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SPECTRAL ASYMPTOTIC BEHAVIOR FOR A CLASS OF SCHRÖDINGER OPERATORS ON 1-DIMENSIONAL FRACTAL DOMAINS*

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Abstract In this paper, we study the spectral asymptotic behavior for a class of Schrödinger operators on 1-dimensional fractal domains. We have obtained, if the potential function is locally constant, the exact second term of the spectral asymptotics. In general, we give a sharp estimate for the second term of the spectral asymptotics.
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1. Introduction

Let Ω be an open set in $R^n (n \geq 1)$, with boundary $\Gamma = \partial\Omega$. We assume that Ω is non-empty and of finite volume (n-dimensional Lebesgue measure). We consider the following eigenvalue problem of Schrödinger operator:

$$\begin{cases} -\Delta u + \lambda V u = \mu u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \tag{P}$$

where $\lambda > 0, V(x) \in C(\bar{\Omega})$, and Δ denotes the Dirichlet Laplacian on Ω. In fact, the scalar μ is said to be an eigenvalue of (P) if there exists $u \neq 0$ in $H_0^1(\Omega)$ which satisfies (P) in the distributional sense. It is well known that, the problem (P) has discrete eigenvalues if λ is given, which can be written in increasing order according to their finite multiplicities:

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \leq \cdots < +\infty \quad \text{with } \mu_r \to \infty \text{ as } r \to \infty \tag{1.1}$$

Let $E > 0, N(E, \lambda)$ denote the "counting function" of (P) associated with E, i.e. $N(E, \lambda) = \#\{k \geq 1, \mu_k \leq \lambda E\}$ is the number of eigenvalues of (P) less than λE, which is counted according to the multiplicities.

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We have known that in one-dimensional case (see [1]),

$$N(E,\lambda)=\left[\frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}}(E-V(x))^{1/2}dx+o(1)\right]\lambda^{1/2},\quad \Omega\subset R^1 \tag{1.2}$$

as $\lambda\to\infty$.

In this paper, we are interested in the sharper asymptotic form of $N(E,\lambda)$ for Ω with fractal boundary. First, let us recall some results on the Weyl conjecture and the Weyl-Berry conjecture, which will help us to understand the main result in this paper.

Consider

$$\begin{cases}-\Delta u=\lambda u & \text{in }\Omega\\ u=0 & \text{on }\Gamma\end{cases} \tag{Q}$$

where Δ denotes the Dirichlet Laplacian defined in Ω. Let $\mu>0, N_0(\mu)=\#\{k\geq 1; 0<\lambda_k\leq\mu\}$, where λ_r is Dirichlet eigenvalue of (Q).

In 1912, H.Weyl[2] proved that if Ω has sufficiently smooth boundary, then

$$N_0(\mu)=\varphi(\mu)(1+o(1))\text{ as }\mu\to+\infty \tag{1.3}$$

where $\varphi(\mu)$ is called the "Weyl term" and is given by

$$\varphi(\mu)=(2\pi)^{-n}B_n|\Omega|_n\mu^{n/2} \tag{1.4}$$

Here, $|\Omega|_n$ denotes the n-dimensional Lebesgue measure of Ω and B_n is the volume of unit ball in R^n. Furthermore, he conjectured that, in the "smooth" case,

$$N_0(\mu)=\varphi(\mu)-C_n|\Gamma|_{n-1}\mu^{(n-1)/2}+o(\mu^{(n-1)/2})\text{ as }\mu\to+\infty \tag{1.5}$$

where C_n is a positive constant depending only on n. (Here $|\Gamma|_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure of Γ).

An important step on the way to the Weyl's conjecture was made by R.T.Seeley[3], and then by Pham The Lai [4]. They showed that for Γ is C^∞ smooth, then

$$N_0(\mu)=\varphi(\mu)+O(\mu^{(n-1)/2})\text{ as }\mu\to+\infty \tag{1.6}$$

Further, V.Ja.Ivrii[5,6] and Melrose[7,8] have established (1.7) under some additional assumption, i.e. the Weyl's conjecture is true under some conditions.

How about the situation if the boundary Γ is non-smooth? The physicist Michael V.Berry[9] made the following conjecture: If the boundary $\Gamma=\partial\Omega$ is "fractal" with Hausdorff dimension $H\in(n-1,n)$ and H-dimensional Hausdorff measure $\mathcal{H}(H;\Gamma)$, then

$$N_0(\mu)=\varphi(\mu)-C_H\mathcal{H}(H;\Gamma)\mu^{H/2}+o(\mu^{H/2})\text{ as }\mu\to+\infty \tag{1.7}$$

where C_H is a positive constant depending only on H. Berry even conjectured that $C_H=(4(4\pi)^{H/2}\Gamma(1+H/2))^{-1}$, where $\Gamma(s)$ denotes the classical gamma function.

Unfortunately, Berry's conjecture was false in general. J.Brossard and R.Carmona[10] gave some counter-exsamples to show Berry's conjecture might be true if the Hausdorff dimension H could be replaced by the Minkowski dimension D. This modified conjecture has been called the Weyl-Berry conjecture (see [11]). Indeed the Minkowski dimension is more appropriate to measure the "roughness" of the boundary Γ. In 1993, Michel L.Lapidus and Carl Pomerance[11] proved that for one-dimensional case, the Weyl-Berry conjecture is true:

$$N_0(\mu) = \varphi(\mu) - C_{1,D}\mathcal{M}(D;\Gamma)\mu^{D/2} + o(\mu^{D/2}) \text{ as } \mu \to +\infty \tag{1.8}$$

where $\mathcal{M}(D;\Gamma)$ denotes the Minkowski measure of Γ and

$$C_{1,D} = 2^{-(1-D)}\pi^{-D}(1-D)(-\zeta(D)) \tag{1.9}$$

Here, we should notice that $\zeta(x)$ is the classical Riemann zeta-function, $D \in (0,1)$ and $\zeta(D) < 0$.

