1)$, we have*

$$\|\mathcal{T}(\tilde{a}) - \mathcal{T}(\tilde{b})\|^2 \leq \|\tilde{a} - \tilde{b}\|^2 - \left[\frac{1-\beta}{\beta}\right] \left\|(I - \mathcal{T})(\tilde{a}) - (I - \mathcal{T})(\tilde{b})\right\|^2.$$

Proposition 2.3. *[16] Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be an averaged operator. If $N : \mathcal{H} \to \mathcal{H}$ is relaxed Lipschitz continuous with constant $c > 0$, then $\mathcal{T}$ is also relaxed Lipschitz continuous with constant $\theta_c > 0$ where $\theta_c = [\beta(1 + c) - 1]$.*

Definition 2.6. A mapping $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be Lipschitz continuous with respect to first argument if there exists a constant $L > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\| \leq L\|x_1 - x_2\| \ \forall \ x_1, x_2, y_1, y_2 \in \mathcal{H}.$$

Similarly, we can define the Lipschitz continuity in other arguments.

3. Problem formulation and algorithm

Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a β-averaged operator where $\beta \in (0, 1)$ and $\mathcal{C}_{I,\lambda}^{M} : \mathcal{H} \to \mathcal{H}$ be the Cayley operator. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued maximal monotone mapping and $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a mapping. Now we consider the following variational inclusion problem involving the averaged operator and the Cayley operator:

Find $\tilde{a} \in \mathcal{H}$ such that

$$0 \in F\left(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a})\right) + M(\tilde{a}). \tag{3.1}$$

For a suitable choice of operators, we can obtain many previously studied variational inclusion problems available in the literature.

If $F\left(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a})\right) \equiv 0$, then (3.1) reduces to the problem of finding $\tilde{a} \in \mathcal{H}$ such that $0 \in M(\tilde{a})$ which is a fundamental inclusion problem in several areas of applied science. The following lemma is a fixed point formulation of problem (3.1).

Lemma 3.1. *The variational inclusion problem (3.1) involving averaged operator and Cayley operator has a solution, if and only if it satisfies the equation:*

$$\tilde{a} = \mathcal{R}_{I,\lambda}^{M}\left(\tilde{a} - \lambda F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right). \tag{3.2}$$

Proof. Let $\tilde{a} \in H$ satisfy equation (3.2). Then we have

$$\tilde{a} = \mathcal{R}_{I,\lambda}^{M}\left[\tilde{a} - \lambda F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right]$$

$$= [I + \lambda M]^{-1}\left[\tilde{a} - \lambda F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right],$$

$$[I + \lambda M](\tilde{a}) = \left[\tilde{a} - \lambda F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right]$$

$$= \left[\tilde{a} - \lambda F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right],$$

$$\lambda M(\tilde{a}) = \lambda\left[F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a}))\right]$$

$$0 \in F\left(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a})) + M(\tilde{a})\right).$$

A similar approach applies to the converse case. □

Applying the fixed-point formulation in (3.2), we produce the following iterative algorithm to solve the problem (3.1):

Algorithm 3.1. Let $\tilde{a}_0 \in \mathcal{H}$, compute the sequence $\{\tilde{a}_n\}$, using the following scheme:

$$\tilde{a}_{n+1} = (1-\rho)\tilde{a}_n + \rho\mathcal{R}_{I,\lambda}^M \left[\tilde{a}_n - \lambda F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))\right],$$

where $\lambda > 0$, $\rho \in [0,1]$ and $n = 0, 1, 2...$

4. Existence and convergence

We have demonstrated both the existence and convergence of solutions to the problem (3.1) in this section.

