Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 6, pp. 1506-1524

DOI: 10.4208/aamm.OA-2016-0049
December 2017

Convergence Analysis for the Chebyshev Collocation Methods to Volterra Integral Equations with a Weakly Singular Kernel

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Received 27 June 2016; Accepted (in revised version) 26 February 2017

Abstract. In this paper, a Chebyshev-collocation spectral method is developed for Volterra integral equations (VIEs) of second kind with weakly singular kernel. We first change the equation into an equivalent VIE so that the solution of the new equation possesses better regularity. The integral term in the resulting VIE is approximated by Gauss quadrature formulas using the Chebyshev collocation points. The convergence analysis of this method is based on the Lebesgue constant for the Lagrange interpolation polynomials, approximation theory for orthogonal polynomials, and the operator theory. The spectral rate of convergence for the proposed method is established in the L^{∞}-norm and weighted L^2-norm. Numerical results are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Chebyshev collocation method, Volterra integral equations, spectral rate of convergence, Hölder continuity.

1 Introduction

Integro-differential equations provide an important tool for modeling physical phenomena in various fields of science and engineering. This work is concerned with applying

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Chebyshev spectral methods to solve Volterra integral equations (VIEs) of second kind with a weakly singular kernel

$$y(t)=g(t)+\int_0^t (t-s)^{-\frac{1}{2}}K(t,s)y(s)ds, \quad 0\le t\le T, \tag{1.1}$$

where the function $y(t)$ is the unknown function whose value is to be determined in the interval $0\le t\le T<\infty$. Here, $g(t)$ is a given smooth function and $K(t,s)$ is a given kernel, which is also assumed to be smooth.

For any positive integer m, if g and K have continuous derivatives of order m, then there exists a function $Z=Z(t,v)$ possessing continuous derivatives of order m, such that the solution of (1.1) can be written as $y(t)=Z(t,\sqrt{t})$, see, e.g., [3, 23, 24]. This implies that near $t=0$ the first derivative of the solution $y(t)$ behaves like $y'(t)\sim t^{-\frac{1}{2}}$. Several methods have been proposed to recover high order convergence properties for (1.1) using collocation type methods, see, e.g., [1, 2, 5, 8, 18, 19, 21, 22] and using multi-step method, see, e.g., [10, 25–27]. For spectral methods, the singular behavior of the exact solution makes the direct application of the spectral approach difficult. More precisely, for any positive integer m, we have $y^{(m)}(t)\sim t^{\frac{1}{2}-m}$, which indicates that $y\notin H^m_\omega(0,T)$, where H^m_ω is a standard Soblev space associated with a weight ω. To overcome this difficulty, we first apply the transformation

$$\tilde{y}(t)=t^{\frac{1}{2}}[y(t)-y(0)]=t^{\frac{1}{2}}[y(t)-g(0)] \tag{1.2}$$

to change (1.1) to the equation

$$\tilde{y}(t)=\tilde{g}(t)+t^{\frac{1}{2}}\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}K(t,s)\tilde{y}(s)ds, \quad 0\le t\le T, \tag{1.3}$$

where

$$\tilde{g}(t)=t^{\frac{1}{2}}[g(t)-g(0)]+t^{\frac{1}{2}}g(0)\int_0^t (t-s)^{-\frac{1}{2}}K(t,s)ds. \tag{1.4}$$

It is easy to see that the solution of (1.3) is a regular function

$$\tilde{y}(t)\in C^m([0,T]). \tag{1.5}$$

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t=\frac{T}{2}(1+x), \quad x=\frac{2}{T}t-1, \tag{1.6}$$

to rewrite the weakly singular problem (1.3) as follows

$$u(x)=f(x)+\left[\frac{T}{2}(1+x)\right]^{\frac{1}{2}}\int_0^{\frac{T}{2}(1+x)} s^{-\frac{1}{2}}\left(\frac{T}{2}(1+x)-s\right)^{-\frac{1}{2}}K\left(\frac{T}{2}(1+x),s\right)\tilde{y}(s)ds, \tag{1.7}$$

where $x\in[-1,1]$, and

$$u(x)=\tilde{y}\left(\frac{T}{2}(1+x)\right),\quad f(x)=\tilde{g}\left(\frac{T}{2}(1+x)\right). \tag{1.8}$$