In this paper, we want to obtain the similar result as the asymptotic formula (1.8) for the eigenvalue problem (P) of the Schrödinger operator. Since in this case we have potential function $V(x)$, the situation would be more complicated.

2. Minkowski Dimension and Measure

The Minkowski dimension D of $\Gamma = \partial\Omega$ is the infimum of such numbers $l \geq 0$ so that the (l-dimentional) upper Minkowski content $\mathcal{M}^*(l;\Gamma) = \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-l)}|\Gamma_\varepsilon \cap \Omega|_n < \infty$, where Γ_ε, the ε-neighborhood of Γ, is defined as the set of all $x \in \boldsymbol{R}^n$ in which the distance between x and Γ will be less than ε. Similarily, $\mathcal{M}_*(l;\Gamma)$, the (l-dimensional) lower Minkowski content of Γ, is defined as similar as $\mathcal{M}^*(l;\Gamma)$, but by means of the lower limit rather than the upper limit.

Let $\mathcal{M}^* = \mathcal{M}^*(D;\Gamma)$ (resp. $\mathcal{M}_* = \mathcal{M}_*(D;\Gamma)$) denote the D-dimensional upper (resp. lower) Minkowski content of Γ. We say that Γ is D-Minkowski measurable if $0 < \mathcal{M}_* = \mathcal{M}^* < +\infty$. In this case, we write $\mathcal{M} = \mathcal{M}(D;\Gamma)$, the Minkowski content of Γ. Observe that

1. The larger D, the more irregular Γ. And it is easy to show that $n-1 \leq D \leq n$. We shall say that Γ is "fractal" if $D \in (n-1, n]$ and "nonfractal" if $D = n-1(n-1$ is the topological dimension of Γ).

2. If Γ is "regular enough" (say, of class C^1), then $H = D = n-1$, where H denotes the Hausdorff dimension of Γ. But in general, we have $n-1 \leq H \leq D \leq n$ and $0 \leq \mathcal{M}_* \leq M^* \leq +\infty$.

Here we shall give some properties of Minkowski dimension and contents without proof (see [11]).

Let Ω be a (non-empty) open subset of $\boldsymbol{R}$, with finite length $|\Omega|$, and with boundary $\Gamma = \partial\Omega$. We write Ω as the union of its connected components:

$$\Omega = \bigcup_{j=1}^{\infty} I_j$$

where the open interval I_j is pairwise disjoint and of length l_j. We can always assume, without loss of generality, that

$$l_1 \geq l_2 \geq \cdots \geq l_j \geq \cdots > 0$$

We call that $(l_j)_{j=1}^{\infty}$ is the sequence associated with Ω.

Proposition 2.1 ([11,Th. 2.4]) *Let Ω be a bounded open subset of R, let $(l_j)_{j=1}^{\infty}$ be the associated sequence, and let $D \in (0,1)$. The following two assertions are equivalent:*

(i) $l_j \asymp j^{-1/D}$ *as* $j \to +\infty$; ($l_j \asymp j^{-1/D}$ *means that* $0 < \liminf_{j\to+\infty} l_j \cdot j^{1/D} \leq \limsup_{j\to+\infty} l_j \cdot j^{+1/D} < +\infty$)

(ii) $\Gamma = \partial\Omega$ *has Minkowski dimension D and*

$$0 < \mathcal{M}_*(D;\Gamma) \leq \mathcal{M}^*(D;\Gamma) < +\infty$$

Proposition 2.2 *The sequence $(l_j)_{j=1}^{\infty}$ associated with Ω satisfies that $l_j \sim Lj^{-1/D}$, as $j \to +\infty$, for $D \in (0,1)$ and $L > 0$ if and only if $\Gamma = \partial\Omega$ is D-Minkowski measurable and has Minkowski dimension $D \in (0,1)$. In addition, we have $\mathcal{M}(D;\Gamma) = \dfrac{2^{1-D}}{1-D}L^D$.*

3. Main Results

Now let us come back to the problem (P). First, we assume that the function $V(x)$ in (1.1) is locally constant and bounded from above. In other words, when $\Omega = \bigcup_{j=1}^{\infty} I_j$ and $(l_j)_{j=1}^{\infty}$ are the associated sequence with Ω, we assume $V(x) \equiv v_j$ in I_j and there is a constant $E > 0$ such that $E > v_j$ for any $j \geq 1$.

Theorem 3.1 *If $D \in (0,1)$ and Γ is D-Minkowski measurable with Minkowski measure $\mu(D;\Gamma) = \dfrac{2^{1-D}}{1-D}L^D$, the locally constant-value function $V(x)$ satisfies:*

$$v_j \to V, \quad as\ j \to +\infty$$

where $V \leq E$ is a constant, then the counting function $N(E,\lambda)$ associated with the problem (P) has the asymptotic form:

$$\begin{aligned} N(E,\lambda) =& \frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}} (E-V(x))^{1/2}dx \cdot \lambda^{1/2} - C_{E,D}\mathcal{M}(D;\Gamma) \\ & \cdot \lambda^{D/2} + o(\lambda^{D/2}), \quad as \lambda \to \infty \end{aligned} \tag{3.1}$$

where $C_{E,D}$ is a non-negative constant given by

$$C_{E,D} = 2^{-(1-D)}\pi^{-D}(1-D)(E-V)^{D/2}(-\zeta(D)) \tag{3.2}$$

Proof

$$N(E,\lambda)=\sum_{j=1}^{\infty}N(E,\lambda;I_j)=\sum_{j=1}^{\infty}\left[\frac{\sqrt{E-v_j}}{\pi}l_j\cdot\lambda^{1/2}\right]$$
$$=\sum_{j=1}^{\infty}\frac{\sqrt{E-v_j}}{\pi}l_j\cdot\lambda^{1/2}-\sum_{j=1}^{\infty}\left\{\frac{\sqrt{E-v_j}}{\pi}l_j\cdot\lambda^{1/2}\right\}$$
$$=\frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}}(E-V(x))^{1/2}dx\cdot\lambda^{1/2}-\sum_{j=1}^{\infty}\left\{\frac{\sqrt{E-v_j}}{\pi}l_j\cdot\lambda^{1/2}\right\}$$
$$=I-II$$