Theorem 4.1. *Let $\mathcal{H}$ be a real Hilbert space and $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a mapping that is α -Lipschitz continuous in the first argument and μ-Lipschitz in the second argument. Let $N : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping, where N is non-expansive and c -relaxed Lipschitz continuous. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone multivalued mapping and $\mathcal{T}$ is a β-averaged operator such that $\mathcal{T}$ is θ_c- relaxed Lipschitz continuous, and $\mathcal{C}_{I,\lambda}^M : \mathcal{H} \to \mathcal{H}$ be a Cayley operator. If the following condition is satisfied:*

$$0 < (1 - \rho + \lambda_R\rho + \lambda_R\lambda\rho\mu\theta_1) + \lambda_R\alpha\lambda\rho\sqrt{\beta_\theta}) < 1, \tag{4.1}$$

where $\beta \in (0,1)$, $0 \le \rho \le 1$, $\beta_\theta = \left[\frac{1-((1-\beta)/\beta)(1+2\theta_c))}{[1+((1-\beta)/\beta)}\right]$ and $\theta_c = [\beta(1+c)-1]$.
Then, the sequence $\{\tilde{a}_n\}$ generated by the iterative algorithm (3.1) converges to $\tilde{a} \in \mathcal{H}$, which is the solution of the variational inclusion problem involving the averaged operator and the Cayley operator.

Proof.   Applying Algorithm 3.1, we evaluate

$$\|\tilde{a}_{n+1} - \tilde{a}_n\| = \|[(1-\rho)\tilde{a}_n - \rho\mathcal{R}_{I,\lambda}^M[\tilde{a}_n - \lambda F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))]]$$

$$-[(1-\rho)\tilde{a}_{n-1} - \rho\mathcal{R}_{I,\lambda}^M[\tilde{a}_{n-1} - \lambda F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))]]\|$$

$$\le (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| - \rho\|[\mathcal{R}_{I,\lambda}^M[\tilde{a}_n - \lambda F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))]$$

$$-[\mathcal{R}_{I,\lambda}^M[\tilde{a}_{n-1} - \lambda F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))]\|. \tag{4.2}$$

Using Lipschitz continuity of resolvent operator $\mathcal{R}_{I,\lambda}^M$ and Lipschitz continutity of F in both arguments, we have

$$\|\tilde{a}_{n+1} - \tilde{a}_n\|$$

$$= (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R\rho\|[\tilde{a}_n - \lambda F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n)]$$

$$-[\tilde{a}_{n-1} - \lambda F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})]\|$$

$$\le (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R\rho\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R\lambda\rho\|F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))$$

$$-F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})) + F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})) - F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))\|$$

$$\le (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R\rho\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R\lambda\rho\|F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))$$

$$-F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})) + F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})) - F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))\|$$

$$\leq (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \rho\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \lambda \rho\|F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_n))$$
$$-F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))\| + \lambda_R \lambda \rho\|F(\mathcal{T}(\tilde{a}_n), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))$$
$$-F(\mathcal{T}(\tilde{a}_{n-1}), \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1}))\|$$
$$\leq (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \rho\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \lambda \rho\mu\|\mathcal{C}_{I,\lambda}^M(\tilde{a}_n) - \mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})\|$$
$$+\lambda_R \lambda \rho\alpha\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|.$$
$$(4.3)$$

Using Lemma 2.1, we have

$$\|(I-\mathcal{T})(\tilde{a}_n) - (I-\mathcal{T})(\tilde{a}_{n-1})\|^2 \leq \|\tilde{a}_n - \tilde{a}_{n-1}\|^2 - 2\langle \tilde{a}_n - \tilde{a}_{n-1}, \mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\rangle$$
$$+\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2. \qquad (4.4)$$

Using relaxed Lipschitz continuity of $\mathcal{T}$ and Lemma 2.1, (4.4) becomes

$$\|(I-\mathcal{T})(\tilde{a}_n) - (I-\mathcal{T})(\tilde{a}_{n-1})\|^2$$
$$\geq \|\tilde{a}_n - \tilde{a}_{n-1}\|^2 + 2\theta_c\|\tilde{a}_n - \tilde{a}_{n-1}\|^2$$
$$+\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2$$
$$\geq (1+2\theta_c)\|\tilde{a}_n - \tilde{a}_{n-1}\|^2 + \|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2$$
$$-\frac{(1-\beta)}{\beta}\|(I-\mathcal{T})(\tilde{a}_n) - (I-\mathcal{T})(\tilde{a}_{n-1})\|^2$$
$$\leq -\frac{(1-\beta)}{\beta}(1+2\theta_c)\|\tilde{a}_n - \tilde{a}_{n-1}\|^2$$
$$-\frac{(1-\beta)}{\beta}\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2.$$

