Journal of Nonlinear Modeling and Analysis
Volume 8, Number 1, February 2026, 217–228
https://global-sci.com/jnma
DOI:10.12150/jnma.2026.217

Cross-Invariant Sets of the Cubic Nonlinear Schrödinger System with Partial Confinement*

Xinlu Zhou[1] and Jian Zhang[1,†]

Received 4 August 2025; Accepted 18 September 2025

Abstract This paper studies the cubic nonlinear Schrödinger system with partial confinement:

$$\begin{cases} -i\varphi_t + (x_1^2+x_2^2)\varphi = \Delta\varphi + \mu_1|\varphi|^2\varphi + \beta|\psi|^2\varphi, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ -i\psi_t + (x_1^2+x_2^2)\psi = \Delta\psi + \mu_2|\psi|^2\psi + \beta|\varphi|^2\psi, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \end{cases}$$

which models the Bose-Einstein condensates with multiple states or the propagation of mutually incoherent wave packets in nonlinear optics. The cross-invariant sets of the evolution flow are obtained by constructing the cross-constrained variational problem. Furthermore, the sharp condition for global existence and blowup of the solutions is derived.

Keywords Bose-Einstein condensates, nonlinear Schrödinger system, cross-invariant sets, sharp condition, global existence

MSC(2010) 35Q55, 35A15, 35B30.

1. Introduction

In this paper, we investigate the following cubic nonlinear Schrödinger systems with partial confinement:

$$\begin{cases} -i\varphi_t + (x_1^2+x_2^2)\varphi = \Delta\varphi + \mu_1|\varphi|^2\varphi + \beta|\psi|^2\varphi, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ -i\psi_t + (x_1^2+x_2^2)\psi = \Delta\psi + \mu_2|\psi|^2\psi + \beta|\varphi|^2\psi, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mu_1, \mu_2, \beta > 0$, $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ are complex-valued wave functions of $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$, $i=\sqrt{-1}$, and Δ is the Laplace operator on $\mathbb{R}^3$. The system (1.1) models Bose-Einstein condensates with multiple states or the propagation of mutually incoherent wave packets in nonlinear optics [1, 8, 9, 19]. The parameters μ_1, μ_2 and β represent the intraspecies and interspecies scattering lengths, describing the interaction between particles of the same component or of different components respectively [6, 8]. In particular, the positive sign of $\mu_j (j = 1, 2)$ (and of β) stands for

†the corresponding author.
Email address: XinluZhou914@163.com(X. Zhou), zhangjian@uestc.edu.cn(J. Zhang)
1School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 611731, China
*The authors were supported by National Natural Science Foundation of China (12271080).

attractive interaction, while the negative sign stands for repulsive interaction [6,8]. In the present paper, we treat (1.1) in the case of positive parameters $\mu_1, \mu_2, \beta > 0$, which is the so-called self-focusing and attractive interaction. For system (1.1), the physically relevant cubic nonlinearity is L^2-supercritical and H^1-subcritical. Cubic nonlinear Schrödinger systems, often referred to as Gross-Pitaevskii equations, are very important in Physics.

In (1.1), we set

$$\varphi(t,x) = u(x)e^{i\lambda_1 t} \quad \text{and} \quad \psi(t,x) = v(x)e^{i\lambda_2 t} \tag{1.2}$$

with $\lambda_1, \lambda_2 > 0$, then (u,v) solves the following elliptic systems:

$$\begin{cases} -\Delta u + \lambda_1 u + (x_1^2 + x_2^2)u = \mu_1 u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\Delta v + \lambda_2 v + (x_1^2 + x_2^2)v = \mu_2 v^3 + \beta u^2 v, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

Hence, (1.2) is a standing wave of the system (1.1). In [16], the existence and asymptotic behavior of least energy solutions of (1.3) are studied in a bounded domain with constant trapping potential. In [17], the asymptotic behavior is studied in $\mathbb{R}^N$ under the influence of nonconstant trapping potentials. In $\mathbb{R}^N$, the least energy and higher energy bound states of (1.3) are investigated in [2,4,15,18,23,24].

On the other hand, the study of normalized solutions for (1.3) has received lots of attention, that is, considering (1.3) with the constraints

$$\int_{\mathbb{R}^3} u^2 dx = a^2 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 dx = b^2. \tag{1.4}$$

When $N = 2$, Guo et al. in [10–12] considered the existence, non-existence, uniqueness and asymptotic behavior of solutions to problem (1.3)-(1.4) with a certain type of trapping potentials. In [22], J. Royo-Letelier addressed both segregation and symmetry breaking for (1.3) in $\mathbb{R}^2$. In [20], B. Noris et al. studied problem (1.3) in bounded domains of $\mathbb{R}^N$, or the problem with trapping potentials in the whole space $\mathbb{R}^N$ (the presence of a trapping potential makes the two problems essentially equivalent) with $N \leq 3$. In both cases, they proved the existence of positive solutions with small masses a and b, and the orbital stability of the associated solitary waves. Gou [13] treated (1.3)-(1.4) with partial confinements when $N = 3$, and showed the compactness of minimizing sequences up to translations in x_3. As a by-product, they also obtained the orbital stability of the set of global minimizers. Jia et al. [14] obtained the existence of stable standing waves for (1.3)-(1.4) when $N = 3$.

