On the Numerical Solution of Solidification of Water*

Abdellatif Boureghda[1,2,†] and Soumaya Belabbes[1]

Abstract The objective of this paper is to find an economical computer algorithm using a third-degree polynomial for the free boundary problem corresponding to phase change problem. We solve the problem to obtain the approximate analytical expression for the temperature distribution and the movement of the interface using the constrained integral method. Results are validated against existing numerical methods, demonstrating strong agreement with other techniques.

Keywords Constrained integral method, melting and solidification, moving boundary problem, Stefan problem, phase change, ice bank

MSC(2010) 35R35, 35R37, 80A22, 65M06, 65N06.

1. Introduction

Many more phenomena can be described by a Stefan problem. For instance, the decrease of oxygen in a muscle in the vicinity of a clotted blood vessel, the freezing food, lubrication, diffusion of gas, liquids, liquid-gas. Moving boundary problems (MBPs) of Stefan-Type arise in a range of physical applications, notably in the analysis of melting and solidification processes in solids [1, 11, 27].

Problems in which the solution of a differential equation must satisfy specific conditions on the boundary are called boundary-value problems. In a Stefan problem, this boundary is an unknown function of time, which must be determined concurrently with the solution of the differential equation. In these cases, we need two boundary conditions: one to determine the position of the moving boundary, and the other to complete the solution of differential equation.

We know that this kind of problem is called a Stefan problem, in reference to the work of J.Stefan, who around 1889 studied the melting of the polar ice cap [26].

†the corresponding author.

Emailaddress:abdellatif.boureghda@univ-setif.dz(A.Boureghda),

soumaya.belabbes@univ-setif.dz (S. Belabbes)

¹Laboratory of Fundamental and Numerical Mathematics, Faculty of Sciences, Ferhat Abbas University Setif 19137, Algeria.

²Department of Mathematics, IRIMAS Haute Alsace University, Faculty of Sciences and Techniques 6, rue des Frères Lumière 68093 Mulhouse Cedex France.

We note that the only difference between various Stefan problems lies in their governing equations; the fundamental idea of the solution methods remains the same. Moving boundary problems have received significant attention in recent years due to their practical importance in engineering and science, see J.R.Ockendon and W.R.Hodgkins [23] and [1,20,21]. These problems are inherently non-linear because of the moving boundary's presence, see Crank [12], Belabbes & Boureghda [2] and Boureghda [3–9]; and for this reason their analytical solutions are different. Many other authors have dealt with the moving boundary problems by various methods [15,22,28].

More references relating to the phase change problem involving a moving boundary can be found in the popular solidification problem, originally discussed by Rubinstein [25], selected for the analysis and in one of the oldest variable time step methods is due to Douglas and Gallie [14]. The same problem has been solved by George and Damle [17] using the method of lines, by Finn and Varoglu [16] using finite elements based on variational formulations, and by Gupta and Banik [18] using the Constrained integral method. Conducting exhaustive research on moving boundaries across several areas presents the challenge of fostering strong interdisciplinary collaboration between mathematics and other fields. For more details, see [10,11,13,19,23].

The aim of this work is to provide a detailed study of solving a moving boundary problem using the conventional integral method. This method yields two simultaneous first-order differential equations, which implicitly determine the position of the moving boundary and the value of an unknown additional parameter. The computed values from the expressions seem to be in very good agreement with those obtained by earlier authors using different numerical techniques.

2. Physical model and boundary conditions

The focus of this work is the problem concerning heat transfer in an ice-water medium occupying the region $0 \leq x \leq 1.0$. For simplicity, at any time t, the water, undergoing phase change, is contained in the region $s_1(t) \leq x \leq s_2(t)$ and the rest of the region outside is occupied by ice.

