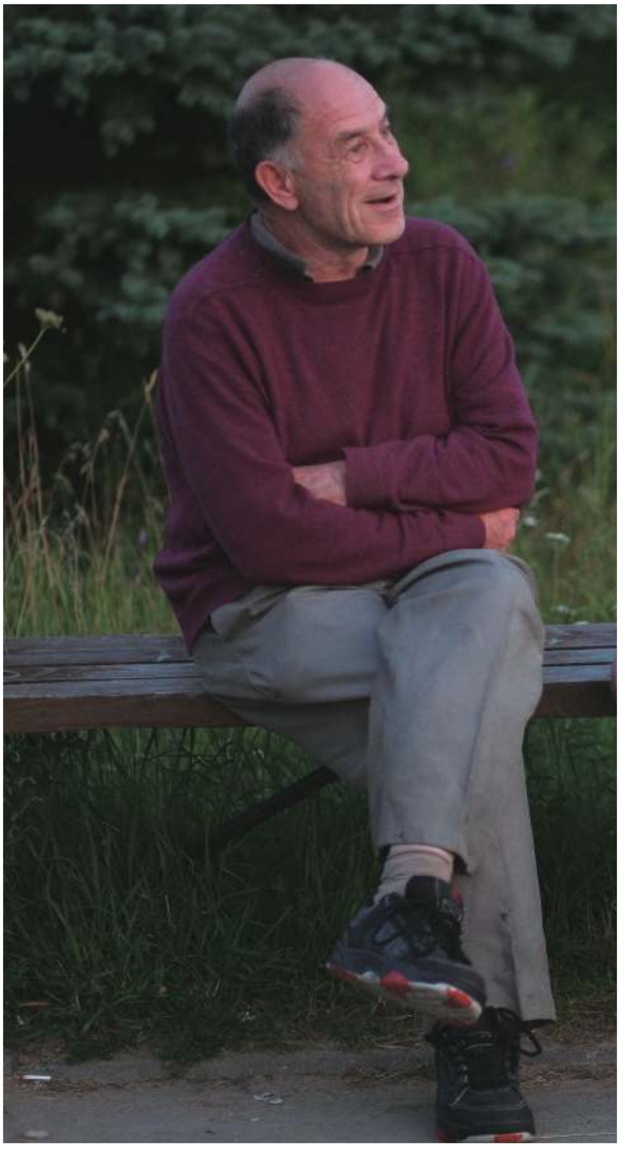
论数学教育[1]



Vladimir Arnold / 文
欧阳顺湘 / 译注

[1] 作者为著名的俄罗斯数学家弗拉基米尔 • 阿诺德（Vladimir Igorevich Arnold, 1937-2010）。他是 20 世纪最伟大的数学家之一，曾获克拉福德奖（1982）、沃尔夫奖（2001）和邵逸夫奖（2008）等奖。此文为 1997 年 3 月 7 日作者在法国巴黎探索文化宫（Palais de la Découverte，为法国科学教育中心博物馆）所发表演讲的扩充版，以俄文发表于 Uspekhi Mat. Nauk 53 (1998), no. 1 (319), 229-234; 英译见 Russian Math. Surveys 53 (1998), no. 1,229-236。英译网页版本可参 http://pauli.uni-muenster.de/~munsteg/arnold.html。亦有中译流传于网络。然而此中译不仅有笔误、漏译等不完善处，错译也不少。因该中译译者佚名，现遵《数学文化》主编之托，参考该中译，根据英译重译该文。英译中也有不清晰处，是故徐佩教授也帮助参考了俄文。译文中尽可能为初学者可能不熟悉的部分、较易说清楚的内容加了些注释并配图以方便阅读。翻译本非易事，本文涉及的数学内容又很广，错误、不恰当处请读者批评。最后，关于阿诺德的故事以及与本文类似的观点的更多阐述，有兴趣的读者可阅读他于 1999 年春意外受伤后养病期间撰写的书 *Yesterday and Long Ago*（原书为俄文，2006 年出版；英译 2007 年由 Springer 出版）。

这篇文章反映了阿诺德对布尔巴基的批判，对庞加莱（Jules Henri Poincaré, 1854-1912）直觉主义的支持。值得指出的是，今年恰为庞加莱逝世 100 周年。

图 1 获 2008 年邵逸夫数学科学奖时的阿诺德（图片来源：http://shawprize.org）

数学是物理学的一部分。物理学是一门实验科学，是自然科学的一部分。而数学乃是物理学中实验代价较小的部分。

雅可比恒等式（蕴涵垂心定理：三角形的三条高相交于一点）[2] 如同地球是圆的（即同胚于球体）一样，是一个实验事实，只不过发现前者不那么昂贵。

20 世纪中叶，人们试图割裂物理学与数学。其后果已被证明是灾难性的。整整几代数学家在对其所从事科学之另

[2] 阿诺德后来有文章更具体地谈到这个观点。例如，在他写的 *A. N. Kolmogorov and natural science* (Russian Mathematical Surveys, 2004, 59:1, 27–46）一文中，即明确提到由雅可比恒等式 $[(a, b)], c]+[(b, c), a]+[(c, a), b] =0$ 可得出三角形三条高相交于垂心，其中 $[\cdot\ ,\cdot]$ 为向量积。(因此中学课程就应该学习该恒等式，不必要等到学李代数等较高等课程时才学习。）更多介绍可以参考《美国数学月刊》上 Nikolai V. Ivanov 的文章 *Arnol′d, the Jacobi Identity, and Orthocenters,* The American Mathematical Monthly, 2011, 118:1, 41–65。顺便提一件轶事。爱因斯坦的传记中记述过他小时两件“奇迹”。一是 5，6 岁时对指南针感兴趣，二是 12 岁时对欧几里得几何着迷，特别提到垂心定理。

Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.

The Jacobi identity (which forces the heights of a triangle to cross at one point) is an experimental fact in the same way as that the Earth is round (that is, homeomorphic to a ball). But it can be discovered with less expense.

In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun).