As a motivation, we recall the single nonlinear Schrödinger equation with a partial confinement

$$i\varphi_t + \Delta\varphi - (x_1^2 + x_2^2)\varphi + |\varphi|^{p-1}\varphi = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.5}$$

where $\varphi = \varphi(t,x)$ is a complex-valued wave function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$, $i = \sqrt{-1}$, Δ is the Laplace operator on $\mathbb{R}^N$, and $1 < p < \frac{N+2}{(N-2)^+}$ (we use the convention: $\frac{N+2}{(N-2)^+} = \infty$ when $N = 1, 2$, and $(N-2)^+ = N-2$ when $N \geq 3$). For equation (1.5), the authors in [5] studied the existence and stability of standing waves for the nonlinear Schrödinger equations with partial confinement in $\mathbb{R}^3$. In [3] and [21], the authors studied the scattering and strong instability of standing waves for nonlinear

Schrödinger equations with partial confinement in $\mathbb{R}^N$ respectively. Zhang [28] studied the sharp threshold for global existence of (1.5) on mass in $\mathbb{R}^N$. Moreover, Wang and Zhang [25] proved the sharp thresholds for global existence and blow-up in $\mathbb{R}^3$. In particular, in [25], the authors obtained the result by using the cross-constrained variational method, which was proposed by Zhang in [26, 27].

To our knowledge, the research of sharp thresholds for global existence and blow-up to a system with a partial confinement is still open. Thus inspired by the aforementioned works, we are going to consider (1.1) by exploiting the cross-constrained variational method. Our aim is to establish cross-invariant sets of the evolution flow, then the sharp condition for global existence and blow-up of the solutions is derived.

The rest of this paper is organized as follows. In Section 2, we state some preliminaries. In Section 3, we construct cross-constrained variational problem and establish the invariant sets for the Cauchy problem (1.1) and (2.1). In Section 4, we prove the sharp condition of global existence and blowup for the Cauchy problem (1.1) and (2.1).

2. Preliminaries

We impose the initial data of (1.1) as follows

$$\varphi(0,x) = \varphi_0(x), \quad \psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^3. \tag{2.1}$$

For (1.1), we define the energy space in the course of nature as

$$H := \Big\{ \phi \in H^1(\mathbb{R}^3) \Big| \int_{\mathbb{R}^3} (x_1^2 + x_2^2)|\phi|^2 dx < \infty \Big\}. \tag{2.2}$$

H becomes a Hilbert space, continuously embedded in $H^1(\mathbb{R}^3)$, when equipped with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}^3} \nabla u \nabla \overline{v} + u\overline{v} + (x_1^2 + x_2^2) u\overline{v} dx, \tag{2.3}$$

whose associated norm is denoted by $\| \cdot \|_H$.

Firstly, from [3, 7], we have the following local well-posedness result for the Cauchy problem (1.1) and (2.1).

Proposition 2.1. *For any $(\varphi_0, \psi_0) \in H \times H$, there exist $T_{max} = T_{max}(\varphi_0, \psi_0) \in (0, \infty]$ and a unique solution $(\varphi(t,x), \psi(t,x)) \in C([0, T_{max}); H \times H)$ of the Cauchy problem (1.1) and (2.1). The solution $(\varphi(t,x), \psi(t,x))$ exists globally in the sense of $T_{max} = +\infty$. Otherwise, the solution $(\varphi(t,x), \psi(t,x))$ blows up in the sense that $T_{max} < \infty$, then*

$$\lim_{t \to T_{max}} \big(\|\varphi(t,\cdot)\|_H + \|\psi(t,\cdot)\|_H \big) = \infty.$$

Furthermore, the solution $(\varphi(t,x), \psi(t,x))$ satisfies the conservation laws

$$\int_{\mathbb{R}^3} |\varphi(t,x)|^2 dx = \int_{\mathbb{R}^3} |\varphi_0(x)|^2 dx, \quad \int_{\mathbb{R}^3} |\psi(t,x)|^2 dx = \int_{\mathbb{R}^3} |\psi_0(x)|^2 dx, \tag{2.4}$$

$$
\begin{aligned}
E(\varphi,\psi) =&\frac{1}{2}\int_{\mathbb{R}^3}\Big[|\nabla\varphi(t,x)|^2+|\nabla\psi(t,x)|^2+(x_1^2+x_2^2)(|\varphi(t,x)|^2+|\psi(t,x)|^2)\Big]dx\\
&-\frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|\varphi(t,x)|^4+\mu_2|\psi(t,x)|^4+2\beta|\varphi(t,x)|^2|\psi(t,x)|^2)dx\\
=&E(\varphi_0,\psi_0)
\end{aligned}
\tag{2.5}
$$

for all $t\in[0,T_{max})$.