Furthermore, to transfer the integral interval $[0,\frac{T}{2}(1+x)]$ to the interval $[-1,x]$, we make a linear transformation:

$$s=\frac{T}{2}(1+\tau),\quad \tau\in[-1,x]. \tag{1.9}$$

Then Eq. (1.7) becomes

$$u(x)=f(x)+\left[\frac{T}{2}(1+x)\right]^{\frac{1}{2}}\int_{-1}^{x}(1+\tau)^{-\frac{1}{2}}(x-\tau)^{-\frac{1}{2}}\widetilde{K}(x,\tau)u(\tau)d\tau, \tag{1.10}$$

where $x\in[-1,1]$, and

$$\widetilde{K}(x,\tau)=K\left(\frac{T}{2}(1+x),\frac{T}{2}(1+\tau)\right).$$

In [19], a Legendre-collocation method is proposed to solve the Volterra integral equations of the second kind whose kernel and solutions are sufficiently smooth. The main purpose of this work is to use Chebyshev collocation methods to numerically solve the VIE (1.10). We will provide a rigorous error analysis which theoretically justify the spectral rate of convergence of the proposed method. This paper is organized as follows. In Section 2, we introduce the Chebyshev-collocation spectral approaches for (1.10). Some preliminaries and useful lemmas are provided in Section 3. The convergence analysis is given in Section 4. We prove the error estimates in L^∞ norm and weighted L^2 norm for the method. The numerical experiments are carried out in Section 5, which will be used to verify the theoretical results obtained in Section 4. The final section contains concluding remarks.

Throughout the paper C will denote a generic positive constant that is independent of N but which will depend on T and on the bounds for the given functions g and K.

2 Chebyshev-collocation methods

Let $\omega(x)=(1-x^2)^{-\frac{1}{2}}$ be a weight function in the usual sense. As defined in [4,17], the set of Chebyshev polynomials $\{T_n(x)\}_{n=0}^{\infty}$ forms a complete $L^2_\omega(-1,1)$-orthogonal system, where $L^2_\omega(-1,1)$ is a weighted space defined by

$$L^2_\omega(-1,1)=\left\{v:\ v \text{ is measurable and } ||v||_{L^2_\omega(-1,1)}<\infty\right\},$$

equipped with the norm

$$||v||_{L^2_\omega(-1,1)}=\left(\int_{-1}^{1}|v(x)|^2\omega(x)dx\right)^{\frac{1}{2}}.$$

and the inner product

$$(u,v)_\omega=\int_{-1}^{1}u(x)v(x)\omega(x)dx, \quad \forall u,v\in L^2_\omega(-1,1).$$

Now, for given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points, and by $\{w_i\}_{i=0}^N$ the corresponding weights. Let $\mathcal{P}_N$ denote the space of all polynomials of degree $\le N$. For any $v\in C[-1,1]$, from [4,17], we can define the Lagrange interpolating polynomial $I_N v\in\mathcal{P}_N$, satisfying

$$I_N v(x_i)=v(x_i), \quad 0\le i\le N.$$

It can be written as an expression of the form

$$I_N v(x)=\sum_{i=0}^{N}v(x_i)F_i(x),$$

where $\{F_i(x)\}$ is the Lagrange interpolation polynomial associated with $\{x_i\}_{i=0}^N$.

Firstly, the Eq. (1.10) holds at the collocation points $\{x_i\}_{i=0}^N$ on $[-1,1]$, namely,

$$u(x_i)=f(x_i)+\left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}}\int_{-1}^{x_i}(1+\tau)^{-\frac{1}{2}}(x_i-\tau)^{-\frac{1}{2}}\widetilde{K}(x_i,\tau)u(\tau)d\tau \tag{2.1}$$

for $0\le i\le N$.

In order to obtain high order accuracy of the approximated solution for the equation (1.10), we use the Gauss-type quadrature formula relative to the Chebyshev weight to compute the integral term in (2.1). Based on this idea, we need to transfer the integral interval $[-1,x_i]$ to a fixed interval $[-1,1]$

$$\int_{-1}^{x_i}(1+\tau)^{-\frac{1}{2}}(x_i-\tau)^{-\frac{1}{2}}\widetilde{K}(x_i,\tau)u(\tau)d\tau=\int_{-1}^{1}(1-\theta^2)^{-\frac{1}{2}}\widetilde{K}(x_i,\tau_i(\theta))u(\tau_i(\theta))d\theta, \tag{2.2}$$

by using the following variable change

$$\tau=\tau_i(\theta)=\frac{1+x_i}{2}\theta+\frac{x_i-1}{2}, \quad \theta\in[-1,1]. \tag{2.3}$$

Next, using a $(N+1)$-point Gauss quadrature formula relative to the Chebyshev weight $\{w_i\}_{i=0}^N$, the integral term in (2.2) can be approximated by

$$\int_{-1}^{1}(1-\theta^2)^{-\frac{1}{2}}\widetilde{K}(x_i,\tau_i(\theta))u(\tau_i(\theta)))d\theta\sim\sum_{k=0}^{N}\widetilde{K}(x_i,\tau_i(\theta_k))u(\tau_i(\theta_k))w_k, \tag{2.4}$$

where the set $\{\theta_k\}_{k=0}^N$ coincides with the collocation points $\{x_i\}_{i=0}^N$ on $[-1,1]$. We use u_i, $0\le i\le N$, to indicate the approximate value for $u(x_i)$, and use

$$u^N(x)=\sum_{j=0}^N u_j F_j(x) \tag{2.5}$$

to approximate the function $u(x)$, namely,

$$u(x_i)\sim u_i, u(x)\sim u^N(x), u(\tau_i(\theta_k))\sim \sum_{j=0}^N u_j F_j(\tau_i(\theta_k)).$$