Since scholastic mathematics that is cut off from physics is fit neither for teaching nor for application in any other science, the result was the universal hate towards mathematicians - both

图 2 法国巴黎探索皇宫

图 3 法国巴黎高等师范学院

一半极其无知的情况下成长，遑论其他科学了。这些人先是把他们丑陋的学院式伪数学传给他们的弟子，接着这些丑陋的伪数学又被教给中小学校里的孩子们（他们浑然忘却了哈代的警告：丑陋的数学在世上无永存之地[3]）。

学院式数学脱离物理，既于教学无益，又对其他科学无用武之地，其后果是人们对数学家的普遍怨恨。这样的人有学校里那些可怜的孩子们（他们当中有的可能还会成为将来的部长），也有应用这些数学的人。

由那些既无法掌握物理学又困倦于自卑中的半桶水式数学家们所拉起来的丑陋建筑，总使人想起"奇数的严格公理化理论"。显然，完全可以创造一种能够使得学生们称赞其完美无暇、内部结构和谐统一的理论（例如，可定义奇数个项的和以及任意多个因子的乘积）。按此狭隘观点，偶数要么被认为是"异端"，要么以后被当作"理想"对象补充入该理论，以此来应付物理与现实世界的需要。

不幸的是，数十年来，正是如上述这样丑陋扭曲的数学结构充斥着我们的数学教育。它肇始自法国，很快传染到基础数学教学，先是毒害大学生，接着中小学生也难免此灾（而灾区最先是法国，接着是其他国家，包括俄罗斯）。

倘若你问法国小学生，"2＋3 等于几"，他会这样回答："3+2，因为加法适合交换律"。他不知道其和为几，甚至不

[3] 哈代（G. H. Hardy, 1877-1947）为英国著名数学家。该引语出自他的名著《一个数学家的自白》（*A Mathematician's Apology*, Cambridge University Press, 1994）。整段表达为：正像画家和诗人的模式一样，数学家的模式也必须是优美的；正像色彩和文字一样，数学家的思想也必须和谐一致。优美是第一关：丑陋的数学在世上无永存之地。（The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.）

on the part of the poor schoolchildren (some of whom in the meantime became ministers) and of the users.

The ugly building, built by undereducated mathematicians who were exhausted by their inferiority complex and who were unable to make themselves familiar with physics, reminds one of the rigorous axiomatic theory of odd numbers. Obviously, it is possible to create such a theory and make pupils admire the perfection and internal consistency of the resulting structure (in which, for example, the sum of an odd number of terms and the product of any number of factors are defined). From this sectarian point of view, even numbers could either be declared a heresy or, with passage of time, be introduced into the theory supplemented with a few "ideal" objects (in order to comply with the needs of physics and the real world).

Unfortunately, it was an ugly twisted construction of mathematics like the one above which predominated in the teaching of mathematics for decades. Having originated in France, this pervertedness quickly spread to teaching of foundations of mathematics, first to university students, then to school pupils of all lines (first in France, then in other countries, including Russia).

To the question "what is 2 + 3" a French primary school pupil replied: "3 + 2, since addition is commutative". He did not know what the sum was equal to and could not even understand what he was asked about!

Another French pupil (quite rational, in my opinion) defined mathematics as follows: "there is a square, but that still has to be proved".

Judging by my teaching experience in France, the university students' idea of mathematics (even of those taught mathematics

能理解你的问题是什么！

还有的法国小学生会如下阐述数学（我认为很有可能）：“存在一个正方形，但仍需证明。”

根据我本人在法国的教学经验，大学生们对数学的认知与这些小学生们同样糟糕（甚至包括那些在“高等师范学校”［高师］[4] 里学习数学的学生——我为这些明显很聪明但却被毒害颇深的孩子们感到极度的惋惜）。

譬如，这些学生从未见过一个抛物面。诸如描述由方程 $xy = z^2$ 所确定曲面之形状这样的问题即可使高师的数学家们发怵。作出平面上由参数方程（如 $x = t^3 - 3t, y = t^4 - 2t^2$）刻画的曲线是学生（甚至或许对大多数法国数学教授）完全无法做的问题。

从洛必达的第一部微积分教科书（名字即为“用于理解曲线的微积分”）[5] 开始，大致到古尔萨写的课本 [6]，解决这些问题的能力（和熟悉单位数乘法表一样）一直都被认为是每一个数学家应当具备的基本技能。

弱智的“抽象数学”的狂热者将几何（由此物理和现实的联系能常在数学中反映）统统摒除于教学之外。由古尔萨、埃尔米特、皮卡等人写的微积分教程被认为是过时而有害的，最近差点被巴黎第 6 和第 7 大学的学生图书馆当垃圾丢掉，只是在我的干预下才得以保存。

高师的学生，听完所有微分几何与代数几何课程（由有名望的数学家所教），却既不熟悉由椭圆曲线 $y^2 = x^3 + ax + b$ 决定的黎曼曲面，事实上也不知道曲面的拓扑分类（更不用提第一类椭圆积分以及椭圆曲线的群性质，即欧拉 - 阿贝尔加法定理了）。他们仅仅学到了霍奇结构与雅可比簇！

这样的事情怎么会在法国发生呢？！这可是为我们这个世界贡献了拉格朗日与拉普拉斯、柯西与庞加莱、勒雷与托姆这样的大家的国度啊！我觉得一个合理的解释出自彼德罗夫斯基 [7]。他在 1966 年曾教导过我：真正的数学家决不会拉帮结派，唯有弱者才会结党营生。他们可能因各种原因而联

[4] 高师是法国最好的大学，法国最具选拔性和挑战性的高等教育研究机构。校友中迄今已有施瓦茨、托姆以及新近的吴宝珠、维拉尼等共 10 位菲尔兹奖得主。

[5] 洛必达（Guillaume François Antoine, Marquis de l'Hôpital, 1661-1704），法国数学家。所提他撰写的教材原文名为 *Analyse des infiniment petits pour l'intelligence des lignes courbes*，1696 年出版。著名的洛必达法则即出现在此书中。

[6] 古尔萨（Edouard Goursat, 1858-1936），法国数学家。他的课本指 1902 年开始陆续出版的三卷本《数学分析教程》（*Cours d'analyse mathématique*）。中译最早有 19 世纪 30 年代王尚济的《解析数学讲义》以及 40 年代刘景芳翻译的《数学分析教程》。

[7] 彼德罗夫斯基（Ivan Georgievich Petrovsky, 1901-1973）研究偏微分方程等，对希尔伯特第 19 和第 16 问题有重要贡献。曾任莫斯科国立大学校长（期间阿诺德被录取）。

图 4 古尔萨

at the École Normale Supérieure - I feel sorry most of all for these obviously intelligent but deformed kids) is as poor as that of this pupil.

For example, these students have never seen a paraboloid and a question on the form of the surface given by the equation $xy = z^2$ puts the mathematicians studying at ENS into a stupor. Drawing a curve given by parametric equations (like $x = t^3 - 3t, y = t^4 - 2t^2$) on a plane is a totally impossible problem for students (and, probably, even for most French professors of mathematics).

Beginning with l'Hôpital's first textbook on calculus (“calculus for understanding of curved lines”) and roughly until Goursat's textbook, the ability to solve such problems was considered to be (along with the knowledge of the times table) a necessary part of the craft of every mathematician.

Mentally challenged zealots of “abstract mathematics” threw all the geometry (through which connection with physics and reality most often takes place in mathematics) out of teaching. Calculus textbooks by Goursat, Hermite, Picard were recently dumped by the student library of the Universities Paris 6 and 7 (Jussieu) as obsolete and, therefore, harmful (they were only rescued by my intervention).