Let

$$h_j=\frac{\sqrt{E-v_j}}{\pi}\cdot l_j \tag{3.3}$$

Easily we can check that, from Prop. 2.2,

$$h_j\sim\left(\frac{\sqrt{E-V}}{\pi}L\right)\cdot j^{-1/D}$$

Thus we have the following result:

Lemma 3.1 (See [11], Th.4.2) *Suppose that*

$$h_1\geq h_2\geq\cdots>0, h_j\sim L'j^{-1/D}\ \textit{for}\ L'>0\ \textit{and}\ D\in(0,1),\ \textit{as}\ j\to+\infty. \tag{3.4}$$

Let $\delta(x)=\sum_{j=1}^{\infty}\{h_jx\}$, *then*

$$\delta(x)\sim-\zeta(D)L'^D\cdot x^D,\quad as\ x\to+\infty \tag{3.5}$$

where $\zeta=\zeta(x)$ *denotes the classical Riemann zeta-function.*

Proof For $\varepsilon>0$, we define

$$J(\varepsilon)=\max\{j\geq1;h_j\geq\varepsilon\} \tag{3.6}$$

From (3.4), we can know that

$$J(\varepsilon)\sim L'^D\cdot\varepsilon^D,\ \text{as}\ \varepsilon\to0^+ \tag{3.7}$$

Let $k\geq2$ be an arbitrary fixed integer

$$\delta(x)=\sum_{j>J(1/x)}\{h_jx\}+\sum_{j\leq J(k/x)}q\{h_jx\}+\sum_{q=2}^{k}\sum_{j=J(q/x)+1}^{J((q-1)/x)}\{h_jx\}$$
$$=\left(\sum_{j>J(1/x)}h_j\right)x+\sum_{j\leq J(k/x)}\{h_jx\}+\sum_{q=2}^{k}\sum_{j=J(q/x)+1}^{J((q-1)/x)}((h_jx)-(q-1))$$
$$=\sum_{j\leq J(k/x)}\{h_jx\}+x\sum_{j>J(k/x)}h_j-\sum_{q=2}^{k}(q-1)\left(J\left(\frac{q-1}{x}\right)-J\left(\frac{q}{x}\right)\right)$$

$$= x \sum_{j>J(k/x)} h_j + \sum_{j\le J(k/x)} \{h_j x\} - \sum_{q=1}^{k-1} J\left(\frac{q}{x}\right) + (k-1)J\left(\frac{k}{x}\right)$$

We rewrite

$$\delta(x) = A + B + C$$

where

$$A = x \sum_{j>J(k/x)} h_j, B = kJ\left(\frac{k}{x}\right) - \sum_{q=1}^{k-1} J\left(\frac{q}{x}\right), C = \sum_{j\le J(k/x)} (\{h_j x\} - 1)$$

Since $-1 \le \{h_j x\} - 1 < 0$, we have

$$-J(k/x) \le c \le 0$$

so we can get this estimate

$$0 \le -\left(L'x\right)^{-D} C \le \left(L'x\right)^{-D} J(k/x) \to k^{-D}, \text{ as } x \to +\infty \tag{3.8}$$

From (3.7), we can easily deduce that

$$(L'x)^{-D} B \to k^{1-D} - \sum_{q=1}^{k-1} q^{-D}, \text{ as } x \to +\infty \tag{3.9}$$

Further, we can estimate $(L'x)^{-D}A$. According to (3.4) and (3.7), for $\forall \varepsilon > 0$, there exist $x_0 > 0$, such that $\forall x \ge x_0$, we have $h_j \in ((L'-\varepsilon)\cdot j^{-1/D}, (L'+\varepsilon)\cdot j^{-1/D})$ for all $j > J(k/x)$.

Thus for all $x \ge x_0$,

$$A < x \sum_{j>J(k/x)} [(L'+\varepsilon)\cdot j^{-1/D}] < x(L'+\varepsilon)\int_{J(k/x)}^{+\infty} t^{-\frac{1}{D}} dt$$
$$= x(L'+\varepsilon)\frac{D}{1-D} J(k/x)^{1-1/D}$$

Similarily, we get that

$$A > x \sum_{j>J(k/x)} [(L'-\varepsilon)\cdot j^{-1/D}] > x(L'-\varepsilon)\frac{D}{1-D}(J(k/x)+1)^{1-1/D}$$

Using (3.7), we have

$$A \le (1+o(1))k^{1-D}\frac{D}{1-D}(L'+\varepsilon)L'^{D-1}x^D, \text{ as } x \to +\infty$$
$$A \ge (1+o(1))k^{1-D}\frac{D}{1-D}(L'-\varepsilon)L'^{D-1}x^D, \text{ as } x \to +\infty$$

Noting the arbitraryness of $\varepsilon > 0$, we find that

$$(L'x)^{-D}A \to k^{1-D}\frac{D}{1-D}, \text{ as } x \to +\infty \tag{3.10}$$

So for fixed k, we get that

$$(L'x)^{-D}\delta(x) = (L'x)^{-D}(A+B) + (L'x)^{-D}C$$

$$(L'x)^{-D}(A+B) \to \frac{1}{1-D}k^{1-D} - \sum_{q=1}^{k-1} q^{-D} = f_k(D) + \frac{1}{1-D}, \text{ as } k \to +\infty$$

where $f_k(s) = \int_1^k (t^{-s} - [t]^{-s})dt$.

The sequence of entire functions $\{f_k(s)\}_{k=1}^{\infty}$ converges uniformly as $k \to +\infty$ on every compact subset of $\mathrm{Re}s > 0$ to the function

$$f(s) = \int_1^{+\infty} (t^{-s} - [t]^{-s})dt = \frac{-1}{1-s} - \zeta(s)$$

So, we have $f_k(D) + \frac{1}{1-D} \to -\zeta(D)$ as $k \to +\infty$. By variating $k \to +\infty$, we can get that,

$$(L'x)^{-D}\delta(x) = (L'x)^{-D}(A+B+C) \to -\zeta(D), \text{ as } x \to +\infty \tag{3.11}$$

which implies that Lemma 3.1 is proved. Observe that from (3.3) we cannot ensure that $h_1 \geq h_2 \geq \cdots > 0$, although we have that $l_1 \geq l_2 \geq \cdots > 0$. In order to use Lemma 3.1, we need the following result.

