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ESS in a River Network with Two Branches*

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Abstract In a recent work by Jiang et al. [Bull. Math. Biol., 2020, Paper No. 131, 42pp], to discuss the impact of differing topology of river networks on the evolution of dispersal, the authors proposed three different models and found the winning of slower/faster diffuser or evolutionarily singular strategies. However, it is unknown whether there is an evolutionarily stable strategy (ESS, a central concept in evolution game theory) for these three models. In this paper, focusing on the Model (III) proposed by them (the most complicated one as they stated), we give a confirmed answer to this issue, that is, there does exist ESS.

Keywords Evolution of dispersal, evolutionarily stable strategy, advective patchy environment, selection gradient

MSC(2010) 34D20, 92D15, 92D40.

1. Introduction

The issue of the evolution of dispersal, a central topic in spatial ecology and evolutionary biology, has been extensively studied by both mathematicians and mathematical biologists in the past few decades. A well accepted result due to Hastings [11] (see also Dockery et al. [8]) says that in spatially heterogeneous and temporally constant environments, slower diffusion is selected for provided the populations take random diffusion only. This issue was later further explored in different biological situations, see, e.g., in temporally varying (time periodic) environments [12], in the situation where organisms take both random diffusion and some conditional dispersal like movement upward along the resource gradient [4–6, 9, 15, 16], in the advective environments such as rivers, streams where individuals are subject to passive downstream movement [7, 18, 22, 25, 26], and in the space-discrete environments with a finite number of patches located in a straight line [1–3, 10, 19, 21, 23]. To mention but a few.

Recently, considering river populations in a patchy environment, Jiang et al. [13] proposed a different angle to pursue further the above issue. Specifically, starting

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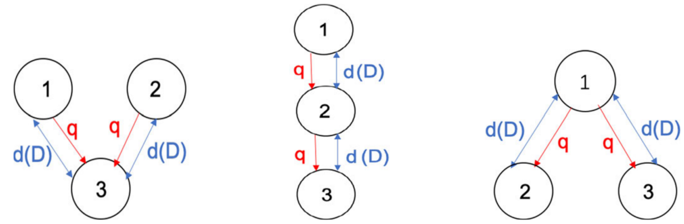
with the case of three patches, they considered three differing river network modules (not necessarily located in a line, see Fig. 1 below) and proposed three different models based on each module. Compared with the two-patch case [10,13,20,23], beside the phenomenon "the slower/faster diffuser wins", they also found the existence of evolutionarily singular strategies for the three-patch case.



(a) Model (I) **(b)** Model (II) **(c)** Model (III)

Figure 1. Three different river network modules proposed by Jiang et al. [13, Fig. 2]. Here, the two-way blue arrows reflect the random diffusion between connected patches and the one-way red arrows reflect the unidirectional drift.

Motivated by Jiang et al. [13], a natural question that deserves further investigation is whether there exists an evolutionarily stable strategy (abbreviated below as ESS), which is a core concept in evolutionary biology proposed by Maynard-Smith and Price [24]. Roughly speaking, a strategy is said to be an ESS (resp. EBP, i.e, evolutionarily branching point) if a resident population adopting it cannot (resp. can) be invaded by a mutant using a different strategy when rare; another related concept is the convergent stable strategy (CSS) and a strategy is called a CSS if the mutant can always invade the resident when the mutant strategy is closer to the CSS than the resident strategy, see the definitions in, e.g., [17]. Note that an evolutionarily singular strategy is not necessarily an ESS, but an ESS must be an evolutionarily singular strategy (see [17]).

For Model (I), a positive answer to the above question was recently given by Jin and Zhou [14]. As pointed out in Jiang et al. [13], Model (III) is the most complicated one and is still less understood. So, in the current paper, we aim to pursue further study on this model by employing some idea developed in [14]. To be more specific, we recall Model (III) as below

$$\begin{cases} u_{1t} = d(u_2 + u_3 - 2u_1) - 2qu_1 + u_1(1 - \frac{u_1+v_1}{K_1}), \\ u_{2t} = d(u_1 - u_2) + qu_1 + u_2(1 - \frac{u_2+v_2}{K_2}), \\ u_{3t} = d(u_1 - u_3) + qu_1 + u_3(1 - \frac{u_3+v_3}{K_3}), \\ v_{1t} = D(v_2 + v_3 - 2v_1) - 2qv_1 + v_1(1 - \frac{u_1+v_1}{K_1}), \\ v_{2t} = D(v_1 - v_2) + qv_1 + v_2(1 - \frac{u_2+v_2}{K_2}), \\ v_{3t} = D(v_1 - v_3) + qv_1 + v_3(1 - \frac{u_3+v_3}{K_3}), \\ u_i(0) = u_{i0} > 0, \quad v_i(0) = v_{i0} > 0, \quad i = 1, 2, 3, \end{cases} \tag{1.1}$$

where $u_i = u_i(t)$ (resp. $v_i = v_i(t)$), $i = 1, 2, 3$, denotes the population density of species u (resp. v) in the patch i with the carrying capacity $K_i > 0$, and the constants $d > 0$ and $D > 0$ are the diffusion rates and $q > 0$ is the advection rate.

To introduce the main result of system (1.1) in Jiang et al. [13], let $E_u =$