Then the Chebyshev collocation method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the following collocation equations:

$$u_i=f(x_i)+\left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}}\sum_{j=0}^N u_j\left(\sum_{k=0}^N \widetilde{K}(x_i,\tau_i(\theta_k))F_j(\tau_i(\theta_k))w_k\right) \tag{2.6}$$

for $0\le i\le N$. We denote the error function by

$$e(x):=(u-u^N)(x),\quad x\in[-1,1]. \tag{2.7}$$

It follows from (1.2) and (1.8) that

$$y(t)=g(0)+\left[\frac{T}{2}(1+x)\right]^{-\frac{1}{2}}u(x). \tag{2.8}$$

Consequently, the approximate solution to (1.1) is given by

$$y^N(t)=g(0)+\left[\frac{T}{2}(1+x)\right]^{-\frac{1}{2}}u^N(x). \tag{2.9}$$

Then the corresponding error functions have the following relations

$$\epsilon(t):=(y-y^N)(t)=\left[\frac{T}{2}(1+x)\right]^{-\frac{1}{2}}e(x)=t^{-\frac{1}{2}}e(x). \tag{2.10}$$

3 Some preliminaries and useful lemmas

In our subsequent analysis, some preliminary results are needed.

The weighted Sobolev norms in which to measure approximation errors for the Chebyshev system involve the Chebyshev weight in the quadratic averages of the error and its derivatives over the interval $(-1,1)$. Thus, for non-negative integer m we set

$$H^m_\omega(-1,1):=\left\{v:\ \partial_x^k v\in L^2_\omega(-1,1),\ 0\le k\le m\right\} \tag{3.1}$$

with the norm

$$||v||_{H^m_\omega(-1,1)} = \left(\sum_{k=0}^{m} |\partial_x^k v|^2_{L^2_\omega(-1,1)} \right)^{\frac{1}{2}}.$$

In bounding some approximation error, only some of the L^2-norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the semi-norms

$$|v|_{H^{m;N}_\omega(-1,1)} = \left(\sum_{k=\min(m,N+1)}^{m} |\partial_x^k v|^2_{L^2_\omega(-1,1)} \right)^{\frac{1}{2}}.$$

We now introduce the orthogonal projection $P_N : L^2_\omega(-1,1) \to \mathcal{P}_N$, which is a mapping such that for any $v \in L^2_\omega(-1,1)$,

$$(v - P_N v, \phi)_\omega = 0, \quad \forall \phi \in \mathcal{P}_N.$$

By using in [4, (5.5.9) and (5.5.22)], we have the estimates

$$||v - P_N v||_{L^2_\omega(-1,1)} \le C N^{-m} |v|_{H^{m;N}_\omega(-1,1)}, \tag{3.2a}$$

$$||v - I_N v||_{L^2_\omega(-1,1)} \le C N^{-m} |v|_{H^{m;N}_\omega(-1,1)}, \tag{3.2b}$$

$$||v - I_N v||_{L^\infty(-1,1)} \le C N^{1/2-m} |v|_{H^{m;N}_\omega(-1,1)}, \tag{3.2c}$$

for any $v \in H^m_\omega(-1,1)$.

Define a discrete inner product, for any continuous functions u, v on $[-1,1]$ by

$$(u,v)_N = \sum_{j=0}^{N} u(x_j) v(x_j) w_j. \tag{3.3}$$

By (3.2a) and (3.2b) we can obtain an estimate for the integration error produced by a Gauss-type quadrature formula relative to the Chebyshev weight.

Lemma 3.1. *If $v \in H^m_\omega(-1,1)$ for some $m \ge 1$ and $\phi \in \mathcal{P}_N$, then we have*

$$|(v,\phi)_\omega - (v,\phi)_N| \le C N^{-m} |v|_{H^{m;N}_\omega(-1,1)} ||\phi||_{L^2_\omega(-1,1)}. \tag{3.4}$$

From the result (9) in [11], we have the following result on the Lebesgue constant for Lagrange interpolation based on the zeros of the Chebyshev polynomials.

Lemma 3.2. *Assume that $\{F_j(x)\}_{j=0}^N$ are Lagrange interpolation polynomials with the Chebyshev Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_j\}$. Then*

$$||I_N||_\infty := \max_{x \in (-1,1)} \sum_{j=0}^{N} |F_j(x)| = \mathcal{O}(\log N). \tag{3.5}$$

In our analysis, we shall apply the Gronwall's lemma. We call such a function $v=v(t)$ locally integrable on the interval $[0,T]$ if for each $t\in[0,T]$, its Lebesgue integral $\int_0^t v(s)ds$ is finite.