We suppose that $s_1(0) = 0.25$ and $s_2(0) = 0.75$ and the temperature of ice is linear in each of the two regions in which it lies. The water temperature is assumed to be equal to the critical phase change temperature, which is zero. Figure 1 shows the zero temperature temporal state of the ice-water regions as well as the positions of the ice/water interface. At the boundaries $x = 0$ and $x = 1$ the temperature is maintained at a unit negative temperature throughout. Because of the symmetry about $x = 0.5$, the problem reduces to finding its solution in the region $0 \leq x \leq 0.5$ only.
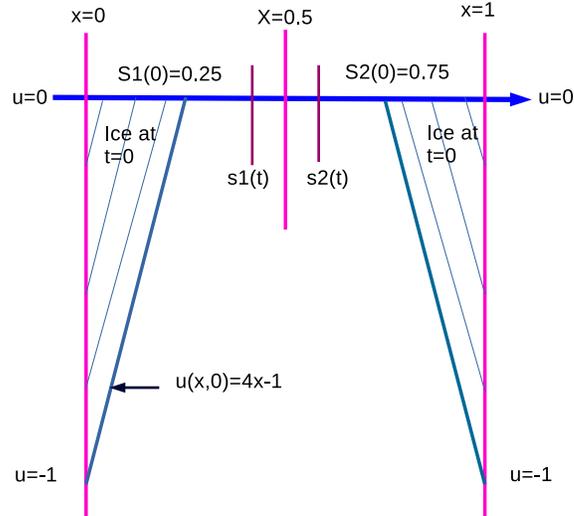
Figure 1. Representation of the initial temperature in ice-water region

Expressed in non-dimensional form, the problem can be defined mathematically by the following differential equation.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < s(t), \qquad t > 0, \tag{2.1}$$

the boundary condition

$$u = 0, \qquad x \geqslant s(t), \qquad t > 0, \tag{2.2}$$

$$u = -1 \qquad at \quad the \quad surface \quad x = 0 \qquad t > 0, \tag{2.3}$$

$$\frac{ds}{dt} = \frac{\partial u}{\partial x} \qquad at \quad the \quad moving \quad boundary \quad x = s(t) \qquad t > 0, \tag{2.4}$$

and the initial conditions at $t = 0$ are

$$s(0) = 0.25, \tag{2.5}$$

$$u(x, 0) = \begin{cases} 4x - 1, \, 0 \leqslant x \leqslant 0.25, \\ 0, \qquad 0.25 \leqslant x \leqslant 0.5. \end{cases} \tag{2.6}$$

We note that $u(x,t)$ is the temperature at a distance x from the fixed surface $x = 0$, at time t, and the location of the moving boundary is $s(t)$.

3. Method of solution

In this paper, let us assume that the temperature profile water satisfying condition (2.2) is given by a new polynomial of the third degree as follows.

3.1. Polynomial profile of third degree for the temperatures in ice

Choosing a polynomial profile of third degree in the spatial variable x containing four unknown parameters which are in general functions of time as follows

$$u(x,t) = \alpha_1 \left(\frac{s-x}{s}\right)^3 + \alpha_2 \left(\frac{s-x}{s}\right)^2 + \alpha_3 \left(\frac{s-x}{s}\right) + \alpha_4, \quad 0 < x < s(t), \quad (3.1)$$

which are determined using (2.2) and (2.3). In order to determine the unknowns, let us derive an extra condition as before at the interface by differentiating (2.2) with respect to t. Using (2.1) and (2.4), this gives

$$\left(\frac{\partial u}{\partial x}(s,t)\right)^2 = -\frac{\partial^2 u}{\partial x^2}(s,t). \quad (3.2)$$

In this problem, the surface gradient is unknown and temperature is prescribed at the fixed surface in accordance with (2.2).

We put a constraint on the gradient on the surface. After substituting the values of the derivatives from (3.1), we get α_1 and α_2 expressed in term of α_3. Thus using conditions (2.3) and (3.2) in (3.1) we get $\alpha_1 = \dfrac{\alpha_3^2 - 2\alpha_3 - 2}{2}, \alpha_2 = \dfrac{-\alpha_3^2}{2}$ and $\alpha_4 = 0$.

Further from (3.1) we see that

$$\frac{\partial u}{\partial x}(0,t) = \frac{1}{2s}\left(-\alpha_3^2 + 4\alpha_3 + 6\right). \quad (3.3)$$

Using (2.6) we get $\alpha_3 = -0.8284$ since $\dfrac{\partial u}{\partial x}$ remains positive at $x = s$ and the temperature profile (3.1) can now be written as

$$u(x,s) = \left(-1 + \frac{\alpha_3^2}{2} - \alpha_3\right)\left(\frac{s-x}{s}\right)^3 - \frac{\alpha_3^2}{2}\left(\frac{s-x}{s}\right)^2 + \alpha_3\left(\frac{s-x}{s}\right), \quad 0 < x < s(t). \quad (3.4)$$

Thus differentiating (3.4) with respect to x at $x = s$ we get from (2.4)

$$\frac{ds}{dt} = \frac{-\alpha_3}{s} \quad (3.5)$$

which after integration and making use of (2.5), gives

$$t = 0.0377\left(16s^2 - 1\right). \quad (3.6)$$

We choose a suitable factor 0.04 instead of 0.0377 on the right side of (3.6) and the corresponding $s - t$ relation, by

$$s(t) = \sqrt{\frac{t + 0.04}{0.64}}. \tag{3.7}$$

with this change the corresponding value of α_3 becomes -0.78125, thus obtaining an empirical relation for the movement of the interface.