Therefore,

$$\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2 \leq \|\tilde{a}_n - \tilde{a}_{n-1}\|^2 - \frac{(1-\beta)}{\beta}(1+2\theta_c)\|\tilde{a}_n - \tilde{a}_{n-1}\|^2$$
$$-\frac{(1-\beta)}{\beta}\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|^2$$
$$\leq \left[\frac{1-((1-\beta)/\beta)(1+2\theta_c))}{[1+((1-\beta)/\beta)}\right]\|\tilde{a}_n - \tilde{a}_{n-1}\|^2$$
$$\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\| \leq \sqrt{\left[\frac{1-((1-\beta)/\beta)(1+2\theta_c))}{[1+((1-\beta)/\beta)}\right]}\|\tilde{a}_n - \tilde{a}_{n-1}\|. \qquad (4.5)$$

Let $\beta_\theta = \left[\frac{1-((1-\beta)/\beta)(1+2\theta_c))}{[1+((1-\beta)/\beta)}\right]$. Therefore, from (4.5) we have,

$$\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\| \leq \sqrt{\beta_\theta}\|\tilde{a}_n - \tilde{a}_{n-1}\|.$$

Using (4.5) and Lipschitz continuity of the Cayley operator, we obtain

$$\|\tilde{a}_{n+1} - \tilde{a}_n\| \leq (1-\rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \rho\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \lambda \rho\mu\|\mathcal{C}_{I,\lambda}^M(\tilde{a}_n)$$
$$-\mathcal{C}_{I,\lambda}^M(\tilde{a}_{n-1})\| + \lambda_R \alpha \lambda \rho\|\mathcal{T}(\tilde{a}_n) - \mathcal{T}(\tilde{a}_{n-1})\|$$

$$\leq (1 - \rho + \lambda_R \rho)\|\tilde{a}_n - \tilde{a}_{n-1}\| + \lambda_R \lambda \rho \mu \theta_1 \|\tilde{a}_n - \tilde{a}_{n-1}\|$$
$$+ \lambda_R \alpha \lambda \rho \sqrt{\beta_\theta} \|\tilde{a}_n - \tilde{a}_{n-1}\|$$
$$\leq (1 - \rho + \lambda_R \rho + \lambda_R \lambda \rho \mu \theta_1 + \lambda_R \alpha \lambda \rho \sqrt{\beta_\theta})\|\tilde{a}_n - \tilde{a}_{n-1}\|.$$

Let $\Theta = (1 - \rho + \lambda_R \rho + \lambda_R \lambda \rho \mu \theta_1) + \lambda_R \alpha \lambda \rho \sqrt{\beta_\theta})$. Therefore

$$\|\tilde{a}_{n+1} - \tilde{a}_n\| \leq \Theta |\tilde{a}_n - \tilde{a}_{n-1}\|.$$

From condition (4.1), we have $0 < \Theta < 1$, and it follows that $\{\tilde{a}_n\}$ is a Cauchy sequence. Since $\mathcal{H}$ is complete, thus $\tilde{a}_n \to \tilde{a}$, which is the solution of the variational inclusion problem involving the averaged operator and the Cayley operator (3.1).

$\square$

5. Numerical example

As validation of Theorem 4.1 and Algorithm 3.1, we construct a numerical example and present its results through a computational table and a convergence graph generated in MATLAB R2024a.