Lemma 3.3. *Assume that a non-negative and locally integrable function v satisfies*

$$v(t)\le b(t)+c(t)\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}v(s)ds,\quad t\in[0,T],$$

where $b(t)\ge 0$ and $c(t)\ge 0$ are upper bounded. Then there exists a constant C such that

$$v(t)\le b(t)+C\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}b(s)ds,\quad t\in[0,T].$$

From now on, for $r\ge 0$ and $\kappa\in[0,1]$, $C^{r,\kappa}([0,T])$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm $||\cdot||_{r,\kappa}$. When $\kappa=0$, $C^{r,0}([0,T])$ denotes the space of functions with r continuous derivatives on $[0,T]$, also denote by $C^r([0,T])$, and with norm $||\cdot||_r$.

We shall make use of a result of Ragozin [13, 14] (see also [7]), which states that, for each non-negative integer r and $\kappa\in[0,1]$, there exists a constant $C_{r,\kappa}>0$ such that for any function $v\in C^{r,\kappa}([0,T])$, there exists a polynomial function $\mathcal{T}_N v\in\mathcal{P}_N$ such that

$$||v-\mathcal{T}_N v||_\infty\le C_{r,\kappa}N^{-(r+\kappa)}||v||_{r,\kappa}, \tag{3.6}$$

where $||\cdot||_\infty$ is the norm of the space $L^\infty(0,T)$, and when the function $v\in C([0,T])$ we also denote $||v||_\infty=||v||_{C([0,T])}$. Actually, as stated in [13, 14], $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}([0,T])$ to $\mathcal{P}_N$. For convenience, we define a linear, weakly singular integral operator $\mathcal{M}$:

$$(\mathcal{M}v)(t)=t^{\frac{1}{2}}\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}K(t,s)v(s)ds,\quad t\in[0,T]. \tag{3.7}$$

We shall need the fact that $\mathcal{M}$ is compact as an operator from $C([0,T])$ to $C^{0,\kappa}([0,T])$ for any $0<\kappa<\frac{1}{2}$.

Lemma 3.4. *Let $\kappa\in(0,\frac{1}{2})$ and $\mathcal{M}$ be defined by (3.7). Then for any function $v(x)\in C([0,T])$, there exists a positive constant C such that*

$$||\mathcal{M}v||_{0,\kappa}\le C||v||_\infty. \tag{3.8}$$

Proof. By the definition of Hölder continuity, we need to prove that

$$\frac{|\mathcal{M}v(t')-\mathcal{M}v(t'')|}{|t'-t''|^\kappa}\le C||v||_\infty \tag{3.9}$$

for every t', $t''\in[0,T]$ and $t'\neq t''$. Without loss of generality, we assume that $0\le t'<t''\le T$. Let

$$k(t,s)=s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}K(t,s). \tag{3.10}$$

Then we have

$$\begin{aligned}
&\frac{|\mathcal{M}v(t')-\mathcal{M}v(t'')|}{(t''-t')^{\kappa}}\\
=&(t''-t')^{-\kappa}\left|\sqrt{t'}\int_0^{t'}k(t',s)v(s)ds-\sqrt{t''}\int_0^{t''}k(t'',s)v(s)ds\right|\\
\le&(t''-t')^{-\kappa}\sqrt{t'}\int_0^{t'}|k(t',s)-k(t'',s)|\,|v(s)|ds\\
&+(t''-t')^{-\kappa}(\sqrt{t''}-\sqrt{t'})\int_0^{t''}|k(t'',s)||v(s)|ds\\
&+(t''-t')^{-\kappa}\sqrt{t'}\int_{t'}^{t''}|k(t'',s)||v(s)|ds\\
=:&E_1+E_2+E_3.
\end{aligned}\tag{3.11}$$

We now estimate the three terms one by one. Observe

$$E_1\le E^{(1)}+E^{(2)},\tag{3.12}$$

where

$$E^{(1)}=(t''-t')^{-\kappa}\sqrt{t'}\int_0^{t'}s^{-\frac{1}{2}}\left[(t'-s)^{-\frac{1}{2}}-(t''-s)^{-\frac{1}{2}}\right]\big|K(t',s)\big||v(s)|ds,\tag{3.13a}$$

$$E^{(2)}=(t''-t')^{-\kappa}\sqrt{t'}\int_0^{t'}s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}}\big|K(t',s)-K(t'',s)\big||v(s)|ds.\tag{3.13b}$$

Recall the definition of the Beta function

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx=B(a,b),\quad a,b>0,\tag{3.14}$$

which gives

$$\int_0^z\tau^{a-1}(z-\tau)^{b-1}d\tau=z^{a+b-1}B(a,b)\tag{3.15}$$

and

$$\int_0^{t'}s^{-\frac{1}{2}}\left[(t'-s)^{-\frac{1}{2}}-(t''-s)^{-\frac{1}{2}}\right]ds=\int_{t'}^{t''}s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}}ds.\tag{3.16}$$