Lemma 3.2 *Let $\{h_j\}_{j=1}^{\infty}$ be an arbitrary positive sequence such that*

$$h_j \sim L \cdot j^{-1/D}, \quad \text{as } j \to +\infty \tag{3.12}$$

for $D \in (0,1)$ and $L > 0$. Then we can get an isomorphism $\sigma : \boldsymbol{N} \to \boldsymbol{N}$, such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \cdots > 0$ and we still have $h_{\sigma(j)} \sim L \cdot j^{-1/D}$, as $j \to +\infty$.

proof From (3.12), we can see that $\{h_j\}_{j=1}^{\infty}$ is a bounded sequence such that $h_j \to 0^+$, as $j \to +\infty$. We can find an isomorphism σ on $\boldsymbol{N}$ satisfying $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \cdots > 0$. Easily we can deduce that $\sigma(j) \to +\infty$ as $j \to +\infty$. Here the isomorphism σ is not necessarily unique.

After choosing an isomorphism σ, we fix a positive constant c satisfying $c < L$. For $\varepsilon > 0$ small enough, there exists a positive integer j_0 such that for all $j \geq j_0$, we have

$$0 < c < L - \varepsilon \leq h_j \cdot j^{1/D} \leq L + \varepsilon$$

(i) We choose $j_1 \in \boldsymbol{N}$, which is large enough and for every $j > j_1$,

$$h_j < h_i, i = 1, 2, \cdots, j_0 \tag{3.13}$$

If an integer k satisfies that

$$j_0 \le k \text{ and } \frac{L-\varepsilon}{L+\varepsilon}\cdot\left(\frac{j}{k}\right)^{1/D} > 1 \tag{3.14}$$

then for arbitrary k' satisfying $j_0 \le k' \le k$, and for arbitrary $j' \ge j$,

$$\frac{h_{k'}}{h_{j'}} = \left(\frac{h_{k'}\cdot k'^{1/D}}{h_{j'}\cdot j'^{1/D}}\right)\cdot\left(\frac{j'^{1/D}}{k'^{1/D}}\right) \ge \frac{L-\varepsilon}{L+\varepsilon}\cdot\left(\frac{j'}{k'}\right)^{1/D} \ge \frac{L-\varepsilon}{L+\varepsilon}\cdot\left(\frac{j}{k}\right)^{1/D} > 1$$

that is $h_{k'} > h_{j'}$.

Using (3.13), we know that for arbitrary $i \le k$, and for arbitrary $j' \ge j$, we have $h_i > h_{j'}$. This means

$$\sigma(j) > k \tag{3.15}$$

Noticing (3.14) and (3.15), we can get that

$$\frac{L-\varepsilon}{L+\varepsilon}\cdot\left(\frac{j}{\sigma(j)}\right)^{1/D} \le 1 \text{ when } j > j_1 \tag{3.16}$$

(ii) Since $\sigma(j) \to +\infty$, as $j \to \infty$, we can choose j large enough such that $\sigma(j) \ge j_0$. We will show that if k is large enough, h_k is less than any one from $\{h_1, h_2, \cdots, h_j\}$.

In fact, if k is an integer satisfying that

$$k \ge j_0 \text{ and } \frac{L+\varepsilon}{L-\varepsilon}\cdot\left(\frac{j}{k}\right)^{1/D} < 1 \tag{3.17}$$

then for any $k' \ge k$, and any integer j' satisfying $j_0 \le j' \le j$,

$$\frac{h_{k'}}{h_{j'}} = \frac{h_{k'}\cdot k'^{1/D}}{h_{j'}\cdot j'^{1/D}}\cdot\frac{j'^{1/D}}{k'^{1/D}} < \frac{L+\varepsilon}{L-\varepsilon}\cdot\left(\frac{j'}{k'}\right)^{1/D}$$

$$\le \frac{L+\varepsilon}{L-\varepsilon}\cdot\left(\frac{j}{k}\right)^{1/D} < 1, \text{ i.e. } h_{k'} < h_{j'}$$

Using (3.13), we know that if $k' \ge k$ and $j' \le j$, then $h_{k'} < h_{j'}$. This means

$$\sigma(j) \le k \tag{3.18}$$

Noticing (3.17) and (3.18), we can get that

$$\frac{L+\varepsilon}{L-\varepsilon}\cdot\left(\frac{j}{\sigma(j)}\right)^{1/D} \ge 1 \text{ when } j > j_0 \tag{3.19}$$

Since ε is arbitrary, from (i) and (ii) above we can deduce that $\left(\frac{j}{\sigma(j)}\right)^{1/D} = 1+o(1)$, as $j \to +\infty$,

$$h_{\sigma(j)}\cdot j^{1/D} = h_{\sigma(j)}\sigma(j)^{1/D}\cdot\left(\frac{j}{\sigma(j)}\right)^{1/D} = L + o(1), \text{ as } j \to +\infty$$

which implies Lemma 3.2 is proved.

