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Optimal Control Approach for Bilateral Elastic Contact Problem with Power-Law Friction

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Abstract We use the control variational technique to examine an elastic contact model, subject to a non-penetration condition in the normal direction and to power-law friction, proving the unique existence of the solution. This method uses optimal control theory to minimize the energy functional of the nonlinear equation. A multivalued equation $f \in \mathcal{F}y + \partial\Phi(y)$ for the displacement field describes the problem in a weak formulation, where a linear mapping is represented by $\mathcal{F}$, and the Clarke's subdifferential of the mapping Φ is indicated by $\partial\Phi$. We employ abstract existence theorems to verify the unique weak solution to the contact model.

Keywords Optimal control approach, multivalued equation, Clarke subdifferential, elastic materials, bilateral contact problem, normal compliance condition, Coulomb dry friction (power-law friction)

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1. Introduction

In everyday life, deformable bodies such as brake pads, tires, and pistons frequently come into contact with one another. This contact also occurs during industrial processes like extrusion and metal shaping. These interactions are characterized by strongly nonlinear elliptic or evolutionary equations due to their complexity.

Contact Mechanics is now the subject of a general mathematical theory thanks to recent developments in modeling, analysis, and simulations. The underlying structures of contact models with various geometries, constitutive laws, and contact conditions are addressed by this theory. The objectives include proving existence, and uniqueness results, establishing a rigorous framework for modeling contact phenomena, and accurately interpreting solutions. Mathematical ideas like multivalued inclusions and variational and hemivariational inequalities are used in this theory. For a comprehensive variational analysis and results, refer to [6–9, 14, 15, 18, 19]. Regarding computational techniques in Contact Mechanics, see [10, 26, 28] and their extensive bibliographies. Additionally, the most recent developments are covered in the proceedings [11, 16, 27] and [17].

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First described in [1, 22], the control variational technique is explained in detail and given many examples and applications in [13]. Its applicability to beam models interacting with a barrier is demonstrated in [21]. By utilizing optimum control theory to minimize system energy, this methodology expands upon classical variational methods. It is significant both theoretically and numerically, and it offers flexibility, potentially offering several solutions for the same problem [24]. It is noteworthy that regularity results are also obtained and fourth-order nonlinear differential equations are substituted with lower-order linear equations [13].

There are two main goals of this article. Firstly, in order to analyze nonlinear equalities involving multivalued operators in real Hilbert space and establish the existence and uniqueness of solutions, it shows how to apply the control variational approach. Secondly, it extends the results to mathematical models of the mechanical interaction between an elastic material and an obstacle, with power-law friction. Bilateral contact, applied normal tension, and power-law friction are all included in our model. We next demonstrate the unique solvability of the corresponding weak model, formulated as a nonlinear inclusion with the displacement as an unknown.

The structure of the paper is as follows: The physical setting and the problem's variational formulation are examined in Section 2. Section 3 investigates the existence and uniqueness of solutions for multivalued nonlinear equations. In Section 4, we extend these findings to a more general mathematical model that characterizes the frictional contact between an obstacle and a linearly elastic body.

2. Problem statement and weak formulation

Let us describe the classical model for the contact problem. Consider an elastic body in the open, bounded domain $\Omega \subset \mathbb{R}^3$. The Lipschitz continuous outer surface $\Gamma = \partial\Omega$ consists of three distinct measurable and nonempty subsets: Γ_1, Γ_2, and Γ_3. It is assumed that Γ_1 has a strictly positive measure. The body is fixed on Γ_1. On Γ_2, surface tractions f_2 are applied, and there is a density of volume forces f_0 throughout Ω. At the contact surface Γ_3, the body might come into contact with a foundation.

Our goal is to investigate the body's equilibrium in the specified context using a mathematical model. To keep the notation simple, the dependence of different functions on $x \in \Omega \cup \Gamma$ is not explicitly indicated here or below. Let $\nu = (\nu_i)$ the outward unit normal vector on Γ. However, the summation with repeated indices is utilized, and the index that comes after a comma stands for a spatial partial derivative with respect to the associated component of $x \in \Omega$. We designate the displacement vector as $u = (u_i)$, the stress tensor as $\sigma = (\sigma_{ij})$, and the linear strain tensor as $\varepsilon(u) = (\varepsilon_{ij}(u))$. The components of this tensor are as follows:

$$\varepsilon_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = (u_{i,j} + u_{j,i}), \quad \forall\, u \in H^1(\Omega)^3 \quad \text{with} \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}. \tag{2.1}$$

The displacement and stress fields, denoted by u and σ respectively, are the unknowns in the contact models under consideration and define the system's state. We refer to the space of 2^{nd} order symmetric tensors on $\mathbb{R}^3$ by $\mathbb{S}^3$. The canonical inner products and norms on $\mathbb{R}^3$ and $\mathbb{S}^3$ are:

$$u \cdot v = u_i v_i, \quad \forall\, u,\, v \in \mathbb{R}^3 \quad \text{with} \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \tag{2.2}$$

$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \forall\, \sigma,\, \tau \in \mathbb{S}^3 \quad \text{with} \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}. \tag{2.3}$$

Any displacement field $u \in H^1(\Omega)^3$ has normal and tangential components on the boundary Γ, denoted here by $u_\nu = u \cdot \nu$ and $u_\tau = u - u_\nu\, \nu$. Similarly, any stress field σ has normal and tangential components on Γ, represented here by σ_ν and σ_τ, respectively, with $\sigma_\nu = \sigma \cdot \nu$ and $\sigma_\tau = \sigma - \sigma_\nu\, \nu$.

