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Multiplicity of Solutions for a Class of Critical Choquard Equation in a Bounded Domain

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Abstract. In this paper, we consider the following critical Choquard equation

$$-\Delta u=\mu f(x)|u|^{p-2}u+\left(\int_\Omega \frac{g(y)|u(y)|^{6-\nu}}{|x-y|^\nu}\mathrm{d}y\right)g(x)|u|^{4-\nu}u,\quad x\in\Omega,$$

where $\mu>0$ is a parameter, $\nu\in(0,3)$, $p\in(4,6)$ and f,g are continuous functions. For μ small enough, by using Lusternik-Schnirelmann category theory, we establish a relationship between the number of solutions and the category of the global maximum set of g.

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1 Introduction

In recentl years, the following nonlinear Choquard equation has been extensively studied in the literature,

$$-\Delta u+V(x)u=\left(|x|^{-\nu}*|u|^p\right)|u|^{p-2}u,\quad \text{in } \mathbb{R}^N.$$

This equation has a rich physical background. In [1], Pekar used it to describe the quantum theory of a polaron at rest. In [2], Choquard exploited it to approximate the Hartree-Fock theory of one component plasma. Penrose [3] derived it as a model of self-gravitating

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matter. These aroused great interest among mathematicians. For $N=3$, $\nu=1$, $p=2$ and V a positive constant, by using rearrangements techniques, Lieb [2] obtained the existence and uniqueness of ground state solution. Then, by using variational approach, Lions [4,5] got a sequence of radially symmetric solutions. The nondegeneracy of the ground states was proved by Wei and Winter [6]. Since then, more and more results have been established about this equation.

Ma and Zhao [7] used the method of moving planes and obtained that all the positive solutions of the following equation must be radially symmetric and monotone decreasing about some fixed point

$$\Delta u - u + 2u\left(\frac{1}{|x|} * |u|^2\right) = 0, \qquad u \in H^1\left(\mathbb{R}^3\right).$$

Gao and Yang [8] generalized the Brézis-Nirenberg problem to Choquard equation

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}y\right) |u|^{2_\mu^*-2} u + \lambda u, \qquad \text{in } \Omega.$$

For different values of the parameter λ, they got the existence of nontrivial solution.

When the potential function is not a constant, Shen et al. [9] investigated the following Choquard equation with potential well

$$-\Delta u + (\lambda V(x) - \beta) u = \left(|x|^{-\mu} * |u|^{2_\mu^*}\right) |u|^{2_\mu^*-2} u, \qquad \text{in } \mathbb{R}^N.$$

Under the assumption that the operator $-\Delta + \lambda V(x) - \beta$ is non-degenerate, they got the existence of ground state solutions. When the parameter λ is large enough, these solutions will localize near the potential well int $V^{-1}(0)$. They also obtained the existence of multiple solutions when the parameter β is small enough.

For the semiclassical states of Choquard equation, Moroz and Van Schaftingen [10] used penalization technique and proved that the following problem has a family of solutions concentrating to the local minimum of $V(x)$

$$-\varepsilon^2 \Delta u_\varepsilon + V u_\varepsilon = \varepsilon^{-\alpha}\left(I_\alpha * |u_\varepsilon|^p\right) |u_\varepsilon|^{p-2} u_\varepsilon, \qquad \text{in } \mathbb{R}^N,$$

for $N \geq 1, \alpha \in (0,N)$, and $\varepsilon > 0$ small enough. In terms of variational methods and energy estimates, Meng and He [11] studied the following Choquard equation

$$-\varepsilon^2 \Delta u + V(x) u = \varepsilon^{-\alpha} Q(x) \left(I_\alpha * |u|^{2_\alpha^*}\right) |u|^{2_\alpha^*-2} u + f(u), \qquad \text{in } \mathbb{R}^N,$$

where $N \geq 3, (N-4)_+ < \alpha < N$. They obtained how the morphology of potential functions $V(x)$ and $Q(x)$ affect the number of solutions. Moreover, the concentration behavior of solutions was also established. For more results about the Choquard equation, we refer to [12–19] and the references therein.