Now let us prove Theorem 3.1:

From Lemma 3.2, we can re-arrange $\{h_j\}_{j=1}^{\infty}$ into $\{h_{\sigma(j)}\}_{j=1}^{\infty}$, satisfying:

$$h_{\sigma(1)} \geq h_{\sigma(2)} \geq \cdots \geq h_{\sigma(j)} \geq \cdots > 0$$

and

$$h_{\sigma(j)} \sim L' \cdot j^{-1/D}, \text{ as } j \to +\infty$$

Then from Lemma 3.1, we have

$$II = \sum_{j=1}^{\infty} \{h_{\sigma(j)} \lambda^{1/2}\} = (-\zeta(D) L'^D + o(1)) \lambda^{D/2}, \text{ as } j \to +\infty$$

Thus we have proved that $N(E, \lambda)$ has the following asymptotic form:

$$\begin{aligned} N(E,\lambda) =& \frac{1}{\pi} \int_{\{x \in \Omega | V(x) < E\}} (E - V(x))^{1/2} dx \cdot \lambda^{1/2} \\ & - C_{E,D} \mathcal{M}(D; \Gamma) \cdot \lambda^{D/2} + o(\lambda^{D/2}) \end{aligned} \tag{3.20}$$

where $C_{E,D} = 2^{-(1-D)} \pi^{-D} (1-D)(E-V)^{D/2}(-\zeta(D))$.

Remark 3.1 In general, even if $V(x)$ is locally constant, v_j may not converge as $j \to +\infty$. But in some cases, we can still use this method to treat with the counting function $N(E, \lambda)$. We briefly state it as follows: if $v_{2k-1} \to V_1$ and $v_{2k} \to V_2$ as $k \to +\infty$, and $V_1 \neq V_2$, then under the condition of Theorem 3.1, we can get that

$$\begin{aligned} N(E,\lambda) =& \frac{1}{\pi} \int_{\{x \in \Omega | V(x) < E\}} (E - V(x))^{1/2} dx \cdot \lambda^{1/2} \\ & + \frac{\zeta(D)}{2} \left[\left(\frac{\sqrt{E - V_1}}{\pi} L \right)^D + \left(\frac{\sqrt{E - V_2}}{\pi} L \right)^D \right] \cdot \lambda^{D/2} + o(\lambda^{D/2}) \end{aligned} \tag{3.21}$$

In Theorem 3.1, the second term of $N(E, \lambda)$ is of the order $D/2$, where D is the Minkowski dimension of $\partial\Omega$. Actually we can construct some examples in which the second term of the counting function $N(E, \lambda)$ will have different order from $D/2$.

Theorem 3.2 *Let $V(x)$ be a locally constant function with upper bound $A > 0$, and $\{l_j\}_{j=1}^{\infty}$ be the associated sequence with Ω satisfying $l_j \sim L \cdot j^{-1/D} (L > 0, D \in (0, 1))$. If $v_j \sim -K \cdot j^{\varepsilon}$, where K is a positive constant and $\varepsilon \geq 0$, a small non-negative constant satisfying $\left(1 + \frac{\varepsilon}{2}\right) D < 1$, then for a constant $E > A$,*

$$\begin{aligned} N(E,\lambda) =& \frac{1}{\pi} \int_{\{x \in \Omega | V(x) < E\}} (E - V(x))^{1/2} dx \cdot \lambda^{1/2} \\ & + \zeta(D_1) \pi^{-D_1} L^{D_1} K^{D_1/2} \cdot \lambda^{D_1/2} + o(\lambda^{D_1/2}) \end{aligned}$$

where

$$D_1 = \frac{2D}{2-\varepsilon D} \in (0,1), D_1 > D$$

Proof

$$\begin{aligned} N(E,\lambda) = \sum_{j=1}^{\infty} N(E,\lambda;I_j) &= \sum_{j=1}^{\infty}\left[\frac{\sqrt{E-v_j}}{\pi}\cdot l_j\lambda^{1/2}\right] \\ &= \sum_{j=1}^{\infty}\left(\frac{\sqrt{E-v_j}}{\pi}\cdot l_j\lambda^{1/2}\right) - \sum_{j=1}^{\infty}\left\{\frac{\sqrt{E-v_j}}{\pi}\cdot l_j\lambda^{1/2}\right\} \\ &= I - II \end{aligned} \tag{3.22}$$

Let $h_j = \frac{\sqrt{E-v_j}}{\pi}\cdot l_j$, then from the conditions in Theorem 3.2, we have

$$\begin{aligned} h_j &= \frac{\sqrt{E-v_j}}{\pi}\cdot l_j = \frac{1}{\pi}\sqrt{\frac{E}{j^\varepsilon}+K+o(1)}\cdot j^{\varepsilon/2}\cdot l_j \\ &= \frac{1}{\pi}\left(\sqrt{K}+o(1)\right)\cdot(L+o(1))\cdot j^{-\frac{1-\varepsilon D/2}{D}}, \text{ as } j\to+\infty \end{aligned} \tag{3.23}$$

We introduce D_1 such that $\frac{1}{D_1} = \frac{1-\varepsilon D/2}{D}$. Since $\left(1+\frac{\varepsilon}{2}\right)D < 1$, we know $D_1 = \frac{2D}{2-\varepsilon D} \in (0,1)$ and $D_1 > D$. Thus we rewrite (3.23) as follows,

$$h_j \sim \frac{1}{\pi}\sqrt{K}L\cdot j^{-1/D_1}(0 < D < D_1 < 1) \text{ as } j\to+\infty \tag{3.24}$$

With (3.24), the first part of $N(E,\lambda)$ (i.e. I) is convergent. As far as II is concerned, we can rearrange h_j into $h_{\sigma(j)}$ satisfying that

$$h_{\sigma(1)} \geq h_{\sigma(2)} \geq \cdots \geq h_{\sigma(j)} \geq \cdots > 0$$

and

$$h_{\sigma(j)} \sim \frac{1}{\pi}\sqrt{K}L\cdot j^{-1/D_1}, \text{ as } j\to+\infty$$

where σ is an isomorphism on $\boldsymbol{N}$.

Then form Lemma 3.1, we can get that

$$II \sim (-\zeta(D_1))K^{D_1/2}L^{D_1}\pi^{-D_1}\cdot\lambda^{D_1/2} \text{ as } j\to+\infty$$

That is to say,

$$\begin{aligned} N(E,\lambda) =& \frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}}(E-V(x))^{1/2}dx\cdot\lambda^{1/2} \\ &+\zeta(D_1)\pi^{-D_1}L^{D_1}K^{D_1/2}\cdot\lambda^{D_1/2}+o(\lambda^{D_1/2}) \end{aligned}$$

Remark 3.2 In fact, by choosing different $\varepsilon > 0, D_1$ in Theorem 3.2 may be chosen as any number belonging to $(D,1)$.