The problem under consideration in this section is classically written as follows:

Problem 2.1. Find a displacement $u : \Omega \to \mathbb{R}^3$, a stress $\sigma : \Omega \to \mathbb{S}^3$ such that

$$\sigma = \mathfrak{F}\varepsilon(u) \qquad in \quad \Omega, \tag{2.4}$$

$$\operatorname{Div} \sigma + f_0 = 0 \qquad in \quad \Omega, \tag{2.5}$$

$$u = 0 \qquad on \quad \Gamma_1, \tag{2.6}$$

$$\sigma\nu = f_2 \qquad on \quad \Gamma_2, \tag{2.7}$$

$$u_\nu = 0 \qquad on \quad \Gamma_3, \tag{2.8}$$

$$\sigma_\tau = -\mu\, \|u_\tau\|^{p-1}\, u_\tau \qquad on \quad \Gamma_3. \tag{2.9}$$

In this case, the linear elastic constitutive law is represented by (2.4), $\mathfrak{F} = (e_{ijkl})$ is the elasticity tensor, and $\sigma_{ij} = e_{ijkl}\varepsilon_{kl}(u)$. The equilibrium equation for a static process is (2.5). The displacement and traction boundary conditions are denoted by equations (2.6) and (2.7), respectively. In relations (2.8)-(2.9), μ stands for the friction coefficient, and $p \in (0, 1]$ is the bilateral contact condition with power-law friction. Equation (2.8) represents a bilateral contact in which there is no loss during the procedure. The tangential shear in (2.9) is directly proportional to the tangential displacement power p. This is significant when a small layer of fluid lubricates the contact surface. See [7, 18] for additional mechanical information about these boundary contact conditions.

Given the boundary condition (2.6), we consider the corresponding closed space:

$$\mathcal{V} = \big\{ v \in H^1(\Omega)^3 : v = 0 \ \text{ on } \ \Gamma_1 \big\}.$$

Given that $meas(\Gamma_1) > 0$, Korn's inequality ensures that $\mathcal{V}$ is a real Hilbert for Euclidian norm $\|\cdot\|_{\mathcal{V}}$, related to the usual scalar product over $\mathcal{H} = \big\{ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \big\}$, that is

$$(u, v)_{\mathcal{V}} = \big(\varepsilon(u), \varepsilon(v)\big)_{\mathcal{H}} = \int_\Omega \varepsilon_{ij}(u)\, \varepsilon_{ij}(v)\, dx, \quad \forall\, u,\, v \in \mathcal{V}.$$

The dual space of $\mathcal{V}$ can be represented by $\mathcal{V}^*$, where the duality pairing is $\langle \cdot, \cdot \rangle_{\mathcal{V}^*, \mathcal{V}}$. It can be observed that the embeddings $\mathcal{V} \subset L^2(\Omega)^3 := L^2(\Omega, \mathbb{R}^3) \subset \mathcal{V}^*$ are dense and compact. According to (2.8), we introduce a subset of admissible displacements $\mathcal{K}$ in $\mathcal{V}$, specified by

$$\mathcal{K} := \big\{ v \in \mathcal{V} : v_\nu = 0 \ \text{ over } \ \Gamma_3 \big\}. \tag{2.10}$$

We notice that $\mathcal{K}$ is nonempty, closed and convex. Additionally, we consider the functional

$$j : \mathcal{K} \to \mathbb{R}, \quad j(v) = \frac{\mu}{p+1}\, \|v_\tau\|^{p+1}.$$

Also, take into account the mapping

$$\Phi : \mathcal{V} \to (-\infty, +\infty]\,, \quad \Phi(v) = \begin{cases} \displaystyle\int_{\Gamma_3} j(v)\, da & \text{for} \quad v \in \mathcal{K}, \\ +\infty & \text{otherwise} \end{cases}. \tag{2.11}$$

The mapping Φ is evidently lower semi-continuous, proper, and convex, where

$$\Phi < +\infty \quad \text{on} \quad \mathcal{K}. \tag{2.12}$$

Let $\mu \in L^\infty(\Gamma_3)$ and assume that μ is a positive function. Directly, if the system (2.8)-(2.9) is satisfied by a pair (u, σ) of regular functions, then:

$$u \in \mathcal{K}, \quad \text{and} \quad \sigma\nu \cdot (v-u) + j(v) - j(u) \geq 0, \ \forall\, v \in \mathcal{K} \quad \text{on} \quad \Gamma_3. \tag{2.13}$$

Remark 2.1. We note that to obtain the above relation, we use to the relation

$$\|v_\tau\|^{p+1} - \|u_\tau\|^{p+1} \geq (p+1)\, \|v_\tau\|^{p-1} u_\tau \cdot (v_\tau - u_\tau),$$

due to the differentiability and the convexity of the function $\|\cdot\|^{p+1} : u \mapsto \|u\|^{p+1}$.

