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Splitting Schemes for Second-Order Backward Stochastic Differential Equations

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Abstract. In this work, we focus on splitting methods for solving second-order backward stochastic differential equations (2BSDEs). By splitting the original d-dimensional 2BSDEs into d 2BSDEs and approximating these split 2BSDEs, we derive the splitting schemes that only require one-dimensional approximations to evaluate the conditional mathematical expectations. Combining the Sinc quadrature rule to approximate the conditional expectations, we further propose the first-order fully discrete splitting schemes. Numerical tests are presented to show the capacity of the schemes.

AMS subject classifications: 65C20, 65C30, 60H35

Key words: Second-order backward stochastic differential equations, splitting methods, splitting scheme, Sinc quadrature rule.

1. Introduction

This paper is concerned with the numerical solution of the following second-order backward stochastic differential equations which is defined on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\begin{cases} Y_t = \varphi(X_T) + \displaystyle\int_t^T f(s, \Theta_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \\ Z_t = Z_0 + \displaystyle\int_0^t A_s \mathrm{d}s + \int_0^t \Gamma_s \mathrm{d}W_s \end{cases} \tag{1.1}$$

for $t \in [0, T]$, where $T > 0$ is the deterministic terminal time; $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ is the natural filtration generated by the standard d-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^d)^\top$; $X_t = X_0 + \sigma W_t$ is a forward diffusion process with the diffusion coefficient σ; $\varphi : \mathbb{R}^d \to \mathbb{R}$ is the terminal condition of the backward stochastic differential equation (BSDE); $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is the generator

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of the BSDE; $\Theta_t = (X_t, Y_t, Z_t, \Gamma_t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^{d\times d}$; the two stochastic integrals with respect to W_s are of the Itô type. An L^2-adapted solution of the 2BSDEs (1.1) is a 4-tuple $(Y_t, Z_t, \Gamma_t, A_t)$, $t \in [0, T]$, which is $\mathcal{F}_t$-adapted, square integrable, and satisfies the 2BSDEs (1.1).

The 2BSDEs (1.1), as an extension of the BSDEs [2, 19], was first introduced by Cheridito *et al.* [6] and further investigated by Soner *et al.* [22]. They showed that the solution of the fully nonlinear parabolic PDE allows for solving the corresponding 2BSDEs. This connection extends the nonlinear Feynman-Kac representations of linear and semi-linear parabolic PDEs to fully nonlinear parabolic PDEs [6, 17, 21, 22], specifically the Hamilton-Jacobi-Bellman equations and the Bellman-Isaacs equations, which are widely used in various fields including stochastic optimal control and stochastic differential games (see, e.g., [4, 5] and references therein).

As the difficulty in obtaining analytical solutions of BSDEs and 2BSDEs, numerical methods for solving BSDEs and 2BSDEs have become highly desired in applications. So far, there has been a significant amount of works on numerical methods for solving BSDEs [1, 3, 7, 10, 15, 16, 18, 25, 26, 30–34]. However, the research on numerical methods for solving 2BSDEs and fully nonlinear second-order parabolic PDEs is still relatively limited due to the complex solution structure [9, 12, 13, 24, 28, 35].

Our objective in this paper is to present new splitting schemes for solving 2BSDEs. It is well known that splitting methods have proved valuable in the approximation of the solutions of multi-dimensional PDEs [8, 20, 27]. Splitting methods solve multi-dimensional equations by reducing multi-dimensional equations into a series of one-dimensional equations, while providing accurate numerical solutions, which are extremely efficient and widely used. The authors in [36] developed splitting schemes for solving BSDEs and rigorously conducted error analysis, demonstrating the first-order convergence rate of the schemes in time in solving BSDEs. However, up to now, the research on splitting methods for solving 2BSDEs and fully nonlinear second-order parabolic PDEs has been limited.

In this paper, we shall propose splitting schemes for solving 2BSDEs. The main contributions of this paper are outlined as follows:

- By splitting the 2BSDEs (1.1) into d 2BSDEs and approximating these split 2BSDEs over each time subinterval, we propose our splitting schemes for solving the original 2BSDEs (1.1). The main advantage of the schemes is that only one-dimensional approximations are needed to calculate the conditional mathematical expectations, making our schemes more efficient and easier to use.

- Our numerical tests show the first-order accuracy and high efficiency of our splitting schemes for solving complex problems, including 2BSDEs and fully nonlinear second-order parabolic PDEs.

The rest of this paper is organized as follows. In Section 2, we give the discretization procedure of 2BSDEs by splitting the original d-dimensional 2BSDEs into d 2BSDEs and approximating these split 2BSDEs. Our splitting schemes for solving 2BSDEs are given