ENS students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out be acquainted neither with the Riemann surface of an elliptic curve $y^2 = x^3 + ax + b$ nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals

图 5 1973 年发行的纪念彼特洛夫斯基的邮票

合（可能为超抽象，反犹主义或为“应用的和工业上的”问题），但其本质总是在一些非数学的社会问题中求生存。

我在此顺便提醒大家温习路易•巴斯德[8]的忠告：从来没有所谓的“应用科学”，有的只是科学的应用（而且非常实际的应用！）。

当时我一直对彼德罗夫斯基的话心存疑虑，但现今我愈来愈坚信：他说的一点没错。可观的超抽象活动最终归结为以工业化的模式无耻地掠夺原创者的成果，然后系统地将这些成果归功于拙劣的推广者。就如美洲没有以哥伦布的名字命名一样，数学结果也几乎从未以它们真正的发现者来命名。

为避免被误引，我须声明，由于某些未知的缘故，我自己的成果从未被上述方式侵占，虽然这样的事情经常发生在我的老师（柯尔莫哥洛夫、彼德罗夫斯基、庞特里亚金、洛赫林）和我的学生身上。M. Berry 教授曾提出如下两个原理：

Arnold 原理：如果某概念出现了某人名，则该人必非发现此概念者。

Berry 原理：阿诺德原理适用于自身。

我们还是回到法国的数学教育上来。

在我为莫斯科国立大学数学与力学系一年级学生时，微积分课教师是集合论拓扑学家 L.A. 图马金。他认真地讲解古尔萨版的古典法式微积分教程。他告诉我们若相应黎曼面是球面，则有理函数沿着代数曲线的积分可以求出来；若相应黎曼

[8] 路易•巴斯德（Louis Pasteur, 1822-1895），法国微生物学家、化学家，微生物学的奠基人之一。巴斯德原文常见英译为“There are no such things as applied sciences, only applications of science”。

of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!

How could this happen in France, which gave the world Lagrange and Laplace, Cauchy and Poincaré, Leray and Thom? It seems to me that a reasonable explanation was given by I.G. Petrovskii, who taught me in 1966: genuine mathematicians do not gang up, but the weak need gangs in order to survive. They can unite on various grounds (it could be super-abstractness, anti-Semitism or “applied and industrial” problems), but the essence is always a solution of the social problem - survival in conditions of more literate surroundings.

By the way, I shall remind you of a warning of L. Pasteur: there never have been and never will be any “applied sciences”, there are only applications of sciences (quite useful ones!).

In those times I was treating Petrovskii's words with some doubt, but now I am being more and more convinced of how right he was. A considerable part of the super-abstract activity comes down simply to industrialising shameless grabbing of discoveries from discoverers and then systematically assigning them to epigons-generalizers. Similarly to the fact that America does not carry Columbus's name, mathematical results are almost never called by the names of their discoverers.

In order to avoid being misquoted, I have to note that my own achievements were for some unknown reason never expropriated in this way, although it always happened to both my teachers (Kolmogorov, Petrovskii, Pontryagin, Rokhlin) and my pupils. Prof. M. Berry once formulated the following two principles:

The Arnold Principle. If a notion bears a personal name, then this name is not the name of the discoverer.

The Berry Principle. The Arnold Principle is applicable to itself.

Let's return, however, to teaching of mathematics in France.

When I was a first-year student at the Faculty of Mechanics and Mathematics of the Moscow State University, the lectures on calculus were read by the set-theoretic topologist L.A. Tumarkin, who conscientiously retold the old classical calculus course of French type in the Goursat version. He told us that integrals of rational functions along an algebraic curve can be taken if the corresponding Riemann surface is a sphere and, generally speaking, cannot be taken if its genus is higher, and that for the sphericity it is enough to have a sufficiently large number of double points on the curve of a given degree (which forces the curve to be unicursal: it is possible to draw its real points on the

图 6 1938 年布尔巴基会议（从左到右分别为：Simone Weil, Charles Pisot, André Weil, Jean Dieudonné, Claude Chabauty, Charles Ehresmann, Jean Delsarte）

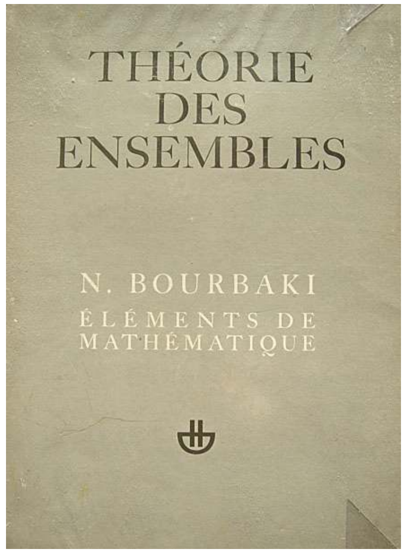


图 7 布尔巴基学派系列作品《数学原理》之一

面亏格更高，则该积分一般来说不可求；此外，给定次数曲线上二重点个数足够多，则对应曲面可为球面（由此知该曲线是有理的：即可以将其实值点在射影平面上一笔画出来）。

这些结果（即使不给出证明）紧紧地抓住了我们的想象，它们表现了更好更正确的现代数学思想，比布尔巴基学派[9]那卷帙浩繁的所有论著不知道好到哪里去了。确实，我们能在这里找到表面上似乎完全不同的事物之间令人惊奇的联系：一方面，积分可否显式表达与相应黎曼面的拓扑有关；另一方面，相应的黎曼面上的亏格与二重点个数之间也有重要的联系。这又和黎曼面的实部分的一笔画性质相关联。

作为数学中最迷人的性质之一，雅可比曾指出：同一个函数控制着用四个平方数之和对整数的表示[10]以及摆的真实运动。

[9] 尼古拉 • 布尔巴基（Nicolas Bourbaki）是二十世纪一群法国数学家为自己作品的集体笔名，其团队的正式称呼是“尼古拉 • 布尔巴基合作者协会”。主页为 http://www.bourbaki.ens.fr。布尔巴基希望在集合论的基础上用公理方法重新构造整个现代数学，致力于数学的严格化与一般化。布尔巴基认为：数学，至少纯粹数学，是研究抽象结构的理论。结构，就是以初始概念和公理出发的演绎系统。有三种基本的抽象结构：代数结构，序结构，拓扑结构。布尔巴基他们自 1935 年开始撰写题为《数学原理》(Éléments de mathématique) 的一系列著作以述说他们的观点。这套书共 9 卷，有七千多页。

[10] 这里的雅可比是卡尔 • 雅可比（Carl Gustav Jacob Jacobi, 1804–1851）。1770 年拉格朗日证明了四平方和定理：每个正整数均可表示为最多四个整数的平方和。例如 $31 = 5^2 + 2^2 + 1^2 + 1^2$。1834 年，雅可比发现了将整数分拆为四个平方数之和的表示方法数的精确公式，这和雅可比椭圆函数有关。可参考华罗庚著《数论导引》第八章。

projective plane with one stroke of a pen).

