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Generalized Variational Inclusion Problem Involving Averaged Operator and Cayley Operator

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Abstract In this paper, we consider and study a variational inclusion problem involving averaged and Cayley operators in the context of a Hilbert space. It is shown that our problem is equivalent to a fixed-point equation. We define an iterative algorithm based on a fixed-point formulation to obtain the solution. Our result is supported by a numerical example, computation tables, and a convergence graph are also provided.

Keywords Cayley operator, averaged operator, variational inclusion, resolved operator

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1. Introduction

In recent years, the study of variational inclusion problems has received significant attention due to its many uses in a variety of disciplines, including engineering, economics, and optimization; for more details, see [6, 11, 13, 14]. Variational inclusions are generalizations of variational inequalities [4, 8]. These problems encompass a variety of mathematical models and provide a unified framework to address various equilibrium problems. By encompassing single-valued and multivalued mappings, variational inclusion extends the classical variational inequality framework, allowing greater flexibility and broader applicability.

In practical terms, variational inclusion has significant implications for the development of algorithms and computational methods. Researchers and practitioners leverage their principles to design efficient algorithms that handle large-scale and high-dimensional problems. As we delve deeper into variational inclusion, we will explore its theoretical properties, practical applications, and the various algorithms developed for its solution. Understanding variational inclusion improves our ability to tackle diverse problems and enriches the mathematical toolbox available for future innovations. For further generalization of variational inclusion, we refer to [1, 2, 9, 12].

Baillon et al. [5] introduced the concept of the averaged operator while dealing with the asymptotic behavior of nonexpansive mappings and semigroups. Average

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operators play a significant role in optimization, variational inequalities, and studying dynamical systems. Average operators are fundamental in studying iterative algorithms for fixed-points, optimization, and variational inequalities. Their properties ensure convergence and stability in various applications [7,10], making them a crucial tool in applied mathematics and computational sciences.

The resolvent operator technique, introduced by Hassouni and Moudafi [1] in 1994, is a powerful tool for analyzing variational inequalities involving maximal monotone mappings and single-valued mappings, termed variational inclusions. The Cayley operator [3], which incorporates a resolvent operator as introduced by Fang and Huang [15] in 2004, provides further insight into this area. Notably, the Cayley transform is renowned for mapping skew-symmetric matrices to special orthogonal matrices, a fundamental concept in linear algebra. In the context of Hilbert spaces, this transform maps linear operators in a manner that preserves essential structural properties. The Cayley transform is a homographic transformation that offers significant applications in various fields, including real analysis, complex analysis, and quaternionic analysis.

This paper explores the theoretical aspects of a generalized variational inclusion problem by establishing the existence and convergence of solutions. We construct a new iterative algorithm involving the averaged operator and the Cayley operator and provide a detailed analysis of its convergence properties. The proposed algorithm is tested through numerical examples to demonstrate its efficacy and practical utility using MATLAB R2024a.

2. Preparatory results

Throughout this paper, $\mathcal{H}$ is a real Hilbert space with its usual norm $||.||$ and the inner product $\langle \cdot, \cdot \rangle$. We denote by $2^{\mathcal{H}}$ the set of all non-empty subsets of $\mathcal{H}$.

Definition 2.1. A mapping $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is,

(i) Lipschitz continuous, if there exists a constant $\lambda > 0$ such that,

$$\|\mathcal{A}(\tilde{a}) - \mathcal{A}(\tilde{b})\| \leq \lambda \|\tilde{a} - \tilde{b}\|, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(ii) non-expansive,

$$\|\mathcal{A}(\tilde{a}) - \mathcal{A}(\tilde{b})\| \leq \|\tilde{a} - \tilde{b}\|, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(iii) monotone if,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \geq 0, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(iv) strictly monotone if $\mathcal{A}$ is monotone and,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle = 0 \text{ iff } \tilde{a} = \tilde{b};$$

(v) strongly monotone if there exists a constant $t > 0$ such that,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \geq t \|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$

(vi) relaxed Lipschitz continuous if there exists a constant $c > 0$ such that,

$$\langle \mathcal{A}\tilde{a} - \mathcal{A}\tilde{b}, \tilde{a} - \tilde{b} \rangle \leq -c \|\tilde{a} - \tilde{b}\|^2, \ \forall \ \tilde{a}, \tilde{b} \in \mathcal{H};$$