4. Conclusions

It is known that moving boundary problems of Stefan-type arise in a range of physical applications in the analysis of melting and solidification processes in solids [12,24,25,27]. Many of these problems have analytical and numerical difficulties such as the determination of a moving boundary. Their complexity arises, in particular, in the heat balance equation at the moving interface that separates the phases, resulting in a non-linear boundary condition.

In this work we observe that an approximate analytical expressions are also obtained for the temperature distribution as well as for the movement of the interface using the integral method and by choosing the third order polynomial.

According to the initial position of the moving boundary $s(0) = 0.25$, condition (2.5) and conditions (3.6) and (3.7), we have calculated the temperature distribution $u(x, s)$ (3.4) at time t and numerical values of s and u have been computed for $\delta t = 0.001$.

Table 1 gives the figures from the present method for temperatures in ice as well as for time required by the ice/water interface $s(t)$ to move to a particular position.

A graph has been plotted in Figure 2 showing the position of the moving boundary as calculated by the present method. Moreover, the temperature distributions with space variable is computed and the results at five times are shown in Figure 3.

The results agree quite well with those found by [16–18, 24, 25], as shown in Table 2. The results computed from various methods show the superiority of the present method over the others. Obviously, the present method can cope with such problems effectively.
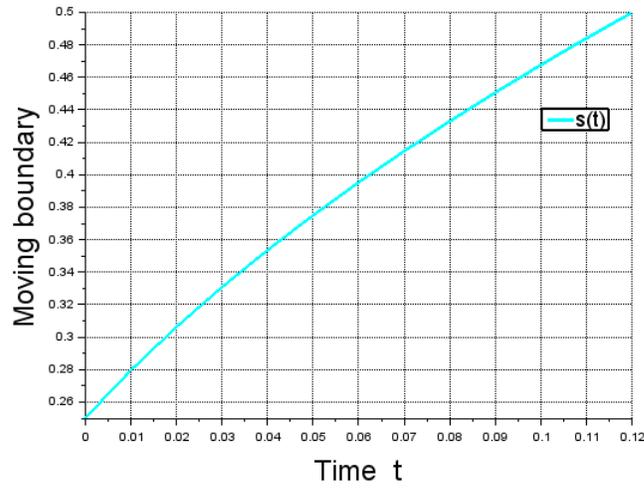
Figure 2. Position of the moving boundary over time computed using the proposed method.
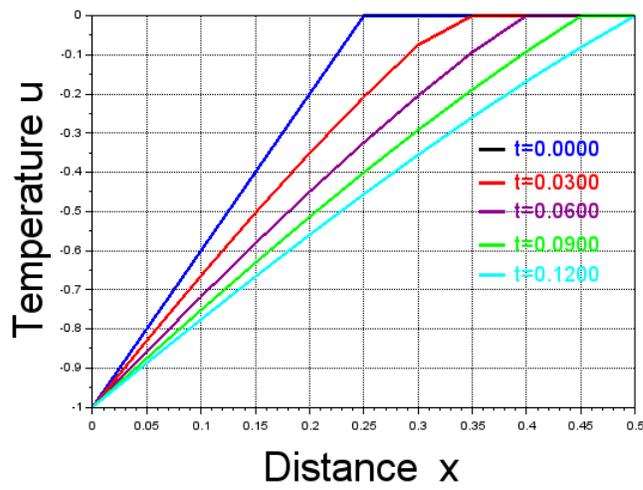

Figure 3. Temperature distributions in ice at various times.

Table 1. Positions of the interface $s(t)$ and temperatures in ice (with minus signs suppressed) with corresponding times computed from (3.7)(3.4) respectively.