Considering all the aspects discussed above, the classical formulation becomes:

Problem 2.2. Find the displacement-stress pair $u : \Omega \to \mathbb{R}^3$ and $\sigma : \Omega \to \mathbb{S}^3$ satisfying

$$\sigma = \mathfrak{F}\varepsilon(u) \qquad \textit{in} \quad \Omega, \tag{2.14}$$

$$\operatorname{Div}\sigma + f_0 = 0 \qquad \textit{in} \quad \Omega, \tag{2.15}$$

$$u = 0 \qquad \textit{on} \quad \Gamma_1, \tag{2.16}$$

$$\sigma\nu = f_2 \qquad \textit{on} \quad \Gamma_2, \tag{2.17}$$

$$u \in \mathcal{K} \ \ \textit{verifying} \ \ \sigma\nu \cdot (v-u) + j(v) - j(u) \geq 0, \quad \forall\, v \in \mathcal{K} \qquad \textit{on} \quad \Gamma_3. \tag{2.18}$$

We now turn to the problem's weak formulation. To this end, we impose the following hypotheses:

<u>($\mathcal{H}_1$)</u> : The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$ satisfies

(a) There exists $L_{\mathfrak{F}} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^3$, one has

$$\|\mathfrak{F}(x, \varepsilon_1) - \mathfrak{F}(x, \varepsilon_2)\| \leq L_{\mathfrak{F}}\, \|\varepsilon_1 - \varepsilon_2\| \quad \text{a.e.} \ \ x \in \Omega.$$

(b) There exists a non-negative constante $m_{\mathfrak{F}} > 0$ such that

$$\mathfrak{F}\, \varepsilon_{ij}\varepsilon_{kl} \geq m_{\mathfrak{F}}\, \|\varepsilon\|^2, \quad \forall\, \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^3.$$

(c) The mapping $x \mapsto \mathfrak{F}(x, \cdot)$ is measurable over Ω.

Clearly, $(\mathcal{H}_1)$ is satisfied, if all the components e_{ijkl} satisfy $e_{ijkl} = e_{jikl} = e_{klij}$, belong to $L^\infty(\Omega)$, and there exists $m_{\mathfrak{F}} > 0$ such that

$$e_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m_{\mathfrak{F}}\|\varepsilon\|^2, \quad \forall\, \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d.$$

<u>($\mathcal{H}_2$)</u> : The body force f_0, and the tractions f_2 verify

$$f_0 \in L^2(\Omega)^3 \ , \quad f_2 \in L^2(\Gamma_2)^3. \tag{2.19}$$

<u>($\mathcal{H}_3$)</u> : The function $\Phi : \mathcal{V} \to (-\infty, +\infty]$ satisfies

(a) There exists c_1, $c_2 > 0$, $\alpha \in]0, 2[$ and $\beta \in \mathbb{R}$, such that

$$\|x\|_{\mathcal{V}} \geq c_2 \implies -c_1\|x\|_V^\alpha + \beta \leq \Phi(x), \quad \forall\, x \in \mathcal{V}.$$

(b) $\Phi : \mathcal{V} \to \mathbb{R}$ is $\mathcal{T}$-lower semi-continuous where $\mathcal{T}$ is the topology over $L^2(\Omega)^3$.

We next define the elasticity tensor $\mathcal{F} : \mathcal{V} \to \mathcal{V}^*$, $u \mapsto \mathcal{F}u$ by

$$\begin{aligned}\langle \mathcal{F}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} &= \int_\Omega \mathfrak{F}\,\varepsilon(u)\cdot\varepsilon(v)\,dx \\ &= \int_\Omega e_{ijkl}\,\varepsilon_{ij}(u)\,\varepsilon_{kl}(v)\,dx, \quad \forall\, v \in \mathcal{V}.\end{aligned} \tag{2.20}$$

We define $f \in \mathcal{V}^*$ by using the Riesz representation theorem.

$$\langle f, v\rangle_{\mathcal{V}^*,\mathcal{V}} = \int_\Omega f_0 \cdot v\,dx + \int_{\Gamma_2} f_2 \cdot v\,da, \quad \text{for all} \quad v \in \mathcal{V}. \tag{2.21}$$

Since Problem (2.14)-(2.18) is satisfied by a pair (u, σ) of smooth functions, it follows directly that

$$u \in \mathcal{K}.$$

Furthermore, utilizing standard arguments based in Green formulas, we get

$$\begin{aligned}&\int_\Omega \sigma\cdot(\varepsilon(v)-\varepsilon(u))\,dx + \int_{\Gamma_3} j(v)\,da - \int_{\Gamma_3} j(u)\,da \\ &\geq \int_\Omega f_0\cdot(v-u)\,dx + \int_{\Gamma_2} f_2\cdot(v-u)\,da, \quad \forall\, v \in \mathcal{K}.\end{aligned}$$

Applying the previously stated notations, we obtain the weak formulation of problem (2.14)-(2.18):

Problem 2.3. Find a displacement $u : \Omega \to \mathbb{R}^3$ satisfying for all $v \in \mathcal{V}$, the inequality:

$$\langle f, v-u\rangle_{\mathcal{V}^*,\mathcal{V}} \leq \langle \mathcal{F}u, v-u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(v) - \Phi(u). \tag{2.22}$$

Remark 2.2. Note that various thermo-elastic, piezoelectric and thermo-piezoelectric contact problems, entirely by passing to the product spaces, lead to inequalities of type (2.22) or to its quasi-static variant consisting of replacing $\Phi(v) - \Phi(u)$ by the expression $\Phi(\tilde{v}, \tilde{u}) - \Phi(\tilde{u}, \tilde{u})$ that includes the additional state unknowns. Subsequently, the results obtained in the remainder of this paper can be extended to piezoelectric and thermo-piezoelectric cases, via suitable choices of functional spaces and necessary adaptations of the calculations, as demonstrated, for example, by [2,3,5,25].