Let $\mathcal{H} = \mathbb{R}^2$ with usual inner product and $M : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be a multi-valued mapping and $\mathcal{T}$ is an averaged operator and N is a single valued mapping such that

$$M(\tilde{a}) = \left\{ \left(\frac{5a_1}{4}, \frac{5a_2}{4} \right) \ \forall \ \tilde{a} = (a_1, a_2) \in \mathbb{R}^2 \right\},$$
$$N(\tilde{a}) = \left(-\frac{a_1}{3}, -\frac{a_2}{3} \right).$$

(i) N is non-expansive that is for all $\tilde{a} = (a_1, a_2)$, $\tilde{b} = (b_1, b_2) \in \mathbb{R}^2$ and using the Euclidean norm, we have

$$\|N(\tilde{a}) - N(\tilde{b})\| = \left\| \left(-\frac{a_1}{3}, -\frac{a_2}{3} \right) - \left(-\frac{b_1}{3}, -\frac{b_2}{3} \right) \right\|$$
$$= \left| -\frac{1}{3} \right| \|(a_1 - b_1), (a_2 - b_2)\|$$
$$\leq \left| -\frac{1}{3} \right| \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$
$$\leq \left| -\frac{1}{3} \right| \|\tilde{a} - \tilde{b}\|$$
$$\leq \|\tilde{a} - \tilde{b}\|.$$

(ii) N is c-relaxed Lipschitz continuous

$$\langle N(\tilde{a}) - N(\tilde{b}), \tilde{a} - \tilde{b} \rangle = \left\langle \left(-\frac{\tilde{a}}{3} - \left(-\frac{\tilde{b}}{3} \right) \right), \tilde{a} - \tilde{b} \right\rangle$$
$$= -\frac{1}{3} \langle \tilde{a} - \tilde{b}, \tilde{a} - \tilde{b} \rangle$$
$$\leq -\frac{1}{4} \|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathbb{R}^2.$$

(iii) For $\beta \in (0,1)$ and $\tilde{a} = (a_1, a_2)$, we calculate averaged operator $\mathcal{T}$ as

$$\begin{aligned}
\mathcal{T}(\tilde{a}) &= [(1-\beta)I + \beta N](\tilde{a}) \\
&= (1-\beta)I(\tilde{a}) + \beta N(\tilde{a}) \\
&= (1-\beta)\tilde{a} - \frac{\beta \tilde{a}}{3} \\
&= \left(1 - \beta - \frac{\beta}{3}\right)(\tilde{a}) \\
&= \left(1 - \frac{4\beta}{3}\right)(\tilde{a}).
\end{aligned}$$

(5.1)

(iv) $\mathcal{T}$ is θ_c-relaxed Lipschitz continous, where $c = \frac{1}{4}$ and $\theta_c = \left(\frac{5\beta}{4} - 1\right)$

$$\begin{aligned}
\langle T(\tilde{a}) - T(\tilde{b}), \tilde{a} - \tilde{b} \rangle &= \left\langle \left(1 - \frac{4\beta}{3}\right)\tilde{a} - \left(1 - \frac{4\beta}{3}\right)\tilde{b}, \tilde{a} - \tilde{b} \right\rangle \\
&= \left(1 - \frac{4\beta}{3}\right)\langle \tilde{a} - \tilde{b}, \tilde{a} - \tilde{b} \rangle \\
&= \left(1 - \frac{4\beta}{3}\right)\|\tilde{a} - \tilde{b}\|^2 \\
&\leq \left(\frac{5\beta}{4} - 1\right)\|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathbb{R}^2.
\end{aligned}$$

(v) For $\lambda = 1$ and $\tilde{a} = (a_1, a_2)$, we calculate the Resolvent operator $\mathcal{R}_{I,\lambda}^{M}$ as:

$$\begin{aligned}
\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) &= [I(\tilde{a}) + \lambda M(\tilde{a})]^{-1} \\
&= [\tilde{a} + \frac{5\tilde{a}}{4}]^{-1} \\
&= \left[\frac{9}{4}\right]^{-1}(\tilde{a}) \\
&= \frac{4\tilde{a}}{9}.
\end{aligned}$$

(5.2)

(vi) For $\tilde{a} = (a_1, a_2)$, we calculate Cayley operator $\mathcal{C}_{I,\lambda}^{M}$ as:

$$\begin{aligned}
\mathcal{C}_{I,\lambda}^{M}(\tilde{a}) &= 2\mathcal{R}_{I,\lambda}^{M}(\tilde{a}) - I(\tilde{a}) \\
&= 2 \times \frac{4}{9}\tilde{a} - \tilde{a} \\
&= -\frac{\tilde{a}}{9}.
\end{aligned}$$

(5.3)