Time \ X	0.05	0.1	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	$s(t)$
0.00	0.8000	0.6000	0.4000	0.2000	0	0	0	0	0	0	0.25
0.01	0.7994	0.6047	0.4189	0.2449	0.0857	0	0	0	0	0	0.2795
0.02	0.8167	0.6380	0.4664	0.3040	0.1531	0.0159	0	0	0	0	0.3061
0.03	0.8301	0.6642	0.5039	0.3511	0.2076	0.0751	0	0	0	0	0.3307
0.04	0.8410	0.6853	0.5344	0.3897	0.2528	0.1250	0.0078	0	0	0	0.3535
0.05	0.8500	0.7029	0.5599	0.4222	0.2911	0.1677	0.0534	0	0	0	0.375
0.06	0.8576	0.7178	0.5816	0.4500	0.3240	0.2048	0.0933	0	0	0	0.3952
0.07	0.8642	0.7307	0.6004	0.4741	0.3528	0.2374	0.1287	0.0278	0	0	0.4145
0.08	0.8699	0.7420	0.6168	0.4953	0.3781	0.2662	0.1603	0.0612	0	0	0.4330
0.09	0.8750	0.7519	0.6314	0.5141	0.4007	0.2921	0.1888	0.0916	0.0012	0	0.4506
0.10	0.8795	0.7608	0.6444	0.5309	0.4210	0.3153	0.2145	0.1192	0.0300	0	0.4677
0.11	0.8836	0.7688	0.6561	0.5461	0.4394	0.3365	0.2380	0.1445	0.0565	0	0.4841
0.12	0.8873	0.7760	0.6667	0.5599	0.4561	0.3557	0.2595	0.1677	0.0810	0	0.5000

Table 2. Comparison of position of the moving boundary $s(t)$ at various times from the present method with other numerical methods

Time	Rubinstein [25]	George and Damle [17]	Finn and Varaglu [16]	Rasmussen [24]	Gupta and Banik [18]	Present 3rd degree
0.01	0.2813	0.2807	0.2806	0.2754	0.2800	0.2795
0.02	0.3079	0.3071	0.3072	0.2986	0.3063	0.3062
0.03	0.3321	0.3312	0.3315	0.3202	0.3304	0.3307
0.04	0.3545	0.3536	0.3541	0.3403	0.3529	0.3535
0.06	0.3955	0.3945	0.3956	0.3775	0.3940	0.3952
0.08	0.4326	0.4316	0.4333	0.4113	0.4312	0.4330
0.10	0.4668	0.4657	0.4682	0.4425	0.4648	0.4677

Author contributions

These authors contributed equally to this work. All authors have read and approved the published version of the manuscript.

Data availability statement

Data are contained within the article.

Acknowledgement

Finally, the author is grateful to the referee for the careful reading of the paper and all useful suggestions and comments.

Conflict of interest

No potential conflict of interest was reported by the authors.

References

[1] M. J. Baines, J. V. Miller and K. W. Morton, *A finite element moving boundary computation with an an adaptive mesh*, Inst. Math. Appl, 1978, 22(4),467–477.
doi:org/10.1093/imamat/22.4.467.

[2] S. Belabbes and A. Boureghda, *Numerical Solution of Oxygen Diffusion Problem in Spherical Cell*, Axioms, 2025, 14(4).
doi:org/10.3390/axioms14010004.

[3] A. Boureghda, *Numerical methods for solving one dimensional problems with a moving boundary*, MA thesis. Department of Computing Science, Glasgow University, Scotland, United Kingdom, 1988.
theses:gla.ac.uk/id/eprint/77737.

[4] A. Boureghda, *Numerical solution of the oxygen diffusion in absorbing tissue with a moving boundary*, Commun. Numer. Methods Eng, 2006, 22(9), 933–942.
doi:org/10.1002/cnm.857.

[5] A. Boureghda, *Numerical solution of the oxygen diffusion problem in cylindrically shaped sections of tissue*, Int. J. Numer. Meth. Fluids, 2008, 56(10), 1945–1960.
doi:org/10.1002/fld.1591.

[6] A. Boureghda, *Moving boundary value problems*, Doctorat en Sciences Mathématiques. PhD thesis. between Department of Mathematics, Ferhat ABBAS University Sétif 1, Algeria and LMIA Haute Alsace University, Mulhouse, France, 2008.
www:learning-center.uha.fr/opac/resource/moving-boundary-value-problems/BUS04580095.

[7] A. Boureghda, *A modified variable time step method for solving ice melting problem*, Journal of Difference Equations and Applications, 2012, 18(9), 1443–1455.
doi:org/10.1080/10236198.2011.561797.

[8] A. Boureghda, *Solution to an ice melting cylindrical problem*, J. Nonlinear Sci. Appl, 2016, 9(4), 1440–1452.
doi:org/10.22436/jnsa.009.04.04.

[9] A. Boureghda, *Solution of an ice melting problem using a fixed domain method with a moving boundary*, Bull. Math. Soc. Sci. Math. Roumanie, 2019, 62(110), no. 4, 341–353.
www:jstor.org/stable/26865968.