The above observation, together with (3.13), yields

$$\begin{aligned}
E_1^{(1)}\le&C\|v\|_\infty(t''-t')^{-\kappa}\int_{t'}^{t''}\left(\frac{t'}{s}\right)^{\frac{1}{2}}(t''-s)^{-\frac{1}{2}}ds\\
\le&C\|v\|_\infty(t''-t')^{-\kappa}\int_{t'}^{t''}(t''-s)^{-\frac{1}{2}}ds\\
\le&C\|v\|_\infty(t''-t')^{\frac{1}{2}-\kappa}\le C\|v\|_\infty.
\end{aligned}\tag{3.17}$$

Furthermore, we have

$$\begin{aligned}
E_1^{(2)} \leq& C||v||_\infty \int_0^{t'} s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}} \frac{|K(t',s)-K(t'',s)|}{(t''-t')^\kappa} ds \\
\leq& C||v||_\infty ||K||_{0,\kappa} \int_0^{t'} s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}} ds \\
\leq& C||v||_\infty \int_0^{t''} s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}} ds \leq C||v||_\infty,
\end{aligned} \tag{3.18}$$

where we have used the fact $t' < t''$ and (3.15). Using the fact that

$$\frac{\sqrt{t''}-\sqrt{t'}}{\sqrt{t''-t'}} \leq C, \quad \forall 0 \leq t' < t'' < T,$$

we have for $\kappa \in (0, \frac{1}{2})$,

$$E_2 \leq C(t''-t')^{\frac{1}{2}-\kappa} ||v||_\infty \int_0^{t''} s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}} ds \leq C||v||_\infty. \tag{3.19}$$

Finally, we have

$$E_3 \leq C||v||_\infty (t''-t')^{-\kappa} \sqrt{t'} \int_{t'}^{t''} s^{-\frac{1}{2}}(t''-s)^{-\frac{1}{2}} ds \leq C||v||_\infty, \tag{3.20}$$

where we have used the estimate for $E_1^{(1)}$, i.e., (3.17). The desired result (3.8) is established by combining (3.11) with the estimates for E_1, E_2 and E_3 above. $\square$

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g., [6, 9, 16]).

Lemma 3.5. *For all measurable function $f \geq 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(\mathcal{K}f)(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_a^b |f(x)|^p v(x) dx \right)^{1/p}$$

holds if and only if

$$\sup_{a<x<b} \left(\int_x^b u(t) dt \right)^{1/q} \left(\int_a^x v^{1-p'}(t) dt \right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \leq q < \infty$. Here, $\mathcal{K}$ is an operator of the form

$$(\mathcal{K}f)(x) = \int_a^x k(x,t) f(t) dt$$

with $k(x,t)$ a given kernel, u, v weight functions, and $-\infty \leq a < b \leq \infty$.

From [12, Theorem 1], we have the following estimate for the Lagrange interpolation associated with the Chebyshev Gaussian collocation points.

Lemma 3.6. *For each bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N \left\| \sum_{j=0}^{N} v(x_j) F_j(x) \right\|_{L^2_\omega(-1,1)} \leq C||v||_\infty,$$

where $F_j(x)$ is the Lagrange interpolation polynomial associated with the Chebyshev collocation points $\{x_j\}_{j=0}^N$.

4 Convergence analysis

The objective of this section is to analyze the approximation scheme (2.6). Firstly, we derive the error estimate in L^∞ norm of the Chebyshev collocation method.

Theorem 4.1. *Let u be the exact solution of the Volterra integral equation* (1.10)*. Assume the approximated solution u^N of the form* (2.5) *is given by the spectral collocation scheme* (2.6) *with the Chebyshev Gauss, or Gauss-Radau, or Gauss-Lobbatto collocation points. If the given data $g(t)$ and $K(t,s)$ in* (1.1) *belong to $C^m([0,T])$, then*

$$||u-u^N||_\infty \leq CN^{1/2-m}|u|_{H^{m;N}_\omega(-1,1)} + CK^* N^{-m}\log N||u||_\infty \tag{4.1}$$

for sufficiently large N, where

$$K^* = \max_{0\leq i\leq N} |\widetilde{K}(x_i,\tau_i(\cdot))|_{H^{m;N}_\omega(-1,1)}. \tag{4.2}$$

Proof. Since the given data $g(t)$ and $K(t,s)$ in (1.3) belong to $C^m([0,T])$, based on the analysis in Section 1 we have $u\in C^m([-1,1])$. Consequently, $u\in H^m_\omega(-1,1)\cap L^\infty(-1,1)$. We first observe that the solution u of (1.10) satisfies (2.1):

$$u(x_i) = f(x_i) + \left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}} \Big(\widetilde{K}(x_i,\tau_i(\cdot)), u(\tau_i(\cdot))\Big)_\omega \tag{4.3}$$

for $0\leq i\leq N$. Using the definition of the discrete inner product (3.3), we set

$$\Big(\widetilde{K}(x_i,\tau_i(\cdot)), \phi(\tau_i(\cdot))\Big)_N = \sum_{k=0}^{N} \widetilde{K}(x_i,\tau_i(\theta_k))\phi(\tau_i(\theta_k))w_k.$$