Next, we derive the following Virial type identity based on Cazenave [7].

Proposition 2.2. *Suppose that* $(\varphi_0,\psi_0)\in H\times H$. *Then the solution* $(\varphi(t,x),\psi(t,x))$ *of the Cauchy problem* (1.1) *and* (2.1) *satisfies* $(\varphi(t,x),\psi(t,x))\in C([0,T_{max});H\times H)$. *Put*

$$
J(t):=\int_{\mathbb{R}^3}|x|^2(|\varphi(t,x)|^2+|\psi(t,x)|^2)dx.
$$

Then one has

$$
\begin{aligned}
J''(t) =&8\int_{\mathbb{R}^3}\Big[|\nabla\varphi(t,x)|^2+|\nabla\psi(t,x)|^2-(x_1^2+x_2^2)(|\varphi(t,x)|^2+|\psi(t,x)|^2)\Big]dx\\
&-6\int_{\mathbb{R}^3}(\mu_1|\varphi(t,x)|^4+\mu_2|\psi(t,x)|^4+2\beta|\varphi(t,x)|^2|\psi(t,x)|^2)dx.
\end{aligned}
\tag{2.6}
$$

3. The cross-invariant sets

For $(u,v)\in H\times H$, we define the following functionals:

$$
\begin{aligned}
I(u,v) :=&\frac{1}{2}\int_{\mathbb{R}^3}\Big[|\nabla u|^2+|\nabla v|^2+(x_1^2+x_2^2)(|u|^2+|v|^2)+\lambda_1|u|^2+\lambda_2|v|^2\Big]dx\\
&-\frac{1}{4}\int_{\mathbb{R}^3}(\mu_1|u|^4+\mu_2|v|^4+2\beta|u|^2|v|^2)dx,
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
S(u,v) :=&\int_{\mathbb{R}^3}\Big[|\nabla u|^2+|\nabla v|^2+(x_1^2+x_2^2)(|u|^2+|v|^2)+\lambda_1|u|^2+\lambda_2|v|^2\Big]dx\\
&-\int_{\mathbb{R}^3}(\mu_1|u|^4+\mu_2|v|^4+2\beta|u|^2|v|^2)dx,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
Q(u,v) :=&\int_{\mathbb{R}^3}\Big[|\nabla u|^2+|\nabla v|^2-(x_1^2+x_2^2)(|u|^2+|v|^2)\Big]dx\\
&-\frac{3}{4}\int_{\mathbb{R}^3}(\mu_1|u|^4+\mu_2|v|^4+2\beta|u|^2|v|^2)dx.
\end{aligned}
\tag{3.3}
$$

Since the Sobolev's embedding theorem, the above functionals are well defined. In addition, we define a cross-manifold as follows:

$$
M:=\{(u,v)\in H\times H),Q(u,v)=0,S(u,v)<0\}. \tag{3.4}
$$

Lemma 3.1. M *is not empty.*

Proof. From [14], we know that there exists $(u,v)\in H\times H\backslash\{(0,0)\}$ such that (u,v) is a solution of (1.3). Multiplying the first equation of (1.3) by $\overline{u}$ and the second equation of (1.3) by $\overline{v}$ respectively, then integrating with respect to x on $\mathbb{R}^3$, one has $S(u,v)=0$. Besides, multiplying the first equation of (1.3) by $x\cdot\nabla\overline{u}$ and the second equation of (1.3) by $x\cdot\nabla\overline{v}$ respectively, then integrating with respect to x on $\mathbb{R}^3$, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx+5\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx+3\int_{\mathbb{R}^3}(\lambda_1|u|^2+\lambda_2|v|^2)dx\\=&\frac{3}{2}\int_{\mathbb{R}^3}\Big(\mu_1|u|^4+\mu_2|v|^4+2\beta|u|^2|v|^2\Big)dx.\end{aligned}\tag{3.5}$$

Combining (3.5) with $S(u,v)=0$, then we can get $Q(u,v)=0$.

Put $u_1=\rho u(x), v_1=\rho v(x)$ for any $\rho>1$. Then $(u_1,v_1)\in H\times H$. From $S(u,v)=0$ and $Q(u,v)=0$, we deduce that $S(u_1,v_1)<0$ and $Q(u_1,v_1)<0$, respectively. Now we let

$$u_1^\lambda(x)=\lambda u_1(\lambda x),\ v_1^\lambda(x)=\lambda v_1(\lambda x)\quad\text{for }\lambda>0.\tag{3.6}$$

Thus, it follows from (3.2) and (3.3) that

$$\begin{aligned}S(u_1^\lambda,v_1^\lambda)=&\lambda\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla v_1|^2)dx+\lambda^{-3}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx\\&+\lambda^{-1}\int_{\mathbb{R}^3}(\lambda_1|u_1|^2+\lambda_2|v_1|^2)dx\\&-\lambda\int_{\mathbb{R}^3}(\mu_1|u_1|^4+\mu_2|v_1|^4+2\beta|u_1|^2|v_1|^2)dx,\end{aligned}\tag{3.7}$$

$$\begin{aligned}Q(u_1^\lambda,v_1^\lambda)=&\lambda\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla v_1|^2)dx-\lambda^{-3}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx\\&-\frac{3}{4}\lambda\int_{\mathbb{R}^3}(\mu_1|u_1|^4+\mu_2|v_1|^4+2\beta|u_1|^2|v_1|^2)dx.\end{aligned}\tag{3.8}$$