[10] A. Boureghda and N. Djellab, *Du Fort-Frankel Finite Difference Scheme for Solving of Oxygen Diffusion Problem Inside One Cell*, J. Comput. Theor. Transp, 2023, 52(5), 363–373.
doi:org/10.1080/23324309.2023.2271229.

[11] J. Crank and R. S. Gupta, *A moving boundary problem arising from the diffusion of oxygen in absorbing tissue*, J. Inst. Maths. Applics, 1972, 10(1), 19–33.
doi:org/10.1093/imamat/10.1.19.

[12] J. Crank, *Free and moving boundary problems*, Clarendon Press, Oxford, 1984.

[13] N. Djellab and A. Boureghda, *A moving boundary model for oxygen diffusion in a sick Cell*, Comput. Meth. Biomech. Biomed. Eng, 2022, 25(12), 1402–1408.
doi:org/10.1080/10255842.2021.2024168.

[14] J. Douglas and T.M Gallie, *On the Numerical Integration of a Parabolic Differential Equation Subject to a Moving Boundary Condition*, Duke. Math. J, 1955, 22(4), 557–571.
doi:org/10.1215/S0012-7094-55-02262-6.

[15] V. N. Egorova, M. C. Casabán, R. Company and L. Jódar, *A random free-boundary diffusive logistic differential model: Numerical analysis, computing and simulation*, Math. Comput. Simu, 2024, 221, 55–78.
doi:org/10.1016/j.matcom.2024.02.016.

[16] W. D. L. Finn and E. Voroglu, *Finite element solution of the Stefan problem*, in: J. R. Whiteman (Ed.), The Mathematics of Finite Elements and Applications, MAFELAP 1978, Academic Press, New York, 1979.

[17] J. George and P.S. Damle, *On the numerical solution of free boundary problems*, Internat. J. Numer. Methods .Engrg, 1975, 9(1), 239–245.
doi:org/10.1002/nme.1620090118.

[18] R.S. Gupta and N.C. BANIK, *Constrained integral method for solving moving boundary problems*, Comput. Meths. Appl. Mech. Engrg, 1988, 67(2), 211–221.
doi:org/10.1016/0045-7825(88)90126-0.

[19] R.S. Gupta and N.C. Banik, *Approximate method for the oxygen diffusion problem*, Int. J. Heat Mass Transfer, 1989, 32(4), 781–783.
doi:org/10.1016/0017-9310(89)90227-5.

[20] E. Hansen and P. Hougaard, *On a moving boundary problem from biomechanics*, IMA Journal of Applied Mathematics, 1974, 13(3), 385–398.
doi:org/10.1093/imamat/13.3.385.

[21] D.F. Mayers and H.O. Dahmardah, *A fourier-series solution of the Crank-Gupta equation*, IMA J. Numer. Anal, 1983, 3(1), 81–86.
doi:org/10.1093/imanum/3.1.81.

[22] M. Miklavčič, *Analytic and numeric solutions of moving boundary problems*, J. Comput. Appl. Math., 2023, 431.
doi:org/10.1016/j.cam.2023.115270.

[23] J.R. Ockendon and W.R. Hodgkins, *eds. Moving Boundary Problems in Heat Flow and Diffusion*, Proc.Oxford Conf.March 1974. Clarendon Press Oxford, 1975.

[24] H. Rasmussen, *An approximate method for solving two-dimensional Stefan problems*, Lett.Heat and Mass, 1977, 4(4), 273–277.
doi:org/10.1016/0094-4548(77)90115-1.

[25] L.I.Rubinstein, *The Stefan problem*, Trans. Math. Monographs 27. (AMS, Providence, RI,1971), 1971.

[26] J. Stefan, *Ueber einige Probleme der Theorie der Waermeleitung*, Sitzungsber Wien, Akad. Mat. Na-tur, 1889, 98, 473–484.
$www.zobodat.at/pdf/SBAWW_98_2a_0471 - 0484.pdf$.

[27] D.A. Tarzia, *A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan and Related Problems*, MAT Ser A 2, 1-297. (Departamento de Matemática - CONICET,Facultad de Ciencias Empresariales, Universidad Austral,Paraguay 1950, S2000FZF Rosario, ARGENTINA.), 2000.
doi:org/10.26422/mat.a.2000.2.tar.

[28] M. Vynnycky, *On boundary immobilization for one-dimensional stefan-type problems with a moving boundary having initially parabolic-logarithmic behaviour*, Appl. Math. Comput, 2023, 444.
doi:org/10.1016/j.amc.2022.127803.