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Optimal Control Approach for Bilateral Elastic Contact Problem with Power-Law Friction

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Abstract We use the control variational technique to examine an elastic contact model, subject to a non-penetration condition in the normal direction and to power-law friction, proving the unique existence of the solution. This method uses optimal control theory to minimize the energy functional of the nonlinear equation. A multivalued equation $f \in \mathcal{F}y + \partial\Phi(y)$ for the displacement field describes the problem in a weak formulation, where a linear mapping is represented by $\mathcal{F}$, and the Clarke's subdifferential of the mapping Φ is indicated by $\partial\Phi$. We employ abstract existence theorems to verify the unique weak solution to the contact model.

Keywords Optimal control approach, multivalued equation, Clarke subdifferential, elastic materials, bilateral contact problem, normal compliance condition, Coulomb dry friction (power-law friction)

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1. Introduction

In everyday life, deformable bodies such as brake pads, tires, and pistons frequently come into contact with one another. This contact also occurs during industrial processes like extrusion and metal shaping. These interactions are characterized by strongly nonlinear elliptic or evolutionary equations due to their complexity.

Contact Mechanics is now the subject of a general mathematical theory thanks to recent developments in modeling, analysis, and simulations. The underlying structures of contact models with various geometries, constitutive laws, and contact conditions are addressed by this theory. The objectives include proving existence, and uniqueness results, establishing a rigorous framework for modeling contact phenomena, and accurately interpreting solutions. Mathematical ideas like multivalued inclusions and variational and hemivariational inequalities are used in this theory. For a comprehensive variational analysis and results, refer to [6–9, 14, 15, 18, 19]. Regarding computational techniques in Contact Mechanics, see [10, 26, 28] and their extensive bibliographies. Additionally, the most recent developments are covered in the proceedings [11, 16, 27] and [17].

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First described in [1, 22], the control variational technique is explained in detail and given many examples and applications in [13]. Its applicability to beam models interacting with a barrier is demonstrated in [21]. By utilizing optimum control theory to minimize system energy, this methodology expands upon classical variational methods. It is significant both theoretically and numerically, and it offers flexibility, potentially offering several solutions for the same problem [24]. It is noteworthy that regularity results are also obtained and fourth-order nonlinear differential equations are substituted with lower-order linear equations [13].

There are two main goals of this article. Firstly, in order to analyze nonlinear equalities involving multivalued operators in real Hilbert space and establish the existence and uniqueness of solutions, it shows how to apply the control variational approach. Secondly, it extends the results to mathematical models of the mechanical interaction between an elastic material and an obstacle, with power-law friction. Bilateral contact, applied normal tension, and power-law friction are all included in our model. We next demonstrate the unique solvability of the corresponding weak model, formulated as a nonlinear inclusion with the displacement as an unknown.

The structure of the paper is as follows: The physical setting and the problem's variational formulation are examined in Section 2. Section 3 investigates the existence and uniqueness of solutions for multivalued nonlinear equations. In Section 4, we extend these findings to a more general mathematical model that characterizes the frictional contact between an obstacle and a linearly elastic body.

2. Problem statement and weak formulation

Let us describe the classical model for the contact problem. Consider an elastic body in the open, bounded domain $\Omega \subset \mathbb{R}^3$. The Lipschitz continuous outer surface $\Gamma = \partial\Omega$ consists of three distinct measurable and nonempty subsets: Γ_1, Γ_2, and Γ_3. It is assumed that Γ_1 has a strictly positive measure. The body is fixed on Γ_1. On Γ_2, surface tractions f_2 are applied, and there is a density of volume forces f_0 throughout Ω. At the contact surface Γ_3, the body might come into contact with a foundation.

Our goal is to investigate the body's equilibrium in the specified context using a mathematical model. To keep the notation simple, the dependence of different functions on $x \in \Omega \cup \Gamma$ is not explicitly indicated here or below. Let $\nu = (\nu_i)$ the outward unit normal vector on Γ. However, the summation with repeated indices is utilized, and the index that comes after a comma stands for a spatial partial derivative with respect to the associated component of $x \in \Omega$. We designate the displacement vector as $u = (u_i)$, the stress tensor as $\sigma = (\sigma_{ij})$, and the linear strain tensor as $\varepsilon(u) = (\varepsilon_{ij}(u))$. The components of this tensor are as follows:

$$\varepsilon_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = (u_{i,j} + u_{j,i}), \quad \forall\, u \in H^1(\Omega)^3 \quad \text{with} \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}. \tag{2.1}$$

The displacement and stress fields, denoted by u and σ respectively, are the unknowns in the contact models under consideration and define the system's state. We refer to the space of 2^{nd} order symmetric tensors on $\mathbb{R}^3$ by $\mathbb{S}^3$. The canonical inner products and norms on $\mathbb{R}^3$ and $\mathbb{S}^3$ are:

$$u \cdot v = u_i v_i, \quad \forall\, u, v \in \mathbb{R}^3 \quad \text{with} \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \tag{2.2}$$