These facts capture the imagination so much that (even given without any proofs) they give a better and more correct idea of modern mathematics than whole volumes of the Bourbaki treatise. Indeed, here we find out about the existence of a wonderful connection between things which seem to be completely different: on the one hand, the existence of an explicit expression for the integrals and the topology of the corresponding Riemann surface and, on the other hand, between the number of double points and genus of the corresponding Riemann surface, which also exhibits itself in the real domain as the unicursality.

Jacobi noted, as mathematics' most fascinating property, that in it one and the same function controls both the presentations of a whole number as a sum of four squares and the real movement of a pendulum.

These discoveries of connections between heterogeneous mathematical objects can be compared with the discovery of the connection between electricity and magnetism in physics or with the discovery of the similarity between the east coast of America and the west coast of Africa in geology.

The emotional significance of such discoveries for teaching is difficult to overestimate. It is they who teach us to search and find such wonderful phenomena of harmony of the Universe.

The de-geometrisation of mathematical education and the divorce from physics sever these ties. For example, not only students but also modern algebro-geometers on the whole do not

发现这些不同种类的数学对象之间联系，就如发现物理学中电与磁之间联系，也类似于地质学上发现美洲大陆的东海岸与非洲大陆的西海岸之间的相似性。

这些发现对于教学的情感意义是难以估量的。正是它们指引着我们去研究和发现宇宙中和谐而精彩的现象。

然而，数学教育的非几何化以及对物理学的背离却割断了这种联系。例如，不仅仅是学生，绝大部分的当代代数几何学家也都不知道如下雅可比事实：第一类椭圆积分表示了相应的哈密顿系统中沿某个椭圆相曲线的运动时间。

套用关于电子与原子的著名说法，可以说圆内旋轮线就如同多项式环中的理想一样是无穷竭的。但若要把理想这一概念教给从不知道圆内旋轮线的学生，就如把分数加法教给从未将蛋糕或苹果等分切割过（至少在脑子里切过）的学生一样令人迷惑。一点也不奇怪孩子们做分数加法时常常分子加分子、分母加分母。

我从法国朋友那里听说这种超抽象的一般化正是他们的传统国民性。我不完全否认这种遗传病的说法，不过我还是愿意强调那个从庞加莱那儿借来的“蛋糕与苹果”的例子[11]。

构造数学理论的方式与在其它自然科学中建立理论的方式完全相同。首先我们要考察某些对象并在特定例子中进行观察。然后我们通过试验找到所得观察结果适用的边界，寻求反例以阻止我们将所得想当然地推广到过于广泛的情形（例如：将连续奇数 1,3,5,7,9 拆为奇数个自然数之和的分拆数[12]给出序列 1,2,4,8,16，但接下来却是 29）。

我们尽可能清晰地将所得经验发现（如费马猜想和庞加莱猜想）表述为结论。之后的阶段将是困难的，因为要检验所得结论在多大程度上可靠。

数学中已对此发展出来一套特别的方法。这种方法，在被应用于现实世界时，有时很有用，但有时也会导致自欺

[11] 庞加莱是直觉主义的先驱，认为直觉是发明的工具。1905 年庞加莱发表《数学中的直觉与逻辑》。英译可参考 http://www-history.mcs.st-and.ac.uk/Extras/Poincare_Intuition.html，中译可参考《科学的价值》（李醒民译，光明日报出版社，1988）。庞加莱在此文中讨论了两类数学家，而迈克尔•阿蒂亚爵士在其演讲《二十世纪的数学》中区分了形式主义传统的代表人物（莱布尼兹—希尔伯特—布尔巴基）和直觉主义传统的代表人物（牛顿—庞加莱—阿诺德）。此处“蛋糕与苹果”的例子，按阿诺德所述，源自庞加莱。阿诺德在一次俄法“Mistral”会议上的演讲（参 *Yersterday and Long Ago*，第 157 页）也提到分数教学，明确提到庞加莱曾说，只有这两种方式来教分数。阿诺德还提到卢梭在其《忏悔录》的回忆：只有在分割矩形后他才理解完全平方和公式 $(a+b)^2 = a^2 + 2ab + b^2$。

[12] 举例来说，5 有如下 7 种整数分拆方式：5 = 5; 5 = 1 + 4; 5 = 2 + 3; 5 = 1 + 2 + 2; 5 = 1 + 1 + 3; 5 =2+1+1+1; 5 = 1+1+1+1+1。其中为奇数之和的分拆数为 4。欧拉曾经证明奇数之和的分拆数与不等数之和的分拆数相等。

图 8 图为法国 1952 年发行的纪念庞加莱邮票

know about the Jacobi fact mentioned here: an elliptic integral of first kind expresses the time of motion along an elliptic phase curve in the corresponding Hamiltonian system.

Rephrasing the famous words on the electron and atom, it can be said that a hypocycloid is as inexhaustible as an ideal in a polynomial ring. But teaching ideals to students who have never seen a hypocycloid is as ridiculous as teaching addition of fractions to children who have never cut (at least mentally) a cake or an apple into equal parts. No wonder that the children will prefer to add a numerator to a numerator and a denominator to a denominator.

From my French friends I heard that the tendency towards super-abstract generalizations is their traditional national trait. I do not entirely disagree that this might be a question of a hereditary disease, but I would like to underline the fact that I borrowed the cake-and-apple example from Poincaré.

The scheme of construction of a mathematical theory is exactly the same as that in any other natural science. First we consider some objects and make some observations in special cases. Then we try and find the limits of application of our observations, look for counter-examples which would prevent unjustified extension of our observations onto a too wide range of events (example: the number of partitions of consecutive odd numbers 1, 3, 5, 7, 9 into an odd number of natural summands gives the sequence 1, 2, 4, 8, 16, but then comes 29).

As a result we formulate the empirical discovery that we made (for example, the Fermat's conjecture or Poincaré's conjecture) as clearly as possible. After this there comes the difficult period of checking the reliability of the conclusions obtained.