3. Existence and uniqueness results

Here, we examine equations with multivalued operators using abstract Hilbert spaces and the control variational technique. We examine a locally Lipschitz function $\Phi : \mathcal{V} \to \mathbb{R}$. Throughout Sect.3, we define $\Phi^0(x; v)$ as the Clarke derivative of Φ at a spatial location $x \in \mathcal{V}$ in the direction $v \in \mathcal{V}$ (details may be found in [4]).

$$\Phi^0(x; v) := \limsup_{y\to x,\,\lambda\downarrow 0} \frac{\Phi(y+\lambda\,v) - \Phi(y)}{\lambda}. \tag{3.1}$$

Furthermore, the application $\partial\Phi : \mathcal{V} \to 2^{\mathcal{V}^*}$, $x \mapsto \partial\Phi(x)$, is the Clarke subdifferential of Φ. At a given point x, the Clarke gradient $\partial\Phi(x)$ of Φ is provided as follows:

$$\partial\Phi(x) := \big\{\varrho \in \mathcal{V}^* : \Phi^0(x; v) \geq \langle \varrho, v\rangle_{\mathcal{V}^*,\mathcal{V}}, \ \forall\, v \in \mathcal{V}\big\} \subset \mathcal{V}^*. \tag{3.2}$$

According to this, for every $v \in \mathcal{V}$, the one-sided directional derivative $\Phi'(x; v)$ must exist and equal $\Phi^0(x; v)$. Hence, we can infer

$$\Phi'(x; v) = \Phi^0(x; v), \quad \forall\, v \in \mathcal{V}.$$

Next, we will address the following nonlinear inclusion in $\mathcal{V}^*$.

$$\mathcal{F}u + \partial\Phi(u) \ni f. \tag{3.3}$$

Remark 3.1. The assumptions $(\mathcal{H}_3)$ on Φ enable the formulation of various hemivariational inequalities in the form (3.3), as demonstrated in [12, 15]. Moreover, if Φ is convex, the inclusion (3.3) can equivalently be reformulated as an elliptic variational inequality of the following type

$$\langle f, v - u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) \leq \langle \mathcal{F}u, v - u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(v), \quad \forall\, x \in \mathcal{V}. \tag{3.4}$$

Recall that, for any $h \in \mathcal{V}$, hypothesis $(\mathcal{H}_1)(b)$ ensures that $\mathcal{F}h = f$ has one solution. The following optimal control problem is obtained by using the control variational technique to (3.3).

Problem 3.1. Find an optimal couple (u, z) in $\mathcal{V} \times \mathcal{V}^*$, such that

$$\min_{u\in V} \big\{\langle z, u\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\langle z, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u)\big\}, \tag{3.5}$$

$$\mathcal{F}u = z - f. \tag{3.6}$$

Remark 3.2. The nonlinear component in the cost (3.5) is separated from the equation (3.3) by problem (3.5)-(3.6), which explains why the solution to (3.3) is constructive and simple. The inverse operator $\mathcal{F}^{-1}$ of $\mathcal{F}$ is all that is needed to solve this problem, and there are no limitations on the approach.

The following theorem describes the relation between the multivalued equation (3.3) and the control problem (3.5)-(3.6).

Theorem 3.1. *If hypothesis $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold, problem* (3.5)-(3.6) *admits at least one optimal pair (u^*, z^*) in $\mathcal{V} \times \mathcal{V}^*$, where u^* stands for a solution of* (3.3)*. Moreover, in the case that* (3.3) *has one solution, problem* (3.5)-(3.6) *can have a unique optimal couple.*

Proof. Using (3.6) and replacing $z = \mathcal{F}u + f$ in (3.5), the optimal control problem will be

$$\min_{u\in V} \big\{\langle \mathcal{F}u, u\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\langle \mathcal{F}u, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle f, u\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\langle f, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u)\big\}. \tag{3.7}$$

By utilizing $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, any sequence that minimizes the function cost in (3.7) is $\mathcal{T}$-relatively compact and weakly convergent in $\mathcal{V}$. Then (3.7) accepts at least one solution u^* of $\mathcal{V}$ because of the $\mathcal{T}$-lower semicontinuity of Φ. Next, (3.6) is used to produce the optimal control, which can be $z^* = \mathcal{F}u^* + f$. Therefore, there is at least one optimum pair (u^*, z^*) for problem (3.5)-(3.6). Furthermore, there could be more than one optimum couple for the control problem (3.5)-(3.6) since Φ is not necessarily convex. Next, we propose the admissible variations over (u^*, z^*) such that $(u^* + \lambda\varrho, z^* + \lambda\delta)$, where

$$\mathcal{F}\,\varrho = \delta, \quad \text{for all} \quad \lambda \in \mathbb{R}_+, \ (\varrho, \delta) \in \mathcal{V} \times \mathcal{V}^*. \tag{3.8}$$

We hence get

$$\begin{aligned} &\langle z^*, u^* \rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u^*) - 3\langle z^*, h\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &\leq \langle z^* + \lambda\delta, u^* + \lambda\,\varrho\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\,\langle z^* + \lambda\delta, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u^* + \lambda\,\varrho). \end{aligned}$$

After dividing by $\lambda > 0$, we impose $\lambda \to 0^+$ to get

$$0 \leq \langle \delta, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle z^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi^0(u^*; \varrho) - 3\,\langle \delta, h\rangle_{\mathcal{V}^*,\mathcal{V}}. \tag{3.9}$$

Now we use the previous relations (3.8) and (3.9) to deduce

$$0 \leq \langle \mathcal{F}\varrho, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle \mathcal{F}u^* + f, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi^0(u^*; \varrho) - 3\,\langle \mathcal{F}\varrho, h\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

Given the arbitrary selection of the element $\varrho \in \mathcal{V}$ and the symmetry of the operator $\mathcal{F}$, the definition of $h \in \mathcal{V}^*$ indicates that $u^* \in \mathcal{V}$ solves (3.3), proving the existence part. Lastly, the unique solvability of (3.3) implies uniqueness. □

Let us discuss the convex context. For that, we impose on $\Phi : \mathcal{V} \to (-\infty, +\infty]$ to verify

$(\mathcal{R}_1)$: Φ is a convex, lower semi-continuous and proper mapping.