Then the numerical scheme (2.6) can be written as

$$u_i = f(x_i) + \left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}} \Big(\widetilde{K}(x_i,\tau_i(\cdot)), u^N(\tau_i(\cdot))\Big)_N \tag{4.4}$$

for $0\le i\le N$, where u^N is defined by (2.5). We now subtract (4.4) from (4.3) to get the error equation

$$\begin{aligned} u(x_i)-u_i&=\left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}}\left(\widetilde{K}(x_i,\tau_i(\cdot)),e(\tau_i(\cdot))\right)_{\omega}+\left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}}I_{i,2}\\ &=\sqrt{t_i}\int_0^{x_i}(1+\tau)^{-\frac{1}{2}}(x_i-\tau)^{-\frac{1}{2}}\widetilde{K}(x_i,\tau)e(\tau)d\tau+\sqrt{t_i}I_{i,2}, \end{aligned} \tag{4.5}$$

for $0\le i\le N$, where $t_i=T(1+x_i)/2$, $e(x)=u(x)-u^N(x)$ is the error function and

$$I_{i,2}=\left(\widetilde{K}(x_i,\tau_i(\cdot)),u^N(\tau_i(\cdot))\right)_{\omega}-\left(\widetilde{K}(x_i,\tau_i(\cdot)),u^N(\tau_i(\cdot))\right)_{N}.$$

In (4.5), the integral transformation (2.2) was used. Applying again the transformation (1.6) and (1.9), we change (4.5) to

$$\begin{aligned} u(x_i)-u_i&=\sqrt{t_i}\int_0^{t_i}(t_i-s)^{-\frac{1}{2}}K(t_i,s)\epsilon(s)ds+\sqrt{t_i}I_{i,2}\\ &=\sqrt{t_i}\int_0^{t_i}s^{-\frac{1}{2}}(t_i-s)^{-\frac{1}{2}}K(t_i,s)\tilde{\epsilon}(s)ds+\sqrt{t_i}I_{i,2}, \end{aligned} \tag{4.6}$$

where $\tilde{\epsilon}(t)=t^{\frac{1}{2}}\epsilon(t)$, and $\epsilon(t)$ was defined in (2.10). Multiplying $F_i(x)$ on both sides of the error equation (4.6) and summing up from $i=0$ to $i=N$ yield

$$I_Nu-u^N=\sum_{i=0}^{N}(\mathcal{M}\tilde{\epsilon})(t_i)F_i(x)+\sum_{i=0}^{N}t_i^{\frac{1}{2}}I_{i,2}F_i(x), \tag{4.7}$$

where $\mathcal{M}$ was defined in (3.7). Consequently, recalling the relation of error function (2.10) to get that

$$\tilde{\epsilon}(t)=t^{\frac{1}{2}}\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}K(t,s)\tilde{\epsilon}(s)ds+I_1+I_2+I_3, \tag{4.8}$$

where

$$I_1=u-I_Nu,\quad I_2=\sum_{i=0}^{N}t_i^{\frac{1}{2}}I_{i,2}F_i(x),\quad I_3=I_N(\mathcal{M}\tilde{\epsilon})-\mathcal{M}\tilde{\epsilon}.$$

From (4.8), we have

$$|\tilde{\epsilon}(t)|\le b(t)+Bt^{\frac{1}{2}}\int_0^t s^{\frac{1}{2}}(t-s)^{-\frac{1}{2}}|\tilde{\epsilon}(s)|ds,\quad t\in[0,T], \tag{4.9}$$

where

$$b(t)=|I_1+I_2+I_3|,\quad B=\max_{0\le s<t\le T}|K(t,s)|. \tag{4.10}$$

Using the generalized Gronwall inequality, i.e., Lemma 3.3, we have

$$|\tilde{\epsilon}(t)| \le b(t) + Bt^{\frac{1}{2}} \int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}|b(s)|ds, \quad t \in [0,T]. \tag{4.11}$$

In fact, from (2.10) we have

$$e(x) = t^{\frac{1}{2}}\epsilon(t) = \tilde{\epsilon}(t). \tag{4.12}$$

Then it follows from (4.11) and (3.15) that

$$||e||_\infty = ||\tilde{\epsilon}||_\infty \le C||b||_\infty \le C\Big(||I_1||_\infty + ||I_2||_\infty + ||I_3||_\infty\Big). \tag{4.13}$$

We now estimate the right-hand-side of (4.13). By (3.2c), we have

$$||I_1||_\infty = ||u - I_N u||_\infty \le CN^{1/2-m}|u|_{H^{m;N}_\omega(-1,1)}. \tag{4.14}$$