4. Lower and Upper Bounds of Second Term

As we have mentioned, $l_j \sim L \cdot j^{-1/D}$ is equivalent to that $\Gamma = \partial\Omega$ has Minkowski dimention D and is Minkowski measurable. How does $N(E,\lambda)$ look like when only knowing Γ has Minkowski dimention D? We can no longer expect to have a precise second-term of the counting function $N(E,\lambda)$, however, we can get sharp estimate for the upper and lower bounds of the second asymptotic term. Actually we have:

Theorem 4.1 *Given $D \in (0,1)$, let $\{l_j\}_{j=1}^{\infty}$ be an arbitrary positive sequence such that $l_j \asymp j^{-1/D}$, as $j \to +\infty$. Let $V(x)$ be a bounded locally constant function on Ω, and we use V_+ (resp. V_-) denoting the superium (resp. inferium). Then for $E > V_+$, the second term of $N(E,\lambda)$ (denoted by $\delta(\lambda^{1/2})$) satisfies that $\delta(\lambda^{1/2}) \asymp \lambda^{D/2}$.*

Before we start our proof of Theorem 4.1, we introduce the following two lemmas:

Lemma 4.1 *Let $\{h_j\}_{j=1}^{\infty}$ be an arbitrary positive sequence such that $h_j \asymp j^{-1/D}$ where $D \in (0,1)$, or more precisely $0 < \alpha_1 = \liminf_{j\to+\infty}(h_j \cdot j^{+1/D}) \le \limsup_{j\to+\infty}(h_j \cdot j^{+1/D}) = \beta_1 < +\infty$. Then we rearrange h_j into $h_{\sigma(j)}$ satisfying that $h_{\sigma(j)}$ is nonincreasing. Besides, we can get that*

$$\begin{aligned} 0 < \frac{\alpha_1^2}{\beta_1} &\le \liminf_{j\to+\infty}(h_{\sigma(j)} \cdot j^{+1/D}) \le \limsup_{j\to+\infty}(h_{\sigma(j)} \cdot j^{+1/D}) \\ &\le \frac{\beta_1^2}{\alpha_1} < +\infty (i.e.\ h_{\sigma(j)} \asymp j^{-1/D}) \end{aligned} \tag{4.1}$$

where σ is an isomorphism on $\mathbf{N}$.

The proof of Lemma 4.1 is similar to the proof of Lemma 3.2, we only need to note that $\liminf_{j\to+\infty}(h_j \cdot j^{+1/D}) \ne \limsup_{j\to+\infty}(h_j \cdot j^{+1/D})$ may be true in this case.

Lemma 4.2 *Let $\{h_j\}_{j=1}^{\infty}$ be an positive non-increasing sequence satisfying*

$$0 < \alpha_2 = \liminf_{j\to+\infty}(h_j \cdot j^{+1/D}) \le \limsup_{j\to+\infty}(h_j \cdot j^{+1/D}) = \beta_2 < +\infty, D \in (0,1)$$

then we have $\delta(x) = \sum_{j=1}^{\infty}\{h_j x\} \asymp x^D$, as $x \to +\infty$.

Proof For $\varepsilon > 0$, we define

$$J(\varepsilon) = \max\{j \ge 1 : h_j \ge \varepsilon\}$$

Since

$$0 < \alpha_2 = \liminf_{j\to+\infty}(h_j \cdot j^{+1/D}) \le \limsup_{j\to+\infty}(h_j \cdot j^{+1/D}) = \beta_2 < +\infty \tag{4.2}$$

we can easily get that

$$\alpha_2^D \le \liminf_{\varepsilon\to 0^+}(J(\varepsilon) \cdot \varepsilon^D) \le \limsup_{\varepsilon\to 0^+}(J(\varepsilon) \cdot \varepsilon^D) \le \beta_2^D \tag{4.3}$$

Then

$$\begin{aligned}\delta(x)=\sum_{j=1}^{\infty}\{h_jx\}&=\sum_{j>J(1/x)}\{h_jx\}+\sum_{j\le J(k/x)}\{h_jx\}+\sum_{q=2}^{k}\sum_{j=J(q/x)+1}^{J((q-1)/x)}\{h_jx\}\\&=\left(\sum_{j>J(1/x)}h_j\right)x+\sum_{j\le J(k/x)}\{h_jx\}+\sum_{q=2}^{k}\sum_{j=J(q/x)+1}^{J((q-1)/x)}(h_jx-(q-1))\\&=\left(\sum_{j>J(k/x)}h_j\right)x+\sum_{j\le J(k/x)}\{h_jx\}-\sum_{q=1}^{k-1}J(x/q)+(k-1)J(k/x)\\&=A+B+C\end{aligned}$$

where

$$A=x\sum_{j>J(k/x)}h_j,\ B=(k-1)J(k/x)-\sum_{q=1}^{k-1}J(x/q),\ C=\sum_{j\le J(k/x)}\{h_jx\}$$

Since $0\le\{h_jx\}<1$, we have $0\le C\le J(k/x)$, and

$$0\le x^{-D}\cdot C\le(\beta_2^D+o(1))\cdot k^{-D},\text{ as }x\to+\infty$$

For B, we can easily see that $B=\sum_{q=1}^{k-1}(J(k/x)-J(x/q))\le 0$. By using (4.3), we can get that

$$\begin{aligned}\alpha_2^D(k-1)k^{-D}-\left(\sum_{q=1}^{k-1}\frac{1}{q^D}\right)\beta_2^D+o(1)&\le x^{-D}B\\&\le\beta_2^Dk^{-D}(k-1)-\left(\sum_{q=1}^{k-1}\frac{1}{q^D}\right)\alpha_2^D+o(1),\text{ as }x\to+\infty\end{aligned}$$