The subdifferential operator defined in the convex case is denoted by $\partial\Phi$, such that

$$\partial\Phi(x) = \big\{\chi \in \mathcal{V}^* : \Phi(x) + \langle \chi, v - x\rangle_{\mathcal{V}^*,\mathcal{V}} \leq \Phi(v) \ \text{ for each } \ v \in \mathcal{V}\big\}. \tag{3.10}$$

We replace hypothesis $(\mathcal{H}_3)$ with assumption $(\mathcal{R}_1)$, in order to get:

Theorem 3.2. *Let us assume that* $(\mathcal{H}_1$-$(b))$ *and* $(\mathcal{R}_1)$ *are satisfied. There is only one solution to the nonlinear equation* (3.3) *and to the control model* (3.5)-(3.6)*. Additionally, if* $(u^*, \mathcal{F}u^* + f)$ *is the optimal couple of* (3.5)-(3.6)*, then* $u^* \in \mathcal{V}$ *solves inclusion* (3.3)*.*

Proof. According to condition $(\mathcal{R}_1)$, the functional Φ is lower-bounded by an affine map. Consequently, the lower semicontinuity of Φ and monotonicity arguments ensure that (3.3) has a unique solution, denoted by u^*. We then deal with the equivalence: we employ the same reasons from Theorem 3.1's proof for one implication, therefore all we have to do is to show the converse implication. Hence, $\mathcal{F}u^* + \chi^* = f$ is verified by the existence of $\chi^* \in \partial\Phi(u^*)$, using the fact that $\mathcal{F}h = f$, which implies:

$$\chi^* - \mathcal{F}h + \mathcal{F}u^* = 0_{\mathcal{V}^*}. \tag{3.11}$$

Let $(\varrho, \delta) \in \mathcal{V} \times \mathcal{V}^*$ satisfy (3.8). Then, employing the inner product of (3.11) with 2ϱ and with $z^* = \mathcal{F}u^* + f$, we obtain

$$\begin{aligned} &2\,\langle \chi^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle u^*, \delta\rangle_{\mathcal{V}^*,\mathcal{V}} - 2\langle \delta, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle z^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle f, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &= 2\,\big(\langle \chi^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle \mathcal{F}h, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle \mathcal{F}u^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}}\big) \\ &= 0. \end{aligned} \tag{3.12}$$

For the control system (3.5)-(3.6), consider a second admissible pair (u, z). Since $\delta = z^* - z$ and $\varrho = u^* - u$ both satisfy (3.8), we may use (ϱ, δ) as a test pair in (3.12). Given the convexity of Φ, (3.12) and (3.10), we have

$$\begin{aligned}&\langle z^*, u^* - u\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle z^* - z, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\,\langle z^* - z, h\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u^*) - \Phi(u)\\ &\le \langle z^*, u^* - u\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle z^* - z, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\,\langle z^* - z, h\rangle_{\mathcal{V}^*,\mathcal{V}} + 2\langle \chi^*, u^* - u\rangle_{\mathcal{V}^*,\mathcal{V}} = 0.\end{aligned} \tag{3.13}$$

Moreover, we see that

$$\begin{aligned}\langle z^*, u\rangle_{\mathcal{V}^*,\mathcal{V}} + &\langle z, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle z^*, u^*\rangle_{\mathcal{V}^*,\mathcal{V}} - \langle z, u\rangle_{\mathcal{V}^*,\mathcal{V}}\\ &= \langle z^* - z, u - u^*\rangle_{\mathcal{V}^*,\mathcal{V}}\\ &= -\langle \delta, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}}\\ &= -\langle \mathcal{F}\varrho, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} \le 0.\end{aligned} \tag{3.14}$$

Then, combine (3.13) and (3.14) to get $(u^*, \mathcal{F}u^* + f)$ as the optimum pair of problem (3.5)-(3.6). □

It should be noted that Theorems 3.1 and 3.2 implicitly guarantee the existence and uniqueness of the multivalued inclusion (3.3). These results are similar to previous results; for example, see [12, 13] and the references therein. Furthermore, when $\Phi = \mathbb{I}_{\mathcal{K}}$ is the indicator mapping of $\mathcal{K}$, Theorem 3.2 is still valid, implicitly including the constraint $u \in \mathcal{K}$ in the control problem.

