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Cross-Invariant Sets of the Cubic Nonlinear Schrödinger System with Partial Confinement*

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Abstract This paper studies the cubic nonlinear Schrödinger system with partial confinement:

$$\begin{cases} -i\varphi_t + (x_1^2 + x_2^2)\varphi = \Delta\varphi + \mu_1|\varphi|^2\varphi + \beta|\psi|^2\varphi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ -i\psi_t + (x_1^2 + x_2^2)\psi = \Delta\psi + \mu_2|\psi|^2\psi + \beta|\varphi|^2\psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

which models the Bose-Einstein condensates with multiple states or the propagation of mutually incoherent wave packets in nonlinear optics. The cross-invariant sets of the evolution flow are obtained by constructing the cross-constrained variational problem. Furthermore, the sharp condition for global existence and blowup of the solutions is derived.

Keywords Bose-Einstein condensates, nonlinear Schrödinger system, cross-invariant sets, sharp condition, global existence

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1. Introduction

In this paper, we investigate the following cubic nonlinear Schrödinger systems with partial confinement:

$$\begin{cases} -i\varphi_t + (x_1^2 + x_2^2)\varphi = \Delta\varphi + \mu_1|\varphi|^2\varphi + \beta|\psi|^2\varphi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ -i\psi_t + (x_1^2 + x_2^2)\psi = \Delta\psi + \mu_2|\psi|^2\psi + \beta|\varphi|^2\psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mu_1, \mu_2, \beta > 0$, $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ are complex-valued wave functions of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $i = \sqrt{-1}$, and Δ is the Laplace operator on $\mathbb{R}^3$. The system (1.1) models Bose-Einstein condensates with multiple states or the propagation of mutually incoherent wave packets in nonlinear optics [1,8,9,19]. The parameters μ_1, μ_2 and β represent the intraspecies and interspecies scattering lengths, describing the interaction between particles of the same component or of different components respectively [6,8]. In particular, the positive sign of $\mu_j (j = 1,2)$ (and of β) stands for

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attractive interaction, while the negative sign stands for repulsive interaction [6,8]. In the present paper, we treat (1.1) in the case of positive parameters $\mu_1, \mu_2, \beta > 0$, which is the so-called self-focusing and attractive interaction. For system (1.1), the physically relevant cubic nonlinearity is L^2-supercritical and H^1-subcritical. Cubic nonlinear Schrödinger systems, often referred to as Gross-Pitaevskii equations, are very important in Physics.

In (1.1), we set

$$\varphi(t,x) = u(x)e^{i\lambda_1 t} \quad \text{and} \quad \psi(t,x) = v(x)e^{i\lambda_2 t} \tag{1.2}$$

with $\lambda_1, \lambda_2 > 0$, then (u,v) solves the following elliptic systems:

$$\begin{cases} -\Delta u + \lambda_1 u + (x_1^2 + x_2^2)u = \mu_1 u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\Delta v + \lambda_2 v + (x_1^2 + x_2^2)v = \mu_2 v^3 + \beta u^2 v, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

Hence, (1.2) is a standing wave of the system (1.1). In [16], the existence and asymptotic behavior of least energy solutions of (1.3) are studied in a bounded domain with constant trapping potential. In [17], the asymptotic behavior is studied in $\mathbb{R}^N$ under the influence of nonconstant trapping potentials. In $\mathbb{R}^N$, the least energy and higher energy bound states of (1.3) are investigated in [2,4,15,18,23,24].

On the other hand, the study of normalized solutions for (1.3) has received lots of attention, that is, considering (1.3) with the constraints

$$\int_{\mathbb{R}^3} u^2 dx = a^2 \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 dx = b^2. \tag{1.4}$$

When $N = 2$, Guo et al. in [10–12] considered the existence, non-existence, uniqueness and asymptotic behavior of solutions to problem (1.3)-(1.4) with a certain type of trapping potentials. In [22], J. Royo-Letelier addressed both segregation and symmetry breaking for (1.3) in $\mathbb{R}^2$. In [20], B. Noris et al. studied problem (1.3) in bounded domains of $\mathbb{R}^N$, or the problem with trapping potentials in the whole space $\mathbb{R}^N$ (the presence of a trapping potential makes the two problems essentially equivalent) with $N \leq 3$. In both cases, they proved the existence of positive solutions with small masses a and b, and the orbital stability of the associated solitary waves. Gou [13] treated (1.3)-(1.4) with partial confinements when $N = 3$, and showed the compactness of minimizing sequences up to translations in x_3. As a by-product, they also obtained the orbital stability of the set of global minimizers. Jia et al. [14] obtained the existence of stable standing waves for (1.3)-(1.4) when $N = 3$.

As a motivation, we recall the single nonlinear Schrödinger equation with a partial confinement

$$i\varphi_t + \Delta\varphi - (x_1^2 + x_2^2)\varphi + |\varphi|^{p-1}\varphi = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.5}$$

where $\varphi = \varphi(t,x)$ is a complex-valued wave function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$, $i = \sqrt{-1}$, Δ is the Laplace operator on $\mathbb{R}^N$, and $1 < p < \frac{N+2}{(N-2)^+}$ (we use the convention: $\frac{N+2}{(N-2)^+} = \infty$ when $N = 1, 2$, and $(N-2)^+ = N - 2$ when $N \geq 3$). For equation (1.5), the authors in [5] studied the existence and stability of standing waves for the nonlinear Schrödinger equations with partial confinement in $\mathbb{R}^3$. In [3] and [21], the authors studied the scattering and strong instability of standing waves for nonlinear