At last, we estimate $x^{-D}A$. According to (4.2) and (4.3), for any small $\varepsilon>0$, there exists $x_0>0$, such that for all $x\ge x_0$,

$$0<\alpha_2-\varepsilon\le h_j\cdot j^{1/D}\le\beta_2+\varepsilon,\text{ for all }j>J(k/x)$$

Then

$$\begin{aligned}A&<x\sum_{j>J(k/x)}(\beta_2+\varepsilon)\cdot j^{-1/D}<x(\beta_2+\varepsilon)\int_{J(k/x)}^{+\infty}t^{-1/D}dt\\&=x(\beta_2+\varepsilon)\frac{D}{1-D}(J(k/x))^{1-\frac{1}{D}}\end{aligned}$$

So

$$x^{-D}A\le(\alpha_2+o(1))^{D-1}k^{1-D}(\beta_2+\varepsilon)\frac{D}{1-D},\text{ as }x\to+\infty$$

Similarily, we can get that

$$x^{-D}A \geq (\alpha_2 - \varepsilon)k^{1-D}(\beta_2 + o(1))^{D-1}\frac{D}{1-D}, \text{ as } x \to +\infty$$

Since ε is arbitrary, we can deduce that

$$\begin{aligned}
&\frac{D}{1-D}\beta_2^{D-1}\alpha_2 k^{1-D} + \alpha_2^D(k-1)k^{-D} - \left(\sum_{q=1}^{k-1}\frac{1}{q^D}\right)\beta_2^D \\
&\leq \liminf_{x\to+\infty} x^{-D}\delta(x) \leq \limsup_{x\to+\infty} x^{-D}\delta(x) \\
&\leq \frac{D}{1-D}\beta_2\alpha_2^{D-1}k^{1-D} + \beta_2^D k^{-D}(k-1) - \sum_{q=1}^{k-1}\frac{1}{q^D} + \beta_2^D k^{-D} \\
&= \left(\frac{D}{1-D}\beta_2\alpha_2^{D-1} + \beta_2^D\right)k^{1-D} - \sum_{q=1}^{k-1}\frac{1}{q^D} \qquad (4.4)
\end{aligned}$$

If α_2 and β_2 are given, we can try to get a sharp estimate by selecting $k \in N$ carefully. For instance, we choose $k = 1$, then (4.4) becomes

$$\begin{aligned}
&0 < \frac{D}{1-D}\beta_2^{D-1}\alpha_2 \leq \liminf_{x\to+\infty} x^{-D}\delta(x) \\
&\leq \limsup_{x\to+\infty} x^{-D}\delta(x) \leq \frac{D}{1-D}\beta_2\alpha_2^{D-1} + \beta_2^D < +\infty \qquad (4.5)
\end{aligned}$$

That means $\delta(x) \asymp x^D$ as $x \to +\infty$.

Now, we turn back to the proof of Theorem 4.1.

Proof

$$\begin{aligned}
N(E,\lambda) &= \sum_{j=1}^{\infty} N_j(E,\lambda) = \sum_{j=1}^{\infty}\left[\frac{\sqrt{E-v_j}}{\pi}l_j \cdot \lambda^{1/2}\right] \\
&= \frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}}(E-V(x))^{1/2}dx \cdot \lambda^{1/2} - \sum_{j=1}^{\infty}\left\{\frac{\sqrt{E-v_j}}{\pi}l_j \cdot \lambda^{1/2}\right\} \\
&= I - \delta(\lambda^{1/2})
\end{aligned}$$

Since $l_j \asymp j^{-1/D}$, there exist two constants α, β to satisfy taht

$$0 < \alpha \leq \liminf_{j\to+\infty} l_j \cdot j^{1/D} \leq \limsup_{j\to+\infty} l_j \cdot j^{1/D} \leq \beta < +\infty$$

For $h_j = \frac{\sqrt{E-v_j}}{\pi}l_j$, we have that

$$\begin{aligned}
&0 < \frac{\sqrt{E-V_+}}{\pi}\alpha \leq \liminf_{j\to+\infty} h_j \cdot j^{1/D} \leq \limsup_{j\to+\infty} h_j \cdot j^{1/D} \\
&\leq \frac{\sqrt{E-V_-}}{\pi}\beta < +\infty.(\text{ i.e. } h_j \asymp j^{-1/D})
\end{aligned}$$

According to Lemma 4.1, we can rearrange h_j into $h_{\sigma(j)}$, which is non-increasing and satisfies

$$\begin{aligned}&\pi^{-1}\beta^{-1}(E-V_-)^{-1/2}(E-V_+)\alpha^2\\&\le \liminf_{j\to+\infty} h_{\sigma(j)}\cdot j^{1/D}\le \limsup_{j\to+\infty} h_{\sigma(j)}\cdot j^{1/D}\\&\le \pi^{-1}\alpha^{-1}(E-V_+)^{-1/2}(E-V_-)\beta^2<+\infty\end{aligned}$$

Then we use Lemma 4.2 to obtain

$$\begin{aligned}0&<\frac{D}{1-D}\beta_3^{D-1}\alpha_3\le \liminf_{j\to+\infty}\lambda^{-D/2}\cdot\delta(\lambda^{1/2})\\&\le \limsup_{j\to+\infty}\lambda^{-D/2}\cdot\delta(\lambda^{1/2})\le \frac{D}{1-D}\beta_3\alpha_3^{D-1}+\beta_3^D<+\infty\end{aligned}$$

where

$$\alpha_3=\pi^{-1}\beta^{-1}(E-V_-)^{-1/2}(E-V_+)\alpha^2$$
$$\beta_3=\pi^{-1}\alpha^{-1}(E-V_+)^{-1/2}(E-V_-)\beta^2$$

Theorem 4.1 is proved.