Since $Q(u_1,v_1)<0$, then there exists $\lambda_0>1$ such that $Q(u_1^{\lambda_0},v_1^{\lambda_0})=0$. On the other hand, from $\lambda_0>1$, $S(u_1,v_1)<0$ and (3.7), we still have $S(u_1^{\lambda_0},v_1^{\lambda_0})<0$. Thus $(u_1^{\lambda_0},v_1^{\lambda_0})\in M$, and we showed M is not empty. □

Now we define the following cross-constrained minimization problem:

$$d_M:=\inf_{(u,v)\in M}I(u,v).\tag{3.9}$$

Lemma 3.2. $d_M>0$.

Proof. Take $(u,v)\in M$. By $S(u,v)<0$, we have $(u,v)\neq(0,0)$. From $Q(u,v)=0$, we get

$$\begin{aligned}I(u,v)=&\frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx+\frac{5}{6}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx\\&+\frac{1}{2}\int_{\mathbb{R}^3}(\lambda_1|u|^2+\lambda_2|v|^2)dx.\end{aligned}\tag{3.10}$$

From (3.10) and $(u,v) \neq (0,0)$, we know that $I(u,v) > 0$ for all $(u,v) \in M$. Therefore, (3.9) implies that $d_M \geq 0$. Next (3.10) can be rewritten as

$$\begin{aligned} I(u,v) =& \frac{1}{6}\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx \\ &+ \frac{1}{3}\int_{\mathbb{R}^3} \Big[2(x_1^2+x_2^2)(|u|^2+|v|^2) + (\lambda_1|u|^2+\lambda_2|v|^2)\Big]dx. \end{aligned} \tag{3.11}$$

Since $S(u,v) < 0$, applying Young's inequality and Sobolev's embedding inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx \\ <& \int_{\mathbb{R}^3} (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta|u|^2|v|^2)dx \\ \leq& c\Big\{\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx\Big\}^2. \end{aligned} \tag{3.12}$$

Here and hereafter, c denotes various positive constants. Thus,

$$\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx \geq c > 0. \tag{3.13}$$

Then, (3.11) and (3.13) yield that $I(u,v) \geq c > 0$ for all $(u,v) \in M$. Therefore $d_M > 0$. □

Now we define another constrained variational problem:

$$d_B := \inf_{(u,v)\in B} I(u,v), \tag{3.14}$$

where the set B is defined by

$$B = \{(u,v) \in H \times H \setminus \{(0,0)\}, S(u,v) = 0\}.$$

Lemma 3.3. *B is not empty and $d_B > 0$.*

Proof. It is clear that B is not empty (see [14]). Let $(u,v) \in B$. Then $S(u,v) = 0$ and $(u,v) \neq (0,0)$. From $S(u,v) = 0$, by the Young's inequality and Sobolev's embedding inequality, one can derive

$$\begin{aligned} &\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx \\ =& \int_{\mathbb{R}^3} (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta|u|^2|v|^2)dx \\ \leq& c\Big\{\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx\Big\}^2. \end{aligned} \tag{3.15}$$

Thus

$$\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx \geq c > 0. \tag{3.16}$$

On the other hand, by $S(u,v) = 0$ again, it follows that

$$I(u,v) = \frac{1}{4}\int_{\mathbb{R}^3} \Big[|\nabla u|^2 + |\nabla v|^2 + (x_1^2+x_2^2)(|u|^2+|v|^2) + \lambda_1|u|^2 + \lambda_2|v|^2\Big]dx. \tag{3.17}$$

We see from (3.16) and (3.17) that $I(u,v) \geq c > 0$. Therefore, (3.14) yields $d_B > 0$ for all $(u,v) \in B$. □

We define

$$d := \min\{d_M, d_B\}. \tag{3.18}$$

From (3.9), (3.14) and (3.18), we have the following theorem.