(vii) Let $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that F is Lipschitz continuous in the first argument and the second argument respectively. We define F as follows:

$$F(\tilde{a}, \tilde{b}) = \tilde{a} + \tilde{b} \text{ where } \tilde{a} = (a_1, a_2), \ \tilde{b} = (b_1, b_2) \in \mathbb{R}^2.$$

Then F is Lipschitz continuous in the first argument, for $\tilde{a} = (a_1, a_2)$, $\tilde{b} = (b_1, b_2)$, $\tilde{t} = (t_1, t_2)$ in $\mathbb{R}^2$ and using (5.1), we evaluate

$$\|F(\mathcal{T}(\tilde{a}), \mathcal{C}_{I,\lambda}^{M}(\tilde{t})) - F(\mathcal{T}(\tilde{b}), \mathcal{C}_{I,\lambda}^{M}(\tilde{t}))\| = \|(\mathcal{T}(\tilde{a}) + \mathcal{C}_{I,\lambda}^{M}(\tilde{t})) - (T(\tilde{b}) + C_{I,\lambda}^{M}(\tilde{t})\|$$
$$\leq \|\mathcal{T}(\tilde{a}) - \mathcal{T}(\tilde{b})\|$$
$$\leq \left\|\left(1 - \frac{4\beta}{3}\right)(\tilde{a}) - \left(1 - \frac{4\beta}{3}\right)(\tilde{b})\right\|$$
$$\leq \left(1 - \frac{4\beta}{3}\right)\|\tilde{a} - \tilde{b}\|, \quad \text{for } \beta \in (0,1).$$

(viii) F is Lipschitz continuous in the second argument, using (5.3) we evaluate

$$\|F(\mathcal{T}(\tilde{t}), \mathcal{C}_{I,\lambda}^{M}(\tilde{a})) - F(\mathcal{T}(\tilde{t}), \mathcal{C}_{I,\lambda}^{M}(\tilde{b}))\| = \|(\mathcal{T}(\tilde{t}) + \mathcal{C}_{I,\lambda}^{M}(\tilde{a})) - (\mathcal{T}(\tilde{t}) + \mathcal{C}_{I,\lambda}^{M}(\tilde{b})\|$$
$$\leq \|\mathcal{C}_{I,\lambda}^{M}(\tilde{a})) - \mathcal{C}_{I,\lambda}^{M}(\tilde{b}))\|$$
$$\leq \left\|\frac{-\tilde{a}}{9} - \left(-\frac{\tilde{b}}{9}\right)\right\|$$
$$\leq \frac{1}{9}\|\tilde{a} - \tilde{b}\|.$$

Now using the definitions of resolvent operator $\mathcal{R}_{I,\lambda}^{M}$, averaged operator $\mathcal{T}$ and Algorithm 3.1, $\tilde{t}_{n+1} = (\tilde{a}_{n+1}, \tilde{b}_{n+1})$ is calculated below:

$$\tilde{t}_{n+1} = (1 - \rho)\tilde{t}_n + \rho\mathcal{R}_{I,\lambda}^{M}[\tilde{t}_n - \lambda F(\mathcal{T}(\tilde{t}_n), \mathcal{C}_{I,\lambda}^{M}(\tilde{t}_n))]$$
$$= (1 - \rho)\tilde{t}_n + \rho\mathcal{R}_{I,\lambda}^{M}[\tilde{t}_n - 1.(\mathcal{T}(\tilde{t}_n) + \mathcal{C}_{I,\lambda}^{M}(\tilde{t}_n))]$$
$$= (1 - \rho)\tilde{t}_n + \rho\mathcal{R}_{I,\lambda}^{M}\left[\tilde{t}_n - \left(\left(1 - \frac{4\beta}{3}\right)\tilde{t}_n - \frac{\tilde{t}_n}{9}\right)\right]$$
$$= (1 - \rho)\tilde{t}_n + \rho\frac{4}{9}\left[\tilde{t}_n - \left(\frac{8}{9} - \frac{4\beta}{3}\right)\tilde{t}_n\right]$$
$$= \left[(1 - \rho) + \rho\frac{4}{9}\left[1 - \left(\frac{8}{9} - \frac{4\beta}{3}\right)\right]\right]\tilde{t}_n$$
$$= \left[(1 - \rho) + \rho\frac{4}{9}\left[\frac{1}{9} + \frac{4\beta}{3}\right]\right]\tilde{t}_n$$
$$= \left[1 - \frac{77}{81}\rho + \frac{16}{27}\rho\beta\right]\tilde{t}_n,$$