图 9 匈牙利 1999 年发行的魏格纳纪念邮票

欺人。这就是“建模”。建模时需做如下理想化：某些只以一定概率或一定的精确性知道的事实，被认为是“绝对”正确的并被当作“公理”来接受。这种“绝对性”的意义，恰在于我们容许自己根据形式逻辑规则来运用这些“事实”，而后把所有从这些事实推导出的结论称为“定理”。

显然在任何现实活动中，要完全依赖于这样的推理是不可能的。原因至少在于所研究现象的参数决不可能被绝对准确地确定，而且参数（例如过程的初始条件）的微小变化能够完全地改变结果。例如，正是因为这个原因使得任何可信赖的长期天气预报都是不可能的，将来也无可能——无论计算机或是记录初始条件的设备有多发达。

与此同理，（不能完全可靠的）公理的一个小小改变，通常能得出与从这些公理推导出来的定理完全不同的结论。推理之链（“证明”）越长越复杂，最后得到的结论的可靠性就越低。

复杂的模型（除了对写论文的人）几乎毫无用处。

数学建模方法忽略这些麻烦，把所得到的模型当成是确切地与现实世界相吻合的。从自然科学的观点来看，这种途径是显然不正确的，但却经常导致很多物理上有用的结果，该事实被称为“数学在自然科学中不合理的有效性”（或叫做“魏格纳原理”）[13]。

我在此提一下盖尔方德的一个观点：还有另一类与魏格纳注意到的数学在物理中不可思议的有效性相仿的现象——即数学在生物学中也同样有不可思议的有效性。

[13] 尤金•魏格纳（Eugene Paul Wigner, 1902-1995） 系匈牙利 - 美国物理学家， 诺贝尔物理学奖获得者（1963）。这一说法参魏格纳的文章：*The unreasonable effectiveness of mathematics in the natural sciences. Richard courant lecture in mathematical sciences delivered at New York University*, Communications on Pure and Applied Mathematics (1960), 13:1, 1–14.

At this point a special technique has been developed in mathematics. This technique, when applied to the real world, is sometimes useful, but can sometimes also lead to self-deception. This technique is called modelling. When constructing a model, the following idealisation is made: certain facts which are only known with a certain degree of probability or with a certain degree of accuracy, are considered to be “absolutely” correct and are accepted as “axioms”. The sense of this “absoluteness” lies precisely in the fact that we allow ourselves to use these “facts” according to the rules of formal logic, in the process declaring as “theorems” all that we can derive from them.

It is obvious that in any real-life activity it is impossible to wholly rely on such deductions. The reason is at least that the parameters of the studied phenomena are never known absolutely exactly and a small change in parameters (for example, the initial conditions of a process) can totally change the result. Say, for this reason a reliable long-term weather forecast is impossible and will remain impossible, no matter how much we develop computers and devices which record initial conditions.

In exactly the same way a small change in axioms (of which we cannot be completely sure) is capable, generally speaking, of leading to completely different conclusions than those that are obtained from theorems which have been deduced from the accepted axioms. The longer and fancier is the chain of deductions (“proofs”), the less reliable is the final result.

Complex models are rarely useful (unless for those writing their dissertations).

The mathematical technique of modelling consists of ignoring this trouble and speaking about your deductive model in such a way as if it coincided with reality. The fact that this path, which is obviously incorrect from the point of view of natural science, often leads to useful results in physics is called “the inconceivable effectiveness of mathematics in natural sciences” (or “the Wigner principle”).

Here we can add a remark by I.M. Gel'fand: there exists yet another phenomenon which is comparable in its inconceivability with the inconceivable effectiveness of mathematics in physics noted by Wigner - this is the equally inconceivable ineffectiveness of mathematics in biology.

“The subtle poison of mathematical education” (in F. Klein's words) for a physicist consists precisely in that the absolutised model separates from the reality and is no longer compared with it. Here is a simple example: mathematics teaches us that the

对物理学家来说，“数学教育隐晦的毒害”（克莱因语）恰好就在于绝对化了的模型与现实分离后却与现实不再相符。举一个简单的例子：数学知识告诉我们马尔萨斯方程 $dx/dt = x$ 的解由初始条件唯一决定（即 (t, x) 平面上相应的积分曲线彼此不相交）。这个数学模型的结论与现实世界几乎没有关系。计算机实验却显示所有积分曲线在 t 负半轴上有公共点。例如，具有初始条件 $x(0) = 0$ 与 $x(0) = 1$ 的曲线，从实践的观点看来，在 $t = -10$ 相交，其实在 $t = -100$ 时，你不可能在他们之间插入一个原子[14]。欧氏几何对这种空间在微小距离下的性质没有任何介绍。唯一性定理在这种情况下的应用显然已经超出了模型所容许的精度。在对模型的实际应用中，务必要注意这种情形，否则可能会导致严重的麻烦[15]。

然而，我还要提到，这个唯一性定理也可解释船只在停泊码头时的靠岸阶段为什么必须要人工操作：倘若机动，设行进的速度是距离的光滑（线性）函数，则整个靠岸的过程将会耗费无穷长的时间。否则只能采取与码头相撞（船与码头之间要有非理想的弹性物体形成缓冲）的方法。值得指出的是，月球和火星探测仪器的着陆以及空间站的对接时，此类问题曾严肃地摆在我们面前——此时唯一性问题在与我们做对。

不幸的是，在现代数学教材里，即便是其中较好者，既没有这样的例子，也没有讨论迷信定理的危险性。我甚至觉得，那些学院派数学家（对物理知之甚少）都对公理化形式的数学与建模的主要差异习以为常，而且他们觉得在自然科学中这是很普遍的，只是需要用后期的实验来控制理论推演。

即使不提及初始公设的相对特征，人们也不会忘记在冗长的论证中犯逻辑错误是不可避免的（例如，宇宙射线或量子振动引发计算机崩溃）。任何做研究的数学家都知道，如果不对自己有所控制（最好的方法是用实例来控制），那么在大约 10 页纸的论述之后，半数公式中的符号会出问题，而数字 2 会从分母跑到分子的位置上。

与如此谬误相抗的技术在任何实验科学中都一样，都是通过实验与观察进行的外部控制。而且这一技术应该一开始就教给所有大学低年级的学生。

试图创造“纯粹”推论式公理化数学的做法，导致了摒弃物理学中研究模式（观察－建模－研究模型－得出结论－

[14] 托马斯•罗伯特•马尔萨斯（Thomas Robert Malthus, 1766-1834）是英国人口学家和政治经济学家。人口学原理的基本思想是：如没有限制，人口呈指数增长。方程 $dx/dt = x$ 的解为指数函数 $x(t) = x(0)\ e^{t}$。在 $x(0) = 1$ 时，$x(-100) = e^{-100} \approx 0{:}37 \times 10^{-43}$。原子半径在 30-300 皮米（1 皮米 = 10^{-12} 米）之间。

[15] 阿诺德在 *Yersterday and Long Ago*（第 36-37 页）中也提到相同的内容，并说唯一性定理与物理现实不符是 M. L. Lidov 教给他的。

solution of the Malthus equation $dx/dt = x$ is uniquely defined by the initial conditions (that is that the corresponding integral curves in the (t, x)-plane do not intersect each other). This conclusion of the mathematical model bears little relevance to the reality. A computer experiment shows that all these integral curves have common points on the negative t-semiaxis. Indeed, say, curves with the initial conditions $x(0) = 0$ and $x(0) = 1$ practically intersect at $t = -10$ and at $t = -100$ you cannot fit in an atom between them. Properties of the space at such small distances are not described at all by Euclidean geometry. Application of the uniqueness theorem in this situation obviously exceeds the accuracy of the model. This has to be respected in practical application of the model, otherwise one might find oneself faced with serious troubles.