Theorem 3.1. *$d > 0$.*

Proof. From Lemma 3.2 and Lemma 3.3, it is clear that $d > 0$. □

Theorem 3.2. *Define*

$$K := \{(u,v) \in H \times H), I(u,v) < d, S(u,v) < 0, Q(u,v) < 0\}. \tag{3.19}$$

Then, K is an invariant manifold of the Cauchy problem (1.1) *and* (2.1). *More precisely, if $(\varphi_0, \psi_0) \in K$, then the solution $(\varphi(t,x), \psi(t,x))$ of the Cauchy problem* (1.1) *and* (2.1) *satisfies $(\varphi(t,\cdot), \psi(t,\cdot)) \in K$ for any $t \in [0, T_{max})$.*

Proof. Let $(\varphi_0, \psi_0) \in K$. By Proposition 2.1, the Cauchy problem (1.1) and (2.1) has a unique solution $(\varphi(t,x), \psi(t,x)) \in C([0, T_{max}); H \times H)$ with $T_{max} \leq \infty$. Employing (2.4) and (2.5), we have

$$I(\varphi(t,\cdot), \psi(t,\cdot)) = I(\varphi_0, \psi_0) \tag{3.20}$$

for any $t \in [0, T_{max})$. Thus, $I(\varphi_0, \psi_0) < d$ yields that $I(\varphi(t,\cdot), \psi(t,\cdot)) < d$ for any $t \in [0, T_{max})$.

Next, we prove that $S(\varphi(t,\cdot), \psi(t,\cdot)) < 0$ for any $t \in [0, T_{max})$. If not, then from the continuity of t, there exists $t_1 \in [0, T_{max})$ such that $S(\varphi(t_1,\cdot), \psi(t_1,\cdot)) = 0$. According to (3.20), $(\varphi(t_1,\cdot), \psi(t_1,\cdot)) \neq (0,0)$. Thus $(\varphi(t_1,\cdot), \psi(t_1,\cdot)) \in B$. By (3.14) and (3.18), we get $I(\varphi(t_1,\cdot), \psi(t_1,\cdot)) \geq d_B \geq d$, which is contradictory with $I(\varphi(t,\cdot), \psi(t,\cdot)) < d$ for any $t \in [0, T_{max})$. Therefore, $S(\varphi(t,\cdot), \psi(t,\cdot)) < 0$ for any $t \in [0, T_{max})$.

Finally, we show that $Q(\varphi(t,\cdot), \psi(t,\cdot)) < 0$ for any $t \in [0, T_{max})$. If not, then from the continuity of t, there exists $t_2 \in [0, T_{max})$ such that $Q(\varphi(t_2,\cdot), \psi(t_2,\cdot)) = 0$. Along with $S(\varphi(t_2,\cdot), \psi(t_2,\cdot)) < 0$, we get $(\varphi(t_2,\cdot), \psi(t_2,\cdot)) \in M$. Thus (3.9) and (3.18) imply that $I(\varphi(t_2,\cdot), \psi(t_2,\cdot)) \geq d_M \geq d$. This is contradictory to $I(\varphi(t,\cdot), \psi(t,\cdot)) < d$ for any $t \in [0, T_{max})$. Hence $Q(\varphi(t,\cdot), \psi(t,\cdot)) < 0$ for any $t \in [0, T_{max})$.

In light of the above, we proved $(\varphi(t,\cdot), \psi(t,\cdot)) \in K$ for any $t \in [0, T_{max})$ and the proof of Theorem 3.2 is completed. □

In the same way as Theorem 3.2, we can get he following results.

Theorem 3.3. *Define*

$$K_+ := \{(u,v) \in H \times H, I(u,v) < d, S(u,v) < 0, Q(u,v) > 0\}, \tag{3.21}$$

$$R_- := \{(u,v) \in H \times H, I(u,v) < d, S(u,v) < 0\}, \tag{3.22}$$

$$R_+ := \{(u,v) \in H \times H, I(u,v) < d, S(u,v) > 0\}. \tag{3.23}$$

Then K_+, R_- and R_+ are all invariant sets of the Cauchy problem (1.1) *and* (2.1).

By the definition of K, K_+ and R_+, as well as (3.9), (3.14) and (3.18), we obtain the following result.

Corollary 3.1.

$$\{(\varphi,\psi)\in H\times H\backslash\{(0,0)\}, I(\varphi,\psi)<d\}=K\cup K_+\cup R_+. \tag{3.24}$$

4. Sharp threshold for global existence and blowup

Theorem 4.1. *Assume* $(\varphi_0,\psi_0)\in K_+\cup R_+$. *Then the solution* $(\varphi(t,x),\psi(t,x))$ *of the Cauchy problem* (1.1) *and* (2.1) *exists globally in* $t\in[0,\infty)$.

Proof. Firstly, we treat the case $(\varphi_0,\psi_0)\in K_+$. In light of Proposition 2.1 and Theorem 3.3, we know that the Cauchy problem (1.1) and (2.1) has a unique solution $(\varphi(t,\cdot),\psi(t,\cdot))\in K_+$ for any $t\in[0,T_{max})$. For fixed $t\in[0,T_{max})$, we denote $(\varphi(t,\cdot),\psi(t,\cdot))=(\varphi,\psi)$. Thus we have $I(\varphi,\psi)<d$, $Q(\varphi,\psi)>0$. It follows from (3.1) and (3.3) that

$$\frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx+\frac{5}{6}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx+\frac{1}{2}\int_{\mathbb{R}^3}(\lambda_1|u|^2+\lambda_2|v|^2)dx<d. \tag{4.1}$$

Then, we always get

$$\int_{\mathbb{R}^3}\Big[|\nabla u|^2+|\nabla v|^2+(x_1^2+x_2^2)(|u|^2+|v|^2)+\lambda_1|u|^2+\lambda_2|v|^2\Big]dx<c. \tag{4.2}$$

Thus, by Proposition 2.1, the solution $(\varphi(t,x),\psi(t,x))$ of the Cauchy problem (1.1) and (2.1) exists globally for all $t\in[0,\infty)$.

