J.Partial Diff. Eqs. 8(1995), 333–340
©International Academic Publishers
Vol.8, No.4

POSITIVE SOLUTION OF ELLIPTIC SYSTEMS INVOLVING CRITICAL GROWTH

Hu Jiuxiang
(Dept. of Naval Arch. and Ocean Engineering, Huazhong Univ. Sci. & Tech., Wuhan 430074, China)

Chen Qingyi
(Dept. of Math., Huazhong Univ. of Sci. & Tech., Wuhan 430074, China)

(Received Oct. 4, 1993; revised Apr. 3, 1995)

Abstract In this paper, we consider the asymptotic behaviour of the positive solution of elliptic systems with critical growth and obtain the growth rate.

Key Words Radiation singular solution; critical growth.
Classification 35K60, 35K55, 35B40.

1. Introduction

In this paper, we consider the eigenvalue problem:

$$-\Delta u = v^q + \mu v, \quad v > 0 \text{ in } B_1 \tag{1.1a}$$

$$-\Delta v = u^p + \nu u, \quad u > 0 \text{ in } B_1 \tag{1.1b}$$

$$u = 0, \ v = 0, \qquad \text{on } \partial B_1 \tag{1.2}$$

in which B_1 is the unit ball in R^N $(N \geq 4)$ with boundary ∂B_1 and

$$p = \frac{N-w}{w}, \quad q = \frac{2+w}{N-2-w}, \quad (\mu, \nu) \in R^2 \tag{1.3}$$

where $w \in ((N-4)/2, N/2)$, $N \geq 4$. Notice that p and q satisfy the relation

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \tag{1.4}$$

p and q satisfying (1.4) are called critical exponents of (1.1), which is described in [1] in detail.

To formulate our results, we shall introduce the linear eigenvalue problem

$$-\Delta u = \lambda_2 v, \quad v > 0 \text{ in } B_1$$

$$-\Delta v = \lambda_1 u, \quad u > 0 \text{ in } B_1$$

$$u = 0,\ v = 0 \qquad \text{on } \partial B_1$$

As was shown in [1], there exists a curve $\mathcal{C}$ of eigenvalue for which there is a solution, this curve is given by

$$\mathcal{C} = \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \lambda_2 > 0 \text{ and } \lambda_1\lambda_2 = \mu_1^2\} \tag{1.5}$$

where μ_1 is the principal eigenvalue of Δ on the unit ball.

By a result of W.C. Troy [2], the solution of the problem (1.1) (1.2) is automatically radially symmetric if $\mu \geq 0$ and $\nu \geq 0$. This enables us to use ODE techniques.

By [1] (Theorem 6), for $\mu \leq 0$ and $\nu \leq 0$, problem (1.1) (1.2) also has no solution with $(u, v) \subset (C^2(\overline{B}_1))^2$. If $\mu \geq \lambda_2$ and $\nu \geq \lambda_1$ there exists no solution. However, the point $(\lambda_2, \lambda_1; 0, 0)$ is a bifurcation point from which emanats an unbounded branch of solution $(\lambda_2, \lambda_1; u, v)$. In this note, we shall be interested in the asymptotic properties of u, v as supermum $|u|_\infty$ of u, $|v|_\infty$ of v tends to infinity.

To formulate our results, we first need to introduce the notation of a *radial singular solution of the problem* (1.1) (1.2). By this we mean that functions $U(x)$, $V(x)$ which satisfy (1.1) in $B_1\backslash\{0\}$ and (1.2) on ∂B_1, have radial symmetry, and behave near the origin as

$$|x|^\alpha U(x) \to A(p, q, N), \quad |x|^\beta V(x) \to B(p, q, N), \text{ as } x \to 0 \tag{1.6}$$

where

$$\alpha = \frac{2q+2}{pq-1}, \quad \beta = \frac{2q+2}{pq-1} \tag{1.7a}$$

$$A(p, q, N) = [\alpha\beta^q(N-2-\alpha)(N-2-\beta)^q]^{1/(pq-1)} \tag{1.7b}$$

$$B(p, q, N) = [\alpha^p\beta(N-2-\alpha)^p(N-2-\beta)]^{1/(pq-1)} \tag{1.7c}$$

Here, we know that $A|x|^{-\alpha}$ and $B|x|^{-\beta}$ solve

$$-\Delta u = v^q, \ \Delta v = u^p, \quad u > 0, \ v > 0 \text{ in } B_1\backslash\{0\}$$

We conject that such solutions are in fact the only possible radial solutions of (1.1) with an isolated singularity at the origin.

Theorem 1 *Suppose p and q satisfy* (1.3). *There exists a unique singular solution pair $(u(x), v(x))$. Its asymptotic behaviour near the origin is given by*

$$u(x) = A(p, q, N)|x|^{-\alpha}(1 + C(p, q, N)|x|^h + o(|x|^h))$$
$$v(x) = B(p, q, N)|x|^{-\beta}(1 + D(p, q, N)|x|^h + o(|x|^h))$$

as $x \to 0$, where $h = \min\{(p-1)\alpha), (q-1)\beta)\}$, $g = h\lambda$, λ is given in Lemma 2.1, and if $p > q$,

$$C(p, q, N) = -\frac{(\alpha\beta)^{(p-q)/(pq-1)}[g(\lambda^2 - (\beta-\alpha)\lambda) - \alpha\beta + g(g+1)]}{[g\lambda^2 - \alpha\beta + g(g-1)]^2 - (\beta-\alpha)^2\lambda^2 - \alpha^2\beta^2 pq}$$