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Several Parameter Type Logarithmic Sobolev Inequalities

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Abstract. In this paper, we obtain a parameter type logarithmic Sobolev inequality with weight and a modified parameter type logarithmic Sobolev inequality with weight on Euclidean space based on the parameter type logarithmic Caffarelli-Kohn-Nirenberg inequality on Euclidean space, respectively. By virtue of the convexity of a special function equivalent to a logarithmic Hölder inequality, and combining the Sobolev inequality and Gagliardo-Nirenberg inequality on Hörmander's vector fields, we also derive a logarithmic Sobolev inequality and a parameter type logarithmic Gagliardo-Nirenberg inequality on Hörmander's vector fields, respectively. In addition, a parameter type logarithmic Sobolev inequality on Hörmander's vector fields is given by suitable stretching transformation.

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1 Introduction

Many important inequalities, such as Caffarelli-Kohn-Nirenberg inequality, Gagliardo-Nirenberg inequality, Hardy-Sobolev inequality, Sobolev inequality are basic research tools in the field of nonlinear functional analysis, partial differential equations, and differential geometry. For one thing, the optimal constants and extreme value functions of these inequalities have important research significance. In recent years, the logarithmic

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nonlinear terms in partial differential equations have been widely used to describe natural phenomena occurring in physics, chemistry, and biology due to their unique structure. For instance, in biology, the Keller-Segel chemotaxis model [1] describes the directed movement of bacterial microorganisms towards or away from certain stimulatory chemicals in their environment, the sensing function $B(v)=\log|v|$ in this model aligns with Weber-Fechner's law [2], which describes human perception of external physical stimuli. Notably, Wang [3] discovered that the singularity of the logarithmic sensing function $\log|v|$ at $v=0$ is a necessary condition for generating traveling wave solutions in the Keller-Segel chemotaxis model. These singularities present technical hurdles for the theoretical analysis of related models, and investigating them can reveal general or specific biological insights. This paper will establish several parametric logarithmic Sobolev inequalities associated with the singular logarithmic nonlinear term $\log\left(|x|^\alpha u\right)$. These inequalities can be employed to study the existence, blow-up phenomena, regularity, and other characteristics of solutions to elliptic and parabolic equations featuring singular logarithmic nonlinear term $\log\left(|x|^\alpha u\right)$. Furthermore, They are instrumental in revealing the intrinsic essential characteristics embodied in the solutions of these partial differential equations.

Gross [4] established the logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} |u(x)|^2 \log\left(\frac{|u(x)|}{\|u(x)\|_{L^2(\nu)}}\right) \mathrm{d}\nu(x) \le \int_{\mathbb{R}^n} |\nabla u(x)|^2 \mathrm{d}\nu(x),$$

where $\nu(x)$ is Gaussian measure on $\mathbb{R}^n$ and $\mathrm{d}\nu(x)=(2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\mathrm{d}x$. This inequality has a uniform constant independent of the dimension n and allows extension to infinite-dimensional spaces. The remarkable thing about this inequality is that it can lead to the ultracontractivity and hypercontractivity properties of the corresponding Markovian semigroups.

Weissler [5] obtained an analogous Gross's logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} |u(x)|^q \log|u(x)| \mathrm{d}\nu(x) \le \frac{n}{2q} \log\left[\frac{q^2}{2\pi n e(q-1)} \frac{\mathrm{Re}\left\langle -\Delta u, (\mathrm{sgn} u)|u|^{q-1}\right\rangle}{\|u(x)\|^q_{L^q(\nu)}}\right] + \log\|u(x)\|_{L^q(\nu)}.$$

It is shown to be equivalent to the best norm estimates for a heat-diffusion semigroup as $e^{t\Delta}$, which maps $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1<q<p<+\infty$. This inequality can be reformulated in a form which is optimal under scalings as

$$2\int_{\mathbb{R}^n} |u(x)|^2 \log|u(x)| \mathrm{d}x \le \frac{n}{2} \log\left[\frac{2}{\pi n e} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \mathrm{d}x\right]$$

for any $u \in W^{1,2}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |u(x)|^2 \mathrm{d}x = 1$.