Remark 4.1 If locally constant function $V(x)$ hasnot lower bound, the second asymptotic term of $N(E,\lambda)$ may be different. Especially, if $l_j\asymp j^{-1/D}(D\in(0,1))$, $v_j\asymp -k\cdot j^{\varepsilon}$, where k is a positive constant satisfying $\left(1+\frac{\varepsilon}{2}\right)D<1$. Let $D_1=\frac{D}{1-\varepsilon D/2}(1>D_1>D$ in this case). We can find that the second term of $N(E,\lambda)$ (denoted by $\delta(\lambda^{1/2})$) satisfies $\delta(\lambda^{1/2})\asymp\lambda^{D_1/2}$. The proof of the above result is similar to that of Theorem 4.1 and Theorem 3.2.

Finally what could we obtain if the potential $V(x)$ is not a locally constant function? We have the following result:

Theorem 4.2 *Let $\{l_j\}_{j=1}^{\infty}$ be a non-increasing sequence associated with Ω, and $l_j\sim j^{-1/D}(D\in(0,1),L>0)$. Let $V(x)$ be a bounded function on Ω, i.e. there exists a constant A satisfying $|V(x)|\le A$ for any $x\in\Omega$, then for $E>A$, if there exists $\varepsilon\ge 0$, then $\eta\ge 0$ and $D/2+\varepsilon<1/2$, such that*

$$N_j(E,\lambda)=N(E,\lambda,I_j)=\int_{I_j}\frac{\sqrt{E-V(x)}}{\pi}dx\cdot\lambda^{1/2}+O_j(\lambda^{\varepsilon}(\log\lambda)^{\eta}),\ as\lambda\to+\infty$$

and

$$|O_j(\lambda^{\varepsilon}\log\lambda)|\le C\cdot\lambda^{\varepsilon}(\log\lambda)^{\eta}$$

where $C>0$, independent of j, we have

$$N(E,\lambda)=\int_{\{x\in\Omega|V(x)<E\}}\frac{\sqrt{E-V(x)}}{\pi}dx\cdot\lambda^{1/2}+O(\lambda^{D/2+\varepsilon}(\log\lambda)^{\eta})\qquad(4.6)$$

Proof For a small $\sigma > 0$, there exists $j_0 \in \boldsymbol{N}$ such that for all $j \geq j_0$,

$$l_j \leq (L+\sigma) \cdot j^{-1/D} \tag{4.7}$$

Since $|V(x)| \leq A$, we have

$$\frac{\sqrt{E-A}}{\pi} l_j \cdot \lambda^{1/2} \leq N_j(E,\lambda) \leq \frac{\sqrt{E+A}}{\pi} l_j \cdot \lambda^{1/2} \tag{4.8}$$

Let λ be fixed. If

$$\frac{\sqrt{E+A}}{\pi}(L+\sigma) \cdot j^{-1/D} \lambda^{1/2} < 1, \text{ i.e. } j > (L+\sigma)^D \lambda^{D/2} \left(\frac{\sqrt{E+A}}{\pi}\right)^D$$

we get that

$$N_j(E,\lambda) \leq \frac{\sqrt{E+A}}{\pi} l_j \cdot \lambda^{1/2} \leq \frac{\sqrt{E+A}}{\pi}(L+\sigma) \cdot j^{-1/D} \lambda^{1/2} < 1$$

That means $N_j(E,\lambda) = 0$.

Let

$$J(\lambda) = \left[(L+\sigma)^D \lambda^{D/2} \left(\frac{\sqrt{E+A}}{\pi}\right)^D\right] \tag{4.9}$$

then

$$N(E,\lambda) = \sum_{j=1}^{\infty} N_j(E,\lambda) = \sum_{j=1}^{J(\lambda)} N_j(E,\lambda) = \int_{I_1 \cup \cdots \cup I_{J(\lambda)}} \frac{\sqrt{E-V(x)}}{\pi} dx \cdot \lambda^{1/2} + R(\lambda)$$

where

$$|R(\lambda)| = \left|\sum_{j=1}^{J(\lambda)} O_j(\lambda^{\varepsilon}(\log \lambda)^{\eta})\right| \leq \sum_{j=1}^{J(\lambda)} C \cdot \lambda^{\varepsilon}(\log \lambda)^{\eta} \leq C' \cdot \lambda^{D/2+\varepsilon}(\log \lambda)^{\eta}$$

Furthermore,

$$\begin{aligned} N(E,\lambda) &= \int_{I_1 \cup \cdots \cup I_{J(\lambda)}} \frac{\sqrt{E-V(x)}}{\pi} dx \cdot \lambda^{1/2} + R(\lambda) \\ &= \int_{\{x \in \Omega | V(x) < E\}} \frac{\sqrt{E-V(x)}}{\pi} dx \cdot \lambda^{1/2} + R(\lambda) - I \end{aligned}$$

where

$$I = \int_{I_{J(\lambda)+1} \cup I_{J(\lambda)+2} \cup \cdots} \frac{\sqrt{E-V(x)}}{\pi} dx \cdot \lambda^{1/2} > 0$$

Noting (4.7), (4.8) and (4.9), we have

$$
\begin{aligned}
I &\leq \frac{\sqrt{E+A}}{\pi} \sum_{j=J(\lambda,E)+1}^{\infty} l_j \cdot \lambda^{1/2} \\
&\leq \frac{\sqrt{E+A}}{\pi} \cdot \lambda^{1/2} \sum_{j=J(\lambda,E)+1}^{\infty} [(L+\sigma) \cdot j^{-1/D}] \\
&\leq (L+\sigma) \frac{\sqrt{E+A}}{\pi} \cdot \lambda^{1/2} \frac{D}{1-D} J(\lambda)^{1-1/D} \\
&\leq \left[(L+\sigma)^D \left(\frac{\sqrt{E+A}}{\pi} \right)^D \frac{D}{1-D} + o(1) \right] \cdot \lambda^{D/2} \text{ as } \lambda \to +\infty \qquad (4.10)
\end{aligned}
$$

Thus, we prove that

$$
N(E,\lambda) = \int_{\{x\in\Omega | V(x)<E\}} \frac{\sqrt{E-V(x)}}{\pi} dx \cdot \lambda^{1/2} + O(\lambda^{D/2+\varepsilon} (\log\lambda)^{\eta})
$$

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