for all $n \geq 0$ and $0 \leq \rho \leq 1$ and $\beta \in (0,1)$. Therefore, we have

$$(\tilde{a}_{n+1}, \tilde{b}_{n+1}) = \left[1 - \frac{77}{81}\rho + \frac{16}{27}\rho\beta\right](\tilde{a}_n, \tilde{b}_n).$$

The following tables show the convergence behavior of the iterative sequence with values of $\beta = [0.1, 0.3]$, $\rho = [0.3, 0.7]$ and initial values $(a_0, b_0) = [1, 0.5; 2, 1]$.

Table 1. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (1, 0.5)$ for $\beta = 0.1$ and $\rho = 0.3$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.3	0.1	1	0.5	1	0.732592592592593	0.366296296296296
0.3	0.1	1	0.5	5	0.211014653711299	0.10550732685565
0.3	0.1	1	0.5	10	0.0445271840808995	0.0222635920404497
0.3	0.1	1	0.5	15	0.00939588832957028	0.00469794416478514
0.3	0.1	1	0.5	20	0.00198267012217431	0.000991335061087155

Table 2. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (1, 0.5)$ for $\beta = 0.3$ and $\rho = 0.3$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.3	0.3	1	0.5	1	0.768148148148148	0.384074074074074
0.3	0.3	1	0.5	5	0.267439123027047	0.133719561513524
0.3	0.3	1	0.5	10	0.0715236845254762	0.0357618422627381
0.3	0.3	1	0.5	15	0.0191282314651566	0.00956411573257828
0.3	0.3	1	0.5	20	0.00511563744809985	0.00255781872404992

Table 3. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (1, 0.5)$ for $\beta = 0.1$ and $\rho = 0.7$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.7	0.1	1	0.5	1	0.376049382716049	0.188024691358025
0.7	0.1	1	0.5	5	0.00752011359040731	0.003760056795520365
0.7	0.1	1	0.5	10	$5.65521084126286e^{-5}$	$2.82760542063143e^{-5}$
0.7	0.1	1	0.5	15	$4.25278279039996e^{-7}$	$2.12639139519998e^{-7}$
0.7	0.1	1	0.5	20	$3.1981409659137e^{-9}$	$1.59907048295685e^{-9}$

Table 4. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (1, 0.5)$ for $\beta = 0.3$ and $\rho = 0.7$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.7	0.3	1	0.5	1	0.459012345679012	0.229506172839506
0.7	0.3	1	0.5	5	0.020376136100098	0.010188068050049
0.7	0.3	1	0.5	10	0.000415186922369715	0.000207593461184858
0.7	0.3	1	0.5	15	$8.45990523718613e^{-6}$	$4.22995261859306e^{-6}$
0.7	0.3	1	0.5	20	$1.72380180506836e^{-7}$	8.6190090253418E-08

Table 5. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (2, 1)$ for $\beta = 0.1$ and $\rho = 0.3$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.3	0.1	2	1	1	1.46518518518519	0.732592592592593
0.3	0.1	2	1	5	0.422029307422598	0.211014653711299
0.3	0.1	2	1	10	0.089054368161799	0.0445271840808995
0.3	0.1	2	1	15	0.0187917766591406	0.00939588832957028
0.3	0.1	2	1	20	0.00396534024434862	0.00198267012217431

Table 6. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (2, 1)$ for $\beta = 0.3$ and $\rho = 0.3$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.3	0.3	2	1	1	1.5362962962963	0.768148148148148
0.3	0.3	2	1	5	0.534878246054095	0.267439123027047
0.3	0.3	2	1	10	0.143047369050952	0.0715236845254762
0.3	0.3	2	1	15	0.0382564629303131	0.0191282314651566
0.3	0.3	2	1	20	0.0102312748961997	0.00511563744809985