I would like to note, however, that the same uniqueness theorem explains why the closing stage of mooring of a ship to the quay is carried out manually: on steering, if the velocity of approach would have been defined as a smooth (linear) function of the distance, the process of mooring would have required an infinitely long period of time. An alternative is an impact with the quay (which is damped by suitable non-ideally elastic bodies). By the way, this problem had to be seriously confronted on landing the first descending apparata on the Moon and Mars and also on docking with space stations - here the uniqueness theorem is working against us.

Unfortunately, neither such examples, nor discussing the danger of fetishising theorems are to be met in modern mathematical textbooks, even in the better ones. I even got the impression that scholastic mathematicians (who have little knowledge of physics) believe in the principal difference of the axiomatic mathematics from modelling which is common in natural science and which always requires the subsequent control of deductions by an experiment.

Not even mentioning the relative character of initial axioms, one cannot forget about the inevitability of logical mistakes in long arguments (say, in the form of a computer breakdown caused by cosmic rays or quantum oscillations). Every working mathematician knows that if one does not control oneself (best of all by examples), then after some ten pages half of all the signs in formulae will be wrong and twos will find their way from denominators into numerators.

The technology of combatting such errors is the same external control by experiments or observations as in any experimental science and it should be taught from the very beginning to all juniors in schools.

Attempts to create “pure” deductive-axiomatic mathematics

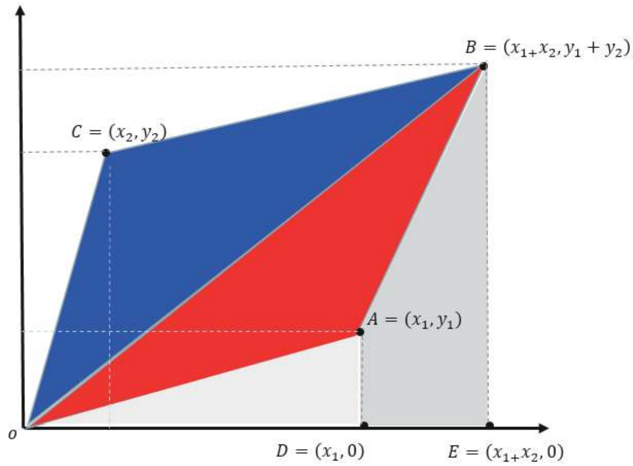


图 10 第一、二列向量分别为 $oA=(x_1, x_2)$, $oC=(y_1, y_2)$ 的矩阵行列式 $x_1y_2-x_2y_1$ 等于平行四边形 $oABC$ 的（有向）面积

再由观察来检验），而代之以定义－定理－证明的模式。人们不可能理解一个没有来由的定义，但这并不能阻止“代数－公理学家”违规。例如，他们会用长乘规则来定义自然数的乘积。这样一来，乘法的交换性变得难以证明，但仍可以从公理中得出这样的定理。这样就可能逼迫可怜的学生来学习该定理及其证明（其目的不外乎是提升这门学科以及教授它的人的地位）。显然，如此定义、如此证明只会伤害教学和实际工作。

要理解乘法的可交换性，只有通过分别按行、列来数方阵里士兵数，或用这两种方式来计算长方形的面积才可能。做不与物理和现实世界交叉的数学的任何企图都属于宗派主义和孤立主义。这必将破坏所有具有合理思维能力的人们眼中数学创造是有用的人类活动的这一美好印象。

我再揭示几个这样的秘密（以此来帮助可怜的学生们）。

一个矩阵的行列式就是一个以矩阵的各列为各边的平行多面体的（有向）体积。如果学生们被告知了这个秘密（在纯粹的代数式的教育中，该秘密被小心地隐藏着），则整个行列式理论都将成为多维线性型理论的一部分。如果用别的方式来定义行列式，则任何聪明人都将会永远怨恨行列式、雅可比式以及隐函数定理这些东西。

什么是一个群呢？代数学家会这样来教学：这是附有两个满足一组令人容易忘却的规则的运算的集合。这个定义会引起自然的抗议：为何需要这一对运算？“哦，该死的数学”——这就是学生们的结论（将来他们中有人可能成为国家科学部长）。

如果我们按照历史发展的顺序，不是由群而是从变换

have led to the rejection of the scheme used in physics (observation - model - investigation of the model - conclusions - testing by observations) and its substitution by the scheme: definition - theorem - proof. It is impossible to understand an unmotivated definition but this does not stop the criminal algebraists-axiomatisators. For example, they would readily define the product of natural numbers by means of the long multiplication rule. With this the commutativity of multiplication becomes difficult to prove but it is still possible to deduce it as a theorem from the axioms. It is then possible to force poor students to learn this theorem and its proof (with the aim of raising the standing of both the science and the persons teaching it). It is obvious that such definitions and such proofs can only harm the teaching and practical work.

It is only possible to understand the commutativity of multiplication by counting and re-counting soldiers by ranks and files or by calculating the area of a rectangle in the two ways. Any attempt to do without this interference by physics and reality into mathematics is sectarianism and isolationism which destroy the image of mathematics as a useful human activity in the eyes of all sensible people.

I shall open a few more such secrets (in the interest of poor students).

The determinant of a matrix is an (oriented) volume of the parallelepiped whose edges are its columns. If the students are told this secret (which is carefully hidden in the purified algebraic education), then the whole theory of determinants becomes a clear chapter of the theory of poly-linear forms. If determinants are defined otherwise, then any sensible person will forever hate all the determinants, Jacobians and the implicit function theorem.

What is a group? Algebraists teach that this is supposedly a set with two operations that satisfy a load of easily-forgettable axioms. This definition provokes a natural protest: why would any sensible person need such pairs of operations? “Oh, curse this maths” —— concludes the student (who, possibly, becomes the Minister for Science in the future).

We get a totally different situation if we start off not with the group but with the concept of a transformation (a one-to-one mapping of a set onto itself) as it was historically. A collection of transformations of a set is called a group if along with any two transformations it contains the result of their consecutive application and an inverse transformation along with every transformation.

