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Well-Posedness of 5th-Order KdV Equation Posed on a Finite Domain with Nonlinear Boundary Values

HOU Wanmei[1] and ZHAO Xiangqing[2,*]

[1] *School of Marxism, Suqian University, Suqian 223800, China;*
[2] *School of Mathematics and Physics, Suqian University, Suqian 223800, China.*

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Abstract. In this article, we investigate the well-posedness of the initial-boundary value problem (I-B-V problem) for the fifth-order KdV equation posed on a finite domain with nonlinear boundary conditions. Firstly, we establish various a priori estimates, including Kato smoothing effects, sharp trace regularity, and nonlinear estimates. Subsequently, we demonstrate that the initial-boundary value problem of the fifth-order KdV equation with quadratic boundary feedbacks is locally well-posed for the appropriately chosen initial value and boundary values.

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1 Introduction

In this paper, we will study a nonlinear dispersive equation

$$\partial_t u+\alpha\partial_x u+\beta\partial_x^3 u+\gamma\partial_x^5 u+u\partial_x u=0,$$

which is used to describe magnetic-acoustic waves in cold collision-free plasma [1] and capillary-gravity waves in shallow water [2]. Since this equation shares much in common with Korteweg-de Vries (KdV) equation

$$\partial_t u+\alpha\partial_x u+\alpha\partial_x^3 u+u\partial_x u=0$$

*Corresponding author. *Email addresses:* `zhao-xiangqing@163.com` (X. Q. Zhao)

both in physical and mathematical view, we call it fifith-order Korteweg-de Vries equation.

As a typical dispersive equation, extensive studies have been conducted on the 5th-order KdV equation over the past two decades.

Regarding the initial value problem (I-V problem) of the 5th-order KdV equation, what people are pursuing is the existence of low-regularity solutions. For instance, in 2005, Cui and Tao [3] demonstrated that the I-V problem is locally well-posed in H^s for $s > \frac{1}{4}$. In 2007, Wang, Cui, and Deng [4] showed that the I-V problem can be locally solved in $H^s(\mathbb{R})$ for $s > -\frac{7}{5}$, and that global solutions exist for $s > -\frac{1}{2}$. From 2010 to 2013, there was substantial progress in improving the global well-posedness of this problem, culminating in the sharp global well-posedness reported in [5–7].

Regarding the I-B-V problem for the 5th-order KdV equation, the handling of different boundary conditions is the focal point of attention, say for example: in 2014, Zhao and Zhang [8] established boundary smoothing effect and sharp trace regularity associate with the boundary value: $u(0,t) = h_1(t)$, $u(1,t) = h_2(t)$, $\partial_x u(0,t) = h_3(t)$, $\partial_x u(1,t) = h_4(t)$, $\partial_x^2 u(1,t) = h_5(t)$; in 2018, Zhao, Zhang [9] verified the well-posedness of the initial boundary value problem for the 5th-order KdV equation under general boundary conditions (where 16 admissible boundary values proposed).

We can't help but ask: Why only study linear boundary conditions? In this article, we will explore the well-posedness of the I-B-V problem for the 5th-order KdV equation under nonlinear boundary conditions. As in Reference [9], without loss of generality, we study the following simplified 5th-order KdV equation posed on finite interval [0,1]:

$$\partial_t u - \partial_x^5 u + u\partial_x u = 0, \qquad x \in [0,1], t > 0.$$

Limited by our analytical methods, we only study quadratic nonlinearity. To illustrate our analytical approach clearly, we investigate the following typical case:

$$\begin{cases} \partial_t u - \partial_x^5 u + u\partial_x u = 0, & 0 < x < 1, t > 0, \\ u(x,0) = \phi(x), & 0 < x < 1, \\ u(0,t) = \partial_x u(0,t) = \partial_x^2 u(1,t) = \partial_x^3 u(1,t) = 0, \partial_x^4 u(1,t) = \frac{1}{3}\left[u(1,t)\right]^2, & t > 0. \end{cases}$$

In order to facilitate the description of the main results, we give some notations first. The initial value belongs to $H_0^s(0,1)$ and the boundary values belong to

$$\mathcal{H}^s(0,T) \equiv H_0^{\frac{s+2}{5}}(0,T) \times H_0^{\frac{s+1}{5}}(0,T) \times H_0^{\frac{s}{5}}(0,T) \times H_0^{\frac{s-1}{5}}(0,T) \times H_0^{\frac{s-2}{5}}(0,T).$$

Thus, function space of the initial value and boundary values can be shorted as

$$X_T^s \equiv H_0^s(0,1) \times \mathcal{H}^s(0,T).$$

Since the solution usually have the Kato smoothing effect, it is enough to introduce the solution space as

$$Y_T^s \equiv C([0,T];H^s(0,1)) \cap L^2([0,T];H^{s+2}(0,1)).$$