Table 7. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (2, 1)$ for $\beta = 0.1$ and $\rho = 0.7$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.7	0.1	2	1	1	0.752098765432099	0.376049382716049
0.7	0.1	2	1	5	0.0150402271808146	0.00752011359040731
0.7	0.1	2	1	10	0.000113104216825257	$5.65521084126286e^{-5}$
0.7	0.1	2	1	15	$8.50556558079992e^{-7}$	$4.25278279039996e^{-7}$
0.7	0.1	2	1	20	$6.39628193182741e^{-9}$	$3.1981409659137e^{-9}$

Table 8. Computational table for initial values $(\tilde{a}_0, \tilde{b}_0) = (2, 1)$ for $\beta = 0.3$ and $\rho = 0.7$

ρ	β	a_0	b_0	No. of iteration	a_n	b_n
0.7	0.3	2	1	1	0.918024691358025	0.459012345679012
0.7	0.3	2	1	5	0.0407522722001959	0.020376136100098
0.7	0.3	2	1	10	0.000830373844739431	0.000415186922369715
0.7	0.3	2	1	15	$1.69198104743723e^{-5}$	$8.45990523718613e^{-6}$
0.7	0.3	2	1	20	$3.44760361013672e^{-7}$	$1.72380180506836e^{-7}$

From the tables above, we observe that the iterative sequence $\{a_n, b_n\}$ converges

to $(0,0)$ after a finite number of iterations in each case; specifically, after the 20th iteration in Tables 1, 2, and 5; the 9th iteration in Table 3; the 14th in Table 4; the 22nd in Table 6; the 13th in Table 7; and the 15th in Table 8.
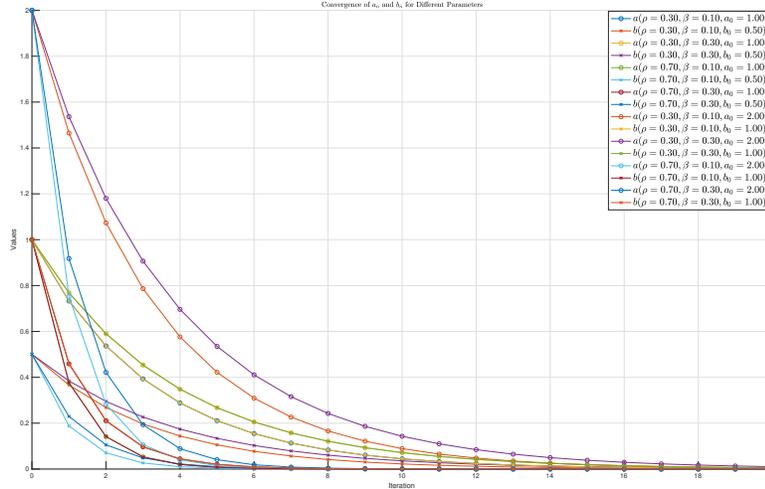
Figure 1. Convergence of the sequence $\{(\tilde{a}_n, \tilde{b}_n)\}$ with different initial values and parameter(ρ, β) represented graphically.

Conclusions

It is well known that the Cayley operator, the resolvent operator, and the systems of variational inclusions have wide applications in various fields. This paper focuses on the study of a generalized variational inclusion problem involving an averaged operator and the Cayley operator in a Hilbert space. We analyze the convergence properties by proposing an iterative algorithm, supported by a numerical example.

The novelty of this work lies in the fact that our results refine and generalize several existing results (see, for instance, [9], [16], [11]). In addition, the findings presented here may serve as a basis for further extensions and could be of interest to other researchers working in related areas.

Declarations

- **Ethics approval and consent to participate**
  Not applicable.

- **Consent for publication**
  Not applicable.

- **Data availability**
  No data were used to support this study.

- **Competing interests**
  The authors declare no competing interests.

- **Funding**
  There is no funding for this paper.

- **Author contribution**
  All the authors contributed equally to prepare the manuscript. All authors reviewed the manuscript.

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