Next, we treat the case $(\varphi_0,\psi_0)\in R_+$. Proposition 2.1 and Theorem 3.3 imply that the Cauchy problem (1.1) and (2.1) has a unique solution $(\varphi(t,\cdot),\psi(t,\cdot))\in R_+$ for any $t\in[0,T_{max})$. Then we have $I(\varphi,\psi)<d$, $S(\varphi,\psi)>0$. By using (3.1) and (3.2), we obtain

$$\int_{\mathbb{R}^3}\Big[|\nabla u|^2+|\nabla v|^2+(x_1^2+x_2^2)(|u|^2+|v|^2)+\lambda_1|u|^2+\lambda_2|v|^2\Big]dx<4d. \tag{4.3}$$

Thus, Proposition 2.1 implies that $(\varphi(t,x),\psi(t,x))$ exists globally in $t\in[0,\infty)$.

Finally, combining the above two cases, we can draw our conclusion. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Assume* $(\varphi_0,\psi_0)\in K$, *then the solution* $(\varphi(t,x),\psi(t,x))$ *of the Cauchy problem* (1.1) *and* (2.1) *blows up in finite time.*

Proof. Let $(\varphi_0,\psi_0)\in K$ and let $(\varphi(t,x),\psi(t,x))$ be the solution of the Cauchy problem (1.1) and (2.1). Then, Proposition 2.1 and Theorem 3.2 imply that $(\varphi(t,\cdot),\psi(t,\cdot))\in K$ for all $t\in[0,T_{max})$. Fixing $t\in[0,T_{max})$, we denote $(\varphi(t,\cdot),\psi(t,\cdot))=(\varphi,\psi)$. Thus (φ,ψ) satisfies that $S(\varphi,\psi)<0, Q(\varphi,\psi)<0$.

Moreover, by the Virial identity (2.6) and (3.3), we have

$$J''(t)=8Q(\varphi,\psi)\quad for\ all\ t\in[0,T_{max}). \tag{4.4}$$

For $\mu>0$, we put $\varphi^\mu=\mu^{\frac{3}{4}}\varphi(\mu x)$, $\psi^\mu=\mu^{\frac{3}{4}}\psi(\mu x)$, then by a direct calculation, we can get

$$\begin{aligned}S(\varphi^\mu,\psi^\mu)=&\mu^{\frac{1}{2}}\int_{\mathbb{R}^3}(|\nabla\varphi|^2+|\nabla\psi|^2)dx+\mu^{-\frac{7}{2}}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx\\&+\mu^{-\frac{3}{2}}\int_{\mathbb{R}^3}(\lambda_1|\varphi|^2+\lambda_2|\psi|^2)dx-\int_{\mathbb{R}^3}(\mu_1|\varphi|^4+\mu_2|\psi|^4+2\beta|\varphi|^2|\psi|^2)dx,\end{aligned} \tag{4.5}$$

$$\begin{aligned} Q(\varphi^\mu,\psi^\mu) =&\mu^{\frac{1}{2}}\int_{\mathbb{R}^3}(|\nabla\varphi|^2+|\nabla\psi|^2)dx-\mu^{-\frac{7}{2}}\int_{\mathbb{R}^3}(x_1^2+x_2^2)(|u|^2+|v|^2)dx\\ &-\frac{3}{4}\int_{\mathbb{R}^3}(\mu_1|\varphi|^4+\mu_2|\psi|^4+2\beta|\varphi|^2|\psi|^2)dx. \end{aligned} \tag{4.6}$$

Since $Q(\varphi,\psi)<0$, it yields that there exists $\mu^*>1$ such that $Q(\varphi^{\mu^*},\psi^{\mu^*})=0$, and $Q(\varphi^\mu,\psi^\mu)<0$ for $\mu\in[1,\mu^*)$. For $\mu\in[1,\mu^*]$, since $S(\varphi,\psi)<0$, $S(\varphi^\mu,\psi^\mu)$ has the following two possibilities:

(i) $S(\varphi^\mu,\psi^\mu)<0$ for $\mu\in[1,\mu^*]$;

(ii) There exists $1<\sigma\le\mu^*$ such that $S(\varphi^\sigma,\psi^\sigma)=0$.