This is all the definition there is. The so-called “axioms” are

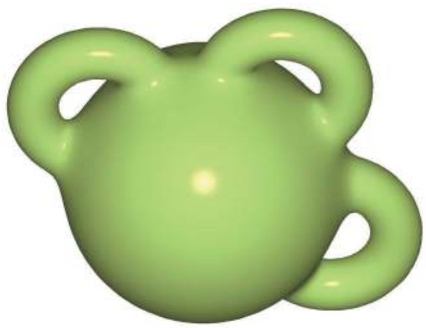


图 11 带三个柄的球面

的观点（一个集合到自身的 1-1 映射）出发，我们就有完全不同的情形。一个集合的一族变换称为一个群，如果其中任何两个变换的复合仍在此族内，并且每个变换的逆变换也是如此。

此即该定义的核心。那些所谓的“公理”事实上仅为变换群（明显）的性质。公理化主义者所称的“抽象群”不过是同构（保持运算的 1-1 映射）意义下不同集合的变换群。正如凯莱所证明的，根本就不存在“更抽象的”群。那么为什么代数学家仍要用抽象的定义来折磨学生呢？

顺便提一下，上世纪 60 年代我曾为莫斯科的中学生们讲授群论。我未使用任何公理，尽可能地贴近物理，半年时间我就能教给学生一般五次方程无根式解的阿贝尔定理（同时还教了复数、黎曼面、基本群以及代数函数的单值群）。这门课程的内容后来由我的一个听众 V. Alekseev 编辑成书出版，书名为 The Abel theorem in problems & solutions[16]。

什么是一个光滑流形？我见到一本美国人最近撰写的书称庞加莱对此概念并不清晰（尽管是由他引入的），其“现代的”定义直到上世纪 20 年代末期才由维布伦给出：一个流形是满足一系列公理的一个拓扑空间。

学生们究竟为何罪孽要在这些扭转曲折中尝试、摸索来寻求其正途？事实上，在庞加莱的《位置分析》(Analysis Situs）中有比现代“抽象”定义更有用，也绝对更清晰的定义。

欧氏空间 $\mathbb{R}^N$ 中的 k 维光滑子流形是 $\mathbb{R}^N$ 的子集，其上每一点都有一个邻域是从 $\mathbb{R}^k$ 到 $\mathbb{R}^{N-k}$ 上的光滑映射的图象（其中 $\mathbb{R}^k$ 和 $\mathbb{R}^{N-k}$ 是坐标子空间 ）。这是平面上大多数普通光滑曲线（如圆周 $x^2+y^2=1$）或三维空间中曲线和曲面的直接的推广。

光滑流形之间的光滑映射可自然地定义。微分同胚是

[16] 此书英译 2004 年由 Kluwer 出版。阿诺德的针对中学生（14-16 岁）的课程由正在进行教学改革（更新欧几里得传统）的柯尔莫哥洛夫组织。参考 *Yersterday and Long Ago* 第 158–163 页。

in fact just (obvious) properties of groups of transformations. What axiomatisators call “abstract groups” are just groups of transformations of various sets considered up to isomorphisms (which are one-to-one mappings preserving the operations). As Cayley proved, there are no “more abstract” groups in the world. So why do the algebraists keep on tormenting students with the abstract definition?

By the way, in the 1960s I taught group theory to Moscow schoolchildren. Avoiding all the axiomatics and staying as close as possible to physics, in half a year I got to the Abel theorem on the unsolvability of a general equation of degree five in radicals (having on the way taught the pupils complex numbers, Riemann surfaces, fundamental groups and monodromy groups of algebraic functions). This course was later published by one of the audience, V. Alekseev, as the book *The Abel theorem in problems*.

What is a smooth manifold? In a recent American book I read that Poincaré was not acquainted with this (introduced by himself) notion and that the “modern” definition was only given by Veblen in the late 1920s: a manifold is a topological space which satisfies a long series of axioms.

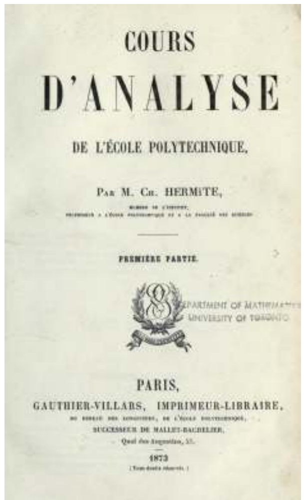
For what sins must students try and find their way through all these twists and turns? Actually, in Poincaré's *Analysis Situs* there is an absolutely clear definition of a smooth manifold which is much more useful than the “abstract” one.

A smooth k-dimensional submanifold of the Euclidean space $\mathbb{R}^N$ is its subset which in a neighbourhood of its every point is a graph of a smooth mapping of $\mathbb{R}^k$ into $\mathbb{R}^{N-k}$ (where $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ are coordinate subspaces). This is a straightforward generalization of most common smooth curves on the plane (say, of the circle $x^2+y^2=1$) or curves and surfaces in the three dimensional space.

Between smooth manifolds smooth mappings are naturally defined. Diffeomorphisms are mappings which are smooth, together with their inverses.

An “abstract” smooth manifold is a smooth submanifold of a Euclidean space considered up to a diffeomorphism. There are no “more abstract” finite-dimensional smooth manifolds in the world (Whitney's theorem). Why do we keep on tormenting students with the abstract definition? Would it not be better to prove them the theorem about the explicit classification of closed two-dimensional manifolds (surfaces)?

It is this wonderful theorem (which states, for example, that any compact connected oriented surface is a sphere with a number


COURS

D'ANALYSE

DE L'ÉCOLE POLYTECHNIQUE,

PAR M. CH. HERMITE,

PREMIÈRE PARTIE.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE,

SUCCESSEUR DE MALLET-BACHELIER,

1873

图 12 1887 年的埃尔米特（1822-1901）；图 13 埃尔米特的微积分教程标题页

其逆也光滑的光滑映射。

“抽象”光滑流形是在微分同胚意义下的欧氏空间的光滑子流形。世上根本无所谓“更抽象的”有限维光滑流形（惠特尼定理）。为什么我们总是要用抽象的定义来折磨学生们呢？把闭二维流形（曲面）的分类定理证给学生看难道不更好吗？

恰是如此精彩的定理（例如，任意紧连通的可定向曲面都是一个带若干柄的球面）使我们对什么是现代数学有了一个正确的印象，而不是那些欧氏空间简单子流形的超抽象的推广，事实上后者根本没有给出任何有新意的东西，不过是被公理化主义者们作为成果用来炫耀的而已。

曲面分类定理是顶级的数学成就，堪比美洲大陆或X射线的发现。这是数学自然科学里一个真正的发现，我们甚至难说该发现是属于物理学的或属于数学的。它对应用以及对发展正确的世界观的意义目前已超越了数学中的其他“成就”——如费马大定理的证明，或对任何充分大的整数都能表示成三个素数之和这类事实的证明。

为了出风头，当代的数学家有时候总要展示一些“运动会式的”成就，并声称那就是他们的学科里盖棺之作。可想而知，这样的做法不仅无助于社会对数学的欣赏，而且适得其反，会使人们产生疑问：对于这样的毫无用处的奇异问题，有必要浪费力量来做这些（仿佛攀岩似的）练习吗？

曲面分类定理应该放到高中数学的课程里（或许不加证明），但由于某些原因甚至在大学的数学课程里也找不到（顺便提一下，在法国近几十年来所有的几何都从大学课程中被删去）。

所有层次的数学教育由学院式腔调全面回归到展示自

of handles) that gives a correct impression of what modern mathematics is and not the super-abstract generalizations of naive submanifolds of a Euclidean space which in fact do not give anything new and are presented as achievements by the axiomatisators.