Remark 3.3. With respect to the variational control method employed in Theorem 3.1, it can be observed that minimizing the cost functional (3.5) under (3.6) is equivalent to minimizing the typical energy functional of problem (3.3), as follows:

$$\begin{aligned}&\langle z, u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) - 3\langle z, h\rangle_{\mathcal{V}^*,\mathcal{V}}\\ =&\langle \mathcal{F}u + f, u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) - 3\,\langle \mathcal{F}u + f, h\rangle_{\mathcal{V}^*,\mathcal{V}}\\ =&\langle \mathcal{F}u, u\rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) - 2\,\langle u, f\rangle_{\mathcal{V}^*,\mathcal{V}} - 3\,\langle f, h\rangle_{\mathcal{V}^*,\mathcal{V}}.\end{aligned}$$

The fact that (3.6) only requires the flexible operator $\mathcal{F}$ makes problem (3.5)-(3.6) preferable to the conventional variational method. Furthermore, the previously mentioned optimal control method might reveal novel characteristics of the solution; for example, see [13, Ch VI].

Now to explore the isotropic materials case, we present the following variant of (3.3).

$$\mathcal{F}_1 u + \mathcal{F}_2 u + \partial\Phi(u) \ni f, \tag{3.15}$$

where $\Phi : \mathcal{V} \to \mathbb{R}$ is locally Lipschitz, $f \in \mathcal{V}^*$ and $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is such that

$(\mathcal{K}_1) : \mathcal{F}_1 : \mathcal{V} \to \mathcal{V}^*$ is linear symmetric continuous and coercive,

$(\mathcal{K}_2) : \mathcal{F}_2 : \mathcal{V} \to \mathcal{V}^*$ is linear symmetric and bounded.

Remark 3.4. It is possible that the sum $\mathcal{F}_1 + \mathcal{F}_2$ is not coercive in the absence of other assumptions on $\mathcal{F}_2$. Furthermore, since $\langle \mathcal{F}_2 u, u\rangle_{\mathcal{V}^*,\mathcal{V}}$ can contradict the hypothesis $(\mathcal{H}_1)$, it is impossible to include $\mathcal{F}_2$ in the subdifferential $\partial\Phi$. Therefore, it is not possible to solve (3.15) using the unique existence result in Theorem 3.1. As a result, we will solve (3.15) using an alternative approach, which is explained below.

Initially, we consider a space $\mathcal{Y}$ of controls, which is a real Hilbert, along with its dual $\mathcal{Y}^*$ and the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{Y}^*,\mathcal{Y}}$. We impose that $\mathcal{S} : \mathcal{V} \to \mathcal{Y}^*$ satisfies

$$\mathcal{S} \text{ is linear and continuous mapping.} \tag{3.16}$$

Assume that the adjoint operator of $\mathcal{S}$ is $\mathcal{S}^* : \mathcal{Y} \to \mathcal{V}^*$. The control parameter is $z \in \mathcal{Y}$, and the unique solution of the linear equation $\mathcal{F}_1 h = f$, as assured by the hypothesis $(\mathcal{K}_1)$, is h_1. (3.15) is related to the optimal control problem that is presented below:

Problem 3.2. Find an optimal couple (u, z) in $\mathcal{V} \times \mathcal{V}^*$, such that

$$\min_{z \in \mathcal{Y}} \big\{ \langle z, \mathcal{S}u \rangle_{\mathcal{Y},\mathcal{Y}^*} - 3 \langle z, \mathcal{S}h_1 \rangle_{\mathcal{Y},\mathcal{Y}^*} + \langle \mathcal{F}_2 u, u \rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) \big\}, \tag{3.17}$$

$$\langle \mathcal{F}_1 u, v \rangle_{\mathcal{V}^*,\mathcal{V}} = \langle z, \mathcal{S}v \rangle_{\mathcal{Y},\mathcal{Y}^*} - \langle f, v \rangle_{\mathcal{V}^*,\mathcal{V}}, \quad \forall\, v \in \mathcal{V}. \tag{3.18}$$

The decoupling of the operators in (3.15) is a key component of the problem (3.17)-(3.18). Further, the state problem may be expressed similarly to (3.18) as

$$\mathcal{F}_1 u = \mathcal{S}^* z - f.$$

It finds extensive utility in optimal control theory. However, in order to get over the practical issues with computing the adjoint operator, we employ the weak formulation (3.18) in the sequel. Take note that the unique existence of a solution to (3.18) is guaranteed by the hypothesis $(\mathcal{K}_1)$. Moreover, we have the following Theorem.

Theorem 3.3. *Let $\Phi : \mathcal{V} \to \mathbb{R}$ be locally Lipschitz and let the conditions $(\mathcal{K}_1)$, $(\mathcal{K}_2)$, and* (3.16) *hold. Then, if $(u^*, z^*) \in \mathcal{V} \times \mathcal{Y}$ is optimal for the problem* (3.17)-(3.18)*, then u^* solves* (3.15).

Proof. Consider $(u^* + \lambda\, \varrho, z^* + \lambda \delta)$ with $\lambda \in \mathbb{R}^*_+$ and $(\varrho, \delta) \in \mathcal{V} \times \mathcal{Y}$ are such that

$$\langle \mathcal{F}_1 \varrho, v \rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \delta, \mathcal{S}v \rangle_{\mathcal{Y},\mathcal{Y}^*} \quad \text{for all} \quad v \in \mathcal{V}. \tag{3.19}$$

We multiply the corresponding expression by $1/\lambda > 0$. Then, using the optimality of the pair (u^*, z^*), we take $\lambda \to 0$ to obtain

$$\begin{aligned} &\langle z^*, \mathcal{S}\varrho \rangle_{\mathcal{Y},\mathcal{Y}^*} + \langle \delta, \mathcal{S}u^* \rangle_{\mathcal{Y},\mathcal{Y}^*} \\ &+ 2 \langle \mathcal{F}_2 u^*, \varrho \rangle_{\mathcal{V}^*,\mathcal{V}} - 3\, \langle \delta, \mathcal{S}h_1 \rangle_{\mathcal{Y},\mathcal{Y}^*} + \Phi^0(u^*; \varrho) \geq 0. \end{aligned}$$