For the case (i), we have $Q(\varphi^{\mu^*},\psi^{\mu^*})=0$ and $S(\varphi^{\mu^*},\psi^{\mu^*})<0$, then $(\varphi^{\mu^*},\psi^{\mu^*})\in M$. From (3.9) and (3.18), one has $I(\varphi^{\mu^*},\psi^{\mu^*})\ge d_M\ge d>I(\varphi,\psi)$. Moreover, we have

$$\begin{aligned} I(\varphi,\psi)-I(\varphi^{\mu^*},\psi^{\mu^*}) =&\frac{1}{2}(1-\mu^{*\frac{1}{2}})\int_{\mathbb{R}^3}\big(|\nabla\varphi|^2+|\nabla\psi|^2\big)dx\\ &+\frac{1}{2}(1-\mu^{*-\frac{7}{2}})\int_{\mathbb{R}^3}\big(x_1^2+x_2^2)(|u|^2+|v|^2\big)dx\\ &+\frac{1}{2}(1-\mu^{*-\frac{3}{2}})\int_{\mathbb{R}^3}\big(\lambda_1|\varphi|^2+\lambda_2|\psi|^2\big)dx, \end{aligned} \tag{4.7}$$

$$\begin{aligned} Q(\varphi,\psi)-Q(\varphi^{\mu^*},\psi^{\mu^*}) =&(1-\mu^{*\frac{1}{2}})\int_{\mathbb{R}^3}(|\nabla\varphi|^2+|\nabla\psi|^2)dx\\ &-(1-\mu^{*-\frac{7}{2}})\int_{\mathbb{R}^3}\big(x_1^2+x_2^2)(|u|^2+|v|^2\big)dx. \end{aligned} \tag{4.8}$$

Noticing that $\mu^*>1$, it follows that

$$I(\varphi,\psi)-I(\varphi^{\mu^*},\psi^{\mu^*})\ge\frac{1}{2}Q(\varphi,\psi)-\frac{1}{2}Q(\varphi^{\mu^*},\psi^{\mu^*})=\frac{1}{2}Q(\varphi,\psi). \tag{4.9}$$

For the case (ii), we have $S(\varphi^\sigma,\psi^\sigma)=0$ and $Q(\varphi^\sigma,\psi^\sigma)\le0$, then $(\varphi^\sigma,\psi^\sigma)\in B$. In light of (3.14) and (3.18), we get $I(\varphi^\sigma,\psi^\sigma)\ge d_B\ge d>I(\varphi,\psi)$. Referring to (4.7) and (4.8), similarly we have

$$I(\varphi,\psi)-I(\varphi^\sigma,\psi^\sigma)\ge\frac{1}{2}Q(\varphi,\psi)-\frac{1}{2}Q(\varphi^\sigma,\psi^\sigma)\ge\frac{1}{2}Q(\varphi,\psi). \tag{4.10}$$

Since $I(\varphi^{\mu^*},\psi^{\mu^*})\ge d$, $I(\varphi^\sigma,\psi^\sigma)\ge d$, it follows from (4.9) and (4.10) that

$$Q(\varphi,\psi)<2[I(\varphi,\psi)-d]. \tag{4.11}$$

Collecting (2.4), (2.5) and (3.1), there is $I(\varphi,\psi)=I(\varphi_0,\psi_0)$. Thus by $(\varphi_0,\psi_0)\in K$ and (4.4), we have

$$J''(t)=8Q(\varphi,\psi)<16[I(\varphi_0,\psi_0)-d]<0. \tag{4.12}$$

It is clear that $J(t)$ can not verify (4.12) for all time t. Therefore, from Proposition 2.1, it must be the case that $T_{max}<\infty$, which implies

$$\lim_{t\to T_{max}}\big(\|\varphi(t,\cdot)\|_H+\|\psi(t,\cdot)\|_H\big)=\infty. \tag{4.13}$$

The proof of Theorem 4.2 is completed. □

Remark 4.1. With the help of Corollary 3.1, Theorem 4.1 and Theorem 4.2 provide a sharp threshold for global existence and blowup of the Cauchy problem (1.1) and (2.1).

Remark 4.2. It is worth noticing that our main results can be easily generalized. By a direction of our proofs, we can extend our results as to cover the general situation where

$$\begin{cases} -i\varphi_t+(x_1^2+\cdots+x_d^2)\varphi=\Delta\varphi+\mu_1|\varphi|^2\varphi+\beta|\psi|^2\varphi, \ (t,x_1,\cdots x_d,x_{d+1},\cdots x_N)\in\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^{N-d}, \\ -i\psi_t+(x_1^2+\cdots+x_d^2)\psi=\Delta\psi+\mu_2|\psi|^2\psi+\beta|\varphi|^2\psi, \ (t,x_1,\cdots x_d,x_{d+1},\cdots x_N)\in\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^{N-d}. \end{cases} \tag{4.14}$$

Actually choosing an appropriate energy space $H\times H$, one can derive the sharp condition for global existence and blow-up of the Cauchy problem (4.14).

References

[1] N. Akhmediev and A. Ankiewicz, *Partially coherent solitons on a finite background*, Physical Review Letters, 1999, 82, 2661.

[2] A. Ambrosetti and E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations*, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, 2006, 342(7), 453–458.

[3] P. Antonelli, R. Carles and J. D. Silva, *Scattering for nonlinear Schrödinger equation under partial harmonic confinement*, Communications in Mathematical Physics, 2015, 334(1), 367–396.

