Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2025-0033

Vol. **18**, No. 4, pp. 845-880
November 2025

Solving the Helmholtz Equation by Sparse Fundamental Solution Neural Network

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Received 21 March 2025; Accepted (in revised version) 7 April 2025

Abstract. A sparse fundamental solution neural network (SFSNN) for solving the Helmholtz equation with constant coefficients and relatively large wave numbers k is proposed. The method combines the strengths of fundamental solution techniques and neural networks by employing a radial basis function neural network, where fundamental solution functions serve as activation functions. Since these functions inherently satisfy the homogeneous Helmholtz equation, SFSNN only requires boundary sampling, significantly accelerating training. To enhance sparsity and generalization, an ℓ_1 regularization term of the weights is introduced into the loss function, reformulating the weight optimization as a least absolute shrinkage and selection operator (Lasso) problem. This not only reduces the number of basis functions but also improves the network's generalization capability. Numerical experiments validate the method's effectiveness for high-wavenumber isotropic Helmholtz equations in two dimensions and three dimensions. The results reveal that when the analytical solution is a linear combination of fundamental solutions, SFSNN accurately identifies their centers. Otherwise, the number of required basis functions scales as $N = \mathcal{O}(k^{(\tau(d-1))})$, where $\tau < 1$ and d is the problem dimension. Moreover, SFSNN has been successfully extended to non-homogeneous and semi-infinite Helmholtz equations, achieving high accuracy. Codes of the examples in this paper are available at `https://github.com/wangzhiwensuda/SFSNN-Helmholtz-problem`.

AMS subject classifications: 35J05, 68T07

Key words: Fundamental solution function, radial basis function neural network, Helmholtz equation, ℓ_1 regularization.

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1. Introduction

In this paper, we consider the Helmholtz equation with Dirichlet boundary conditions as follows:

$$\begin{cases} \Delta u(\boldsymbol{x}) + k^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, k is the wave number, u is the unknown function to be solved (typically representing the wave field), f is the source term, and g is the boundary term. The Helmholtz equation is widely used in various scientific fields and practical applications, such as electromagnetics, acoustics, optics, and seismology. Studying efficient numerical methods for its solution holds significant theoretical and practical importance.

In recent years, a series of deep neural network methods for solving PDEs have been derived. Deep Ritz method (DRM) [28] employs the variational formulation as the loss function. Deep Galerkin method (DGM) [24] and physics-informed neural networks (PINNs) [20] utilize the residual of the PDE as the loss function. Weak adversarial network (WAN) [29] employs the weak formulation to construct the loss function. Miaomiao Chen team [3] further constructed a multi-scale feature fusion mechanism, with core innovations reflected in two aspects: first, they designed a hybrid activation module incorporating triangular basis functions, which effectively captures fluctuations at different scales; second, they proposed a gradient-based dynamic adjustment strategy for loss function weights. Song *et al.* [25] presented a physics-informed neural network based on an adaptive sine activation function (Adaptive Sin-PINN), which significantly enhances the accuracy and convergence speed of solving wave equations by dynamically adjusting the frequency parameters of the activation function.

Radial basis function neural network (RBFNN) represents an attractive alternative to other neural network models [16]. One reason is that it forms a unified connection between function approximation, regularization, noisy interpolation, classification, and density estimation. It is also the case that training a RBFNN is faster than training MLP networks [16]. RBFNN is a simple three-layer feedforward neural network. The first layer is the input layer, representing the input features. The second layer is the hidden layer, composed of radial basis activation functions. Like most neural networks, RBFNN also has the universal approximation property [18]. Meanwhile, compared with deep neural networks, RBFNN avoids the tedious and lengthy calculation of back propagation between the input layer and the output layer, significantly accelerating the training process of the network.

In addition to the aforementioned methods, there is a more common method for solving the homogeneous Helmholtz problem: the method of fundamental solutions (MFS) [1, 9, 10, 15], which uses fundamental solution functions as basis functions. MFS was first introduced by Kupradze and Aleksidze [9], and Mathon and Johnston [15] provided the earliest numerical expressions for it. The main idea of the MFS is to approximate the solution of the original problem using a linear combination of