We have $\langle \mathcal{F}_1 \varrho, g \rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \varrho, f \rangle_{\mathcal{V},\mathcal{V}^*}$ for arbitrary $\varrho \in \mathcal{V}$, then the above relation, with (3.2) and the state equation (3.18), shows that u^* solves the equation (3.15), which completes the proof. □

4. Control variational method for a contact problem

Let us designate the control parameter by $z = (z_i) \in \mathcal{V}^*$, and employ the results from the preceding section. Therefore, we consider the following control problem:

Problem 4.1. Find (u, z) in $\mathcal{V} \times \mathcal{V}^*$, such that

$$\min_{u \in \mathcal{V}} \big\{ \langle z, u \rangle_{\mathcal{V}^*,\mathcal{V}} - 3\, \langle z, h \rangle_{\mathcal{V}^*,\mathcal{V}} + \Phi(u) \big\}, \tag{4.1}$$

$$\mathcal{F}u = z - f. \tag{4.2}$$

The equation $\mathcal{F}h = f$ is solved for $h \in \mathcal{V}$, where $\mathcal{F} : \mathcal{V} \to \mathcal{V}^*$, without involving the linear elasticity tensor. We see that after solving (4.2), the nonlinear component associated with $\Phi(\cdot)$, which needs to be determined, is decoupled by transforming the variational inequality (2.22) into the optimal control model (4.1)-(4.2). Subsequently, $\mathcal{F}$ satisfies $(\mathcal{H}_1)$ according to conditions (2.19), and $(\mathcal{H}_2)$ using the properties of Φ guarantees that $(\mathcal{R}_1)$ holds as well. Thus, the control problem (4.1)-(4.2) and the variational inequality (2.22) both have a unique solution according to Theorem 3.2. Additionally, the constraint $u^* \in \mathcal{K}$ is implicitly taken into account while solving (2.22), provided that the pair $(u^*, \mathcal{F}u^* + f)$ is optimal for the problem (4.1)-(4.2).

In the following, we address the case of isotropic linear materials. Recall that in such a case, the behaviour law is

$$\sigma = 2\mu\, \varepsilon(u) + \lambda\, \mathrm{Tr}(\varepsilon(u))\, \mathbb{I}_3,$$

where $\mu > 0$ and $\lambda > 0$ are the Lamé parameters, $\mathrm{Tr}(\varepsilon(u)) = \varepsilon_{ii}(u)$ is the trace of $\varepsilon(u)$, and $\mathbb{I}_3$ is the identity tensor on $\mathbb{R}^3$; see [19, 20] for more details. Thus, the elasticity tensor is $\mathfrak{F}(\varepsilon(u)) = 2\mu\, \varepsilon(u) + \lambda\, \mathrm{tr}(\varepsilon(u))\, \mathbb{I}_3$, and then we have

$$\langle \mathcal{F}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} = \int_\Omega \big(\lambda\, \varepsilon_{ii}(u)\, \varepsilon_{ii}(v) + 2\mu\, \varepsilon_{ij}(u)\, \varepsilon_{ij}(v)\big)\, dx, \quad \forall\, u,\, v \in \mathcal{V}.$$

Using again Green formulas, the corresponding variational inequality is as below:

$$\begin{aligned} &\int_\Omega \lambda\, \varepsilon_{ii}(u)\, \varepsilon_{ii}(v)\, dx + 2\mu \int_\Omega \varepsilon_{ij}(u)\, \varepsilon_{ij}(v)\, dx + \Phi(v) \\ &\geq \Phi(u) + \int_\Omega f_0 \cdot (v-u)\, dx + \int_{\Gamma_2} f_2 \cdot (v-u)\, da, \quad \forall\, v \in \mathcal{V}. \end{aligned} \tag{4.3}$$

Now, in order to apply Theorem 3.3, with the linear continuous mapping

$$\mathcal{S} : V \to L^2(\Omega)^9\,,\ v \mapsto \mathcal{S}(v) = \nabla v,$$

where ∇v represents the Jacobian matrix of a vector v. Let $\bar{z} \in L^2(\Omega)^9$ be the control parameter and consider the control problem:

Problem 4.2. Find $(u, \bar{z})$ in $\mathcal{V} \times L^2(\Omega)^9$, such that

$$\begin{aligned} \min_{u\in\mathcal{V}} \Big\{ &\int_\Omega \bar{z} \cdot \nabla u\, dx - 3\int_\Omega \bar{z} \cdot \nabla h_1\, dx + \int_\Omega \lambda\, (\mathrm{div}(u))^2\, dx \\ &+ \int_\Omega \mu\, \Big(\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_3}{\partial x_3}\Big)^2 \Big)\, dx \\ &+ 2\mu \int_\Omega \Big(\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \Big)\, dx + 2\,\Phi(u) \Big\}, \end{aligned} \tag{4.4}$$

$$\int_\Omega \mu\, \nabla u \cdot \nabla v\, dx + \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_2} f_2 \cdot v\, da = \int_\Omega \bar{z} \cdot \nabla v\, dx, \quad \forall\, v \in \mathcal{V}. \tag{4.5}$$