[4] T. Bartsch, Z. Q. Wang and J. C. Wei, *Bound states for a coupled Schrödinger system*, Journal of Fixed Point Theory and Applications, 2007, 2(2), 353–367.

[5] J. Bellazzini, N. Boussaid, L. Jeanjean and N. Visciglia, *Existence and stability of standing waves for supercritical NLS with a partial confinement*, Communications in Mathematical Physics, 2017, 353(1), 229–251.

[6] T. Busch and J. R. Anglin, *Dark-bright solitons in inhomogeneous Bose-Einstein Condensates*, Physical Review Letters, 2001, 87(1), 0104011–0104014.

[7] T. Cazenave, *Semilinear Schrödinger Equations*, in: Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematics Sciences, American Mathematics Society, New York Providence, RI, 2003.

[8] B. D. Esry, C. H. Greene, J. P. Burke Jr. and J. L. Bohn, *Hartree-Fock theory for double condensates*, Physical Review Letters, 1997, 78(19), 3594–3597.

[9] D. J. Frantzeskakis, *Dark solitons in atomic Bose-Einstein condensates: from theory to experiments*, Journal of Physics A: Mathematical and Theoretical, 2010, 43(21), 213001.

[10] Y. Guo, S. Li, J. Wei and X. Zeng, *Ground states of two-component attractive Bose-Einstein condensates, I. Existence and uniqueness*, Journal of Functional Analysis, 2019, 276(1), 183–230.

[11] Y. Guo, S. Li, J. Wei and X. Zeng, *Ground states of two-component attractive Bose-Einstein condensates, II. Semi-trivial limit behavior*, Transactions of the American Mathematical Society, 2019, 371(10), 6903–6948.

[12] Y. Guo, X. Zeng and H. Zhou, *Blow-up solutions for two coupled Gross-Pitaevskii equations with attractive interactions*, Discrete and Continuous Dynamical Systems, 2017, 37(7), 3749–3786.

[13] T. Gou, *Existence and orbital stability of standing waves to nonlinear Schrödinger system with partial confinement*, Journal of Mathematical Physics, 2018, 59(7), 071508.

[14] H. F. Jia, G. B. Li and X. Luo, *Stable standing waves for cubic nonlinear Schrödinger systems with partial confinement*, Discrete and Continuous Dynamical Systems, 2020, 40(5), 2739–2766.

[15] T. C. Lin and J. C. Wei, *Ground state of N coupled nonlinear Schrödinger equations in $R^n, n \leq 3$*, Communications in Mathematical Physics, 2005, 255(3), 629–653.

[16] T. C. Lin and J. C. Wei, *Spikes in two coupled nonlinear Schrödinger equations*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 2005, 22(4), 403–439.

[17] T. C. Lin and J. C. Wei, *Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials*, Journal of Differential Equations, 2006, 229(2), 538–569.

[18] L. A. Maia, E. Montefusco and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, Journal of Differential Equations, 2006, 229(2), 743–767.

[19] B. Malomed, in Multi-Component Bose-Einstein Condensates: Theory, Emergent Nonlinear Phenomena in Bose-Einstein Condensation, edited by P. G. Kevrekidis, D. J. Frantzeskakis and R. Carretero-Gonzalez, (Springer-Verlag, Berlin, 2008), pp. 287–305.

[20] B. Noris, H. Tavares and G. Verzini, *Stable solitary waves with prescribed L^2-mass for the cubic Schrödinger system with trapping potentials*, Discrete and Continuous Dynamical Systems, 2015, 35(12), 6085–6112.

[21] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with a partial confinement*, Communications on Pure and Applied Analysis, 2018, 17(4), 1671–1680.

[22] J. Royo-Letelier, *Segregation and symmetry breaking of strongly coupled two-component Bose-Einstein condensates in a harmonic trap*, Calculus of Variations and Partial Differential Equations, 2014, 49(1-2), 103–124.

[23] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in R^n*, Communications in Mathematical Physics, 2007, 271(1), 199–221.

[24] J. C. Wei and T. Weth, *Nonradial symmetric bound states for a system of coupled Schrödinger equations*, Rendiconti Lincei-Matematica e Applicazioni, 2007, 18(3), 279–294.

[25] C. L. Wang and J. Zhang, *Sharp condition for global existence of supercritical nonlinear Schrödinger equation with a partial confinement*, Acta Mathematicae Applicatae Sinica, English Series, 2023, 39(1), 202–210.

[26] J. Zhang, *Cross-constrained variational problem and nonlinear Schrödinger equation*, Foundations of Computational Mathematics, World Scientific Publishing, River Edge, NJ, 2002, 457–469.

[27] J. Zhang, *Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential*, Communications in Partial Differential Equations, 2005, 30(10), 1429–1443.

[28] J. Zhang, *Sharp threshold of global existence for nonlinear Schrödinger equation with partial confinement*, Nonlinear Analysis: Theory, Methods and Applications, 2020, 196, 111832.