The theorem of classification of surfaces is a top-class mathematical achievement, comparable with the discovery of America or X-rays. This is a genuine discovery of mathematical natural science and it is even difficult to say whether the fact itself is more attributable to physics or to mathematics. In its significance for both the applications and the development of correct Weltanschauung it by far surpasses such “achievements” of mathematics as the proof of Fermat's last theorem or the proof of the fact that any sufficiently large whole number can be represented as a sum of three prime numbers.

For the sake of publicity modern mathematicians sometimes present such sporting achievements as the last word in their science. Understandably this not only does not contribute to the society's appreciation of mathematics but, on the contrary, causes a healthy distrust of the necessity of wasting energy on (rock-climbing-type) exercises with these exotic questions needed and wanted by no one.

The theorem of classification of surfaces should have been included in high school mathematics courses (probably, without the proof) but for some reason is not included even in university mathematics courses (from which in France, by the way, all the geometry has been banished over the last few decades).

The return of mathematical teaching at all levels from the scholastic chatter to presenting the important domain of natural science is an espessially hot problem for France. I was astonished that all the best and most important in methodical approach mathematical books are almost unknown to students here (and, seems to me, have not been translated into French). Among these are Numbers and figures by Rademacher and Töplitz, Geometry and the imagination by Hilbert and Cohn Vossen, What is mathematics? by Courant and Robbins, How to solve itand Mathematics and plausible reasoning by Polya, Development of mathematics in the 19th century by F. Klein.

I remember well what a strong impression the calculus course by Hermite (which does exist in a Russian translation!) made on me in my school years.

Riemann surfaces appeared in it, I think, in one of the first lectures (all the analysis was, of course, complex, as it should be). Asymptotics of integrals were investigated by means of

然科学的重要领域，对法国来说是一个极其重要的任务。使我感到异常震惊的是所有那些写得最好而且最重要的阐述数学方法的书在这里却几无学生知晓（在我看来，甚至可能没被译为法文）。这些书有拉德梅彻 - 托普利茨写的《论数与形》、希尔伯特和康福森写的《直观几何》、柯朗和罗宾斯写的《数学是什么》、波利亚写的《如何解题》和《数学合情推理》、克莱因写的《19 世纪数学发展史》[17]。

我清晰地记得，当我在学校求学时，埃尔米特所写的微积分教程[18]（有俄语译本！）给我留下了多么强烈的印象。

我记得在最开始几讲中就出现了黎曼曲面（当然所有分析的内容都是针对复变量的，也本该如此）。而渐近积分理论是在分支点运动下通过黎曼曲面上道路形变的方法来研究（如今，我们称此方法为皮卡 - 莱夫谢茨理论；顺便提一下，皮卡是埃尔米特的女婿——数学能力往往是由女婿来传承：阿达马—莱维—许瓦兹—弗里希王朝就是巴黎科学院中另一范例）。

埃尔米特一百多年前撰写的教程（也许早就被法国大学的学生图书馆扔掉了），已被认为“过时”，但实际上要比那些折磨学生、最令人无聊的微积分课本现代化得多。

如果数学家们再不醒悟，则那些对 (在最恰当意义下的) 现代数学理论仍有需要，同时又对那些毫无用处的公理唠叨具有免疫力（任何具有合理思维的人共有的特点）的消费者终将会拒绝这些中学和大学里面教育不良的学究们所提供的服务。

一位数学教师，倘若至今还未掌握多卷本的朗道和栗弗席兹的教程[19]中的一部分知识，必将成为古董，犹如迄今仍不懂开集与闭集之间差别的人。

致谢：译者感谢陆俊博士对文中代数几何等部分内容的讨论，也特别感谢美国西北大学徐佩教授的阅读校订。

——2012 年 6 月于比勒费尔德

[17]《数学是什么》与《论数与形》的介绍，可参《数学文化》2012 年第三期欧阳顺湘作《最美的数学就如文学》一文。

[18] 埃尔米特的原著 *Cours d´analyse de lécole polytechnique* (1873)，可下载自 http://archive.org/details/coursdanalysedel01hermuoft。

[19] 列夫 • 朗道（Lev Landau, 1908-1968）是前苏联著名物理学家，凝聚态物理学的奠基人。1962 年的诺贝尔物理学奖获得者。栗弗席兹（Evgeny Lifshitz, 1915-1985）为朗道的学生、理论物理学家。这里所提教程为著名的《理论物理学教程》，全书共 10 卷，由朗道、栗弗席兹与皮塔耶夫斯基等人合作完成。

图 14 2008 年阿塞拜疆发行的纪念朗道诞辰 100 周年的邮票

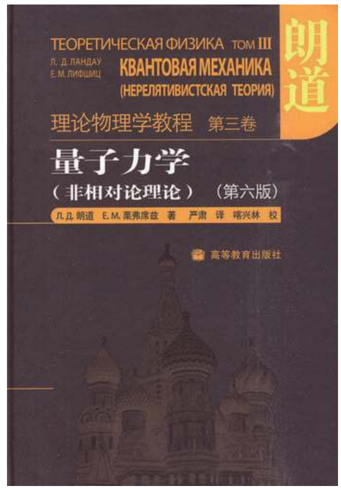


图 15 《理论物理学教程 第三卷：量子力学（非相对论理论）》封面

path deformations on Riemann surfaces under the motion of branching points (nowadays, we would have called this the Picard-Lefschetz theory; Picard, by the way, was Hermite's son-in-law - mathematical abilities are often transferred by sons-in-law: the dynasty Hadamard - P. Levy - L. Schwarz - U. Frisch is yet another famous example in the Paris Academy of Sciences).

The “obsolete” course by Hermite of one hundred years ago (probably, now thrown away from student libraries of French universities) was much more modern than those most boring calculus textbooks with which students are nowadays tormented.

If mathematicians do not come to their senses, then the consumers who preserved a need in a modern, in the best meaning of the word, mathematical theory as well as the immunity (characteristic of any sensible person) to the useless axiomatic chatter will in the end turn down the services of the undereducated scholastics in both the schools and the universities.

A teacher of mathematics, who has not got to grips with at least some of the volumes of the course by Landau and Lifshitz, will then become a relict like the one nowadays who does not know the difference between an open and a closed set.