Next, it follows from Lemma 3.1 that

$$|I_{i,2}| \le CN^{-m}|\widetilde{K}(x_i,\tau_i(\cdot))|_{H^{m;N}_\omega(-1,1)}||u^N(\tau_i(\cdot))||_{L^2_\omega(-1,1)}. \tag{4.15}$$

Hence, by using Lemma 3.2 and (4.15), we have

$$\begin{aligned}
||I_2||_\infty =& \Big\| \sum_{i=0}^N t_i^{\frac{1}{2}} I_{i,2} F_i(x) \Big\|_\infty \le C \max_{0\le i\le N} |I_{i,2}| \max_{x\in(-1,1)} \sum_{j=0}^N |F_j(x)| \\
\le& CK^* N^{-m} \log N \max_{0\le i\le N} ||u^N(\tau_i(\cdot))||_{L^2_\omega(-1,1)} \\
\le& CK^* N^{-m} \log N ||u^N||_\infty \\
\le& CK^* N^{-m} \log N (||e||_\infty + ||u||_\infty)
\end{aligned} \tag{4.16}$$

for sufficiently large N, where K^* is defined by (4.2). We now estimate the third term I_3. Note that

$$I_N p(x) = p(x), \quad (I_N - I)p(x) = 0, \quad \forall p(x) \in \mathcal{P}_N, \tag{4.17}$$

where I denotes the identical operator. It follows from (4.12) that $\tilde{\epsilon} \in C[0,T]$. Consequently, using (3.6) and Lemma 3.4 that

$$||\mathcal{M}\tilde{\epsilon} - \mathcal{T}_N \mathcal{M}\tilde{\epsilon}||_\infty \le CN^{-\kappa}||\tilde{\epsilon}||_\infty, \tag{4.18}$$

where $\mathcal{T}_N \mathcal{M}\tilde{\epsilon} \in \mathcal{P}_N$. It follows from Lemma 3.2, (4.17) and the above estimate that

$$\begin{aligned}
||I_3||_\infty =& ||(I_N - I)\mathcal{M}\tilde{\epsilon}||_\infty = ||(I_N - I)(\mathcal{M}\tilde{\epsilon} - \mathcal{T}_N\mathcal{M}\tilde{\epsilon})||_\infty \\
\le& (1 + ||I_N||_\infty)||\mathcal{M}\tilde{\epsilon} - \mathcal{T}_N\mathcal{M}\tilde{\epsilon}||_\infty \\
\le& CN^{-\kappa}\log N ||\mathcal{M}\tilde{\epsilon}||_{0,\kappa} \le CN^{-\kappa}\log N||\tilde{\epsilon}||_\infty = CN^{-\kappa}\log N||e||_\infty
\end{aligned} \tag{4.19}$$

for any $\kappa \in (0,1/2)$. Finally, the desired estimate (4.1) follows from (4.13)-(4.16) and (4.19). Thus, we complete the proof. □

Our next goal is to derive the error estimate in weighted L^2 norm.

Theorem 4.2. *Let u and u^N be the same as those in Theorem 4.1. If the given data $g(t)$ and $K(t,s)$ in* (1.1) *belong to $C^m([0,T])$, then*

$$\begin{aligned}&||u-u^N||_{L^2_\omega(-1,1)}\\ \le& CN^{1/2-\kappa-m}|u|_{H^{m;N}_\omega(-1,1)}+CK^*N^{-m}\left(||u||_\infty+N^{-1/2}|u|_{H^1_\omega(-1,1)}\right)\end{aligned}\tag{4.20}$$

for sufficiently large N and for any $\kappa\in(0,1/2)$, where K^* is defined by (4.2).

Proof. Using the transformations (1.6) and (1.9), we change (4.11) to

$$|e(x)|\le b(x)+B\left(\frac{T}{2}(1+x)\right)^{\frac{1}{2}}\int_{-1}^{x}(1+\tau)^{-\frac{1}{2}}(x-\tau)^{-\frac{1}{2}}|b(\tau)|d\tau,\quad x\in[-1,1],\tag{4.21}$$

where b and B are defined by (4.10). It follows from the generalized Hardy's inequality Lemma 3.5 that

$$||e||_{L^2_\omega(-1,1)}\le C\Big(||I_1||_{L^2_\omega(-1,1)}+||I_2||_{L^2_\omega(-1,1)}+||I_3||_{L^2_\omega(-1,1)}\Big).\tag{4.22}$$

Firstly, by (3.2b), we see that

$$||I_1||_{L^2_\omega(-1,1)}=||u-I_Nu||_{L^2_\omega(-1,1)}\le CN^{-m}|u|_{H^{m;N}_\omega(-1,1)}.\tag{4.23}$$

Next, it follows from Lemma 3.6 and (4.15) that

$$\begin{aligned}||I_2||_{L^2_\omega(-1,1)}=&\Big\|\sum_{i=0}^{N}t_i^{\frac{1}{2}}I_{i,2}F_i(x)\Big\|_{L^2_\omega(-1,1)}\le C\max_{0\le i\le N}|I_{i,2}|\\ \le& CK^*N^{-m}\max_{0\le i\le N}||u^N(\tau_i(\cdot))||_{L^2_\omega(-1,1)}\le CK^*N^{-m}||u^N||_\infty.\end{aligned}\tag{4.24}$$

