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On the Numerical Solution of Solidification of Water*

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Abstract The objective of this paper is to find an economical computer algorithm using a third-degree polynomial for the free boundary problem corresponding to phase change problem. We solve the problem to obtain the approximate analytical expression for the temperature distribution and the movement of the interface using the constrained integral method. Results are validated against existing numerical methods, demonstrating strong agreement with other techniques.

Keywords Constrained integral method, melting and solidification, moving boundary problem, Stefan problem, phase change, ice bank

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1. Introduction

Many more phenomena can be described by a Stefan problem. For instance, the decrease of oxygen in a muscle in the vicinity of a clotted blood vessel, the freezing food, lubrication, diffusion of gas, liquids, liquid-gas. Moving boundary problems (MBPs) of Stefan-Type arise in a range of physical applications, notably in the analysis of melting and solidification processes in solids [1, 11, 27].

Problems in which the solution of a differential equation must satisfy specific conditions on the boundary are called boundary-value problems. In a Stefan problem, this boundary is an unknown function of time, which must be determined concurrently with the solution of the differential equation. In these cases, we need two boundary conditions: one to determine the position of the moving boundary, and the other to complete the solution of differential equation.

We know that this kind of problem is called a Stefan problem, in reference to the work of J.Stefan, who around 1889 studied the melting of the polar ice cap [26].

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We note that the only difference between various Stefan problems lies in their governing equations; the fundamental idea of the solution methods remains the same. Moving boundary problems have received significant attention in recent years due to their practical importance in engineering and science, see J.R.Ockendon and W.R.Hodgkins [23] and [1,20,21]. These problems are inherently non-linear because of the moving boundary's presence, see Crank [12], Belabbes & Boureghda [2] and Boureghda [3–9]; and for this reason their analytical solutions are different. Many other authors have dealt with the moving boundary problems by various methods [15,22,28].

More references relating to the phase change problem involving a moving boundary can be found in the popular solidification problem, originally discussed by Rubinstein [25], selected for the analysis and in one of the oldest variable time step methods is due to Douglas and Gallie [14]. The same problem has been solved by George and Damle [17] using the method of lines, by Finn and Varoglu [16] using finite elements based on variational formulations, and by Gupta and Banik [18] using the Constrained integral method. Conducting exhaustive research on moving boundaries across several areas presents the challenge of fostering strong interdisciplinary collaboration between mathematics and other fields. For more details, see [10,11,13,19,23].

The aim of this work is to provide a detailed study of solving a moving boundary problem using the conventional integral method. This method yields two simultaneous first-order differential equations, which implicitly determine the position of the moving boundary and the value of an unknown additional parameter. The computed values from the expressions seem to be in very good agreement with those obtained by earlier authors using different numerical techniques.

2. Physical model and boundary conditions

The focus of this work is the problem concerning heat transfer in an ice-water medium occupying the region $0 \leq x \leq 1.0$. For simplicity, at any time t, the water, undergoing phase change, is contained in the region $s_1(t) \leq x \leq s_2(t)$ and the rest of the region outside is occupied by ice.

We suppose that $s_1(0) = 0.25$ and $s_2(0) = 0.75$ and the temperature of ice is linear in each of the two regions in which it lies. The water temperature is assumed to be equal to the critical phase change temperature, which is zero. Figure 1 shows the zero temperature temporal state of the ice-water regions as well as the positions of the ice/water interface. At the boundaries $x = 0$ and $x = 1$ the temperature is maintained at a unit negative temperature throughout. Because of the symmetry about $x = 0.5$, the problem reduces to finding its solution in the region $0 \leq x \leq 0.5$ only.