It should be noted that problem (4.4)-(4.5) is not subject to any constraints. Furthermore, in the minimization problem (4.4), the function h_1 is provided for all $v \in \mathcal{V}$ by:

$$\mu \int_\Omega \nabla h_1 \cdot \nabla v\ dx := \int_\Omega f \cdot v\ dx, \tag{4.6}$$

$\mathcal{F}_1$ is described by three different Laplace equations, the solutions to which are equal to equalities (4.5) or (4.6) that appear in Theorem 3.3. Moreover, $\mathcal{F}_2 : \mathcal{V} \to \mathcal{V}^*$ is specified by

$$\begin{aligned} &\langle \mathcal{F}_2 u, v\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &= \lambda \int_\Omega \big(\operatorname{div}(u)\big)\big(\operatorname{div}(v)\big)dx + \mu \int_\Omega \Big(\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial v_3}{\partial x_3}\Big)\, dx \\ &\quad + 2\mu \int_\Omega \Big(\frac{\partial u_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\frac{\partial v_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial v_3}{\partial x_2}\Big)\, dx, \quad \forall\, u,\, v \in \mathcal{V}. \end{aligned} \tag{4.7}$$

Finally, we possess all the necessary elements to verify the theorem below.

Theorem 4.1. *If the pair* $(u^*, \bar{z}^*) \in \mathcal{V} \times L^2(\Omega)^9$ *is optimal for the problem* (4.4)-(4.5)*, then* u^* *solves the variational inequality* (4.3)*.*

For the proof of Theorem 4.1, we employ an argument that is similar to that used in Theorems 3.1, 3.2, and 3.3. Nevertheless, for convenience, below a proof sketch.

Proof. Consider the optimal pair of problem (4.4)-(4.5) as $(u^*, \bar{z}^*) \in \mathcal{V} \times L^2(\Omega)^9$. Afterwards, we consider that $(u^* + \lambda\varrho, \bar{z}^* + \lambda\delta)$ satisfies the homogeneous version of equation (4.5) for $\lambda \in \mathbb{R}$ and $(\varrho, \delta) \in \mathcal{V} \times L^2(\Omega)^9$. Following the standard calculations, we may obtain

$$\begin{aligned} &\int_\Omega \delta \cdot \nabla u^* dx + \int_\Omega \bar{z}^* \cdot \nabla \varrho\, dx - 3 \int_\Omega \delta\, \nabla h_1\, dx + 2\lambda \int_\Omega \operatorname{div}(u^*) \operatorname{div}(\varrho)\, dx \\ &+ 2\mu \int_\Omega \left(\frac{\partial \varrho_1}{\partial x_1}\frac{\partial u_1^*}{\partial x_1} + \frac{\partial \varrho_2}{\partial x_2}\frac{\partial u_2^*}{\partial x_2} + \frac{\partial \varrho_3}{\partial x_3}\frac{\partial u_3^*}{\partial x_3} + \frac{\partial \varrho_2}{\partial x_1}\frac{\partial u_1^*}{\partial x_2} + \frac{\partial \varrho_1}{\partial x_2}\frac{\partial u_2^*}{\partial x_1} + \frac{\partial \varrho_3}{\partial x_1}\frac{\partial u_1^*}{\partial x_3} \right) dx \\ &+ 2\mu \int_\Omega \left(\frac{\partial \varrho_1}{\partial x_3}\frac{\partial u_3^*}{\partial x_1} + \frac{\partial \varrho_2}{\partial x_3}\frac{\partial u_3^*}{\partial x_2} + \frac{\partial \varrho_3}{\partial x_2}\frac{\partial u_2^*}{\partial x_3} \right) dx + 2\, \langle \chi^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &=0, \end{aligned} \tag{4.8}$$

where $\chi^* \in \partial\Phi(u^*)$. We eliminate $\bar{z}$, δ and h_1 from (4.8), by combining (4.5) with $v = \varrho$, the corresponding relation at a pair $(u^* + \lambda\varrho, \bar{z}^* + \lambda\delta)$, and the definition (4.6) of the map h_1, respectively. Hence, we get

$$\begin{aligned} 0 = 2 &\int_\Omega \big(\lambda\, \varepsilon_{ii}(u^*)\, \varepsilon_{jj}(\varrho) + 2\mu\, \varepsilon_{ij}(u^*)\, \varepsilon_{ij}(\varrho)\big) dx \\ &- 2 \int_\Omega f_0 \cdot \varrho\, dx - 2 \int_{\Gamma_3} f_2 \cdot \varrho\, da + 2\, \langle \chi^*, \varrho\rangle_{\mathcal{V}^*,\mathcal{V}}. \end{aligned}$$

In the above equation, selecting $\varrho = u^* - v$ for any arbitrary element $v \in \mathcal{V}$ is acceptable. Thus, the proof is completed by the definition (3.10), which indicates that u^* satisfies (4.3). □

5. Conclusion and remarks

In this paper, we explore an optimal control problem for an elastic contact model with bilateral contact and power-law friction conditions. We establish the model's weak solvability using abstract theorems on multivalued operators in real Hilbert spaces. Next, applying optimal control techniques, we minimize the energy functional of the nonlinear inclusion, enhancing our understanding of frictional contact problems.

Future research could focus on extending our results to complex frictional contact problems with different materials, such as thermo-piezoelectric materials, with various contact and friction conditions and under different processes (static, quasi-static, and dynamic). This work also provides a foundation for developing numerical approaches using appropriate discretization schemes, facilitating practical implementation, and validating theoretical results through simulations. Developing algorithms to handle the complexity of multivalued operators and nonlinear equations is crucial to advancing this field.

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