By the convergence result in Theorem 4.1, we have

$$||u^N||_\infty\le||e||_\infty+||u||_\infty\le C\left(N^{-1/2}|u|_{H^1_\omega(-1,1)}+||u||_\infty\right),\tag{4.25}$$

which, together with (4.24), gives

$$||I_2||_{L^2_\omega(-1,1)}\le CK^*N^{-m-1/2}||u||_{H^1_\omega(-1,1)}+CK^*N^{-m}||u||_\infty\tag{4.26}$$

for sufficiently large N. Moreover, it follows from (4.17), Lemma 3.6 and (3.6) that

$$\begin{aligned}||I_3||_{L^2_\omega(-1,1)}=&||(I_N-I)\mathcal{M}\tilde{\varepsilon}||_{L^2_\omega(-1,1)}=||(I_N-I)(\mathcal{M}\tilde{\varepsilon}-\mathcal{T}_N\mathcal{M}\tilde{\varepsilon})||_{L^2_\omega(-1,1)}\\ \le&||I_N(\mathcal{M}\tilde{\varepsilon}-\mathcal{T}_N\mathcal{M}\tilde{\varepsilon})||_{L^2_\omega(-1,1)}+||(\mathcal{M}\tilde{\varepsilon}-\mathcal{T}_N\mathcal{M}\tilde{\varepsilon})||_{L^2_\omega(-1,1)}\\ \le&C||\mathcal{M}\tilde{\varepsilon}-\mathcal{T}_N\mathcal{M}\tilde{\varepsilon}||_\infty\le CN^{-\kappa}||\mathcal{M}\tilde{\varepsilon}||_{0,\kappa}\\ \le&CN^{-\kappa}||\tilde{\varepsilon}||_\infty=CN^{-\kappa}||e||_\infty,\end{aligned}\tag{4.27}$$

where in the last step we have used Lemma 3.4. By the convergence result in Theorem 4.1, we obtain that

$$||I_3||_{L^2_\omega(-1,1)} \le CN^{1/2-\kappa-m}|u|_{H^{m;N}_\omega(-1,1)} + CK^* N^{-\kappa-m}\log N||u||_\infty \tag{4.28}$$

for sufficiently large N and for any $\kappa \in (0,1/2)$. The desired estimate (4.20) is obtained by combining (4.22) with (4.23), (4.26) and (4.28). □

Finally, we state the main result of this paper, i.e., the error estimates for the numerical solutions to the VIE (1.1).

Theorem 4.3. *Let y be the exact solution of the Volterra integral equation* (1.1)*. Assume the approximated solution y^N is given by* (2.9) *together with the spectral collocation scheme* (2.5)-(2.6) *with the Chebyshev Gauss, or Gauss-Radau, or Gauss-Lobbatto collocation points. If the given data $g(t)$ and $K(t,s)$ in* (1.1) *belong to $C^m([0,T])$, then*

$$\max_{0\le i\le N+1}\left|y(t_i)-y^N(t_i)\right| \le CN^{3/2-m}|u|_{H^{m;N}_\omega(-1,1)} + CK^* N^{1-m}\log N||u||_\infty \tag{4.29}$$

and

$$\begin{aligned} &||y-y^N||_{L^2_{\tilde\omega}(0,T)} \\ \le& CN^{1/2-\kappa-m}|u|_{H^{m;N}_\omega(-1,1)} + CK^* N^{-m}\left(||u||_\infty + N^{-1/2}|u|_{H^1_\omega(-1,1)}\right) \end{aligned} \tag{4.30}$$

for any $\kappa \in (0,1/2)$, where $u \in C^m[-1,1]$ is defined by (1.2) *and* (1.8)*, K^* is defined by* (4.2)*, and*

$$\tilde\omega(t) := \sqrt{t}. \tag{4.31}$$

Proof. Using (2.10) we have $(y-y^N)(t) = t^{-\frac{1}{2}}(u-u^N)(x)$. Note that

$$\max_{1\le i\le N} t_i^{-\frac{1}{2}} \le \frac{T}{2}\max_{1\le i\le N}(1+x_i)^{-\frac{1}{2}} \le CN.$$

This, together with (4.1), leads to (4.29). The estimate (4.30) is obtained by using (4.20) and the observation

$$\begin{aligned} \int_0^T \tilde\omega(t)(y-y^N)^2 dt =& \sqrt{\frac{2}{T}}\int_{-1}^1 (1+x)^{-\frac{1}{2}}(u-u^N)^2 dx \\ \le& C\int_{-1}^1 (1-x^2)^{-\frac{1}{2}}(u-u^N)^2 dx. \end{aligned}$$

This completes the proof of Theorem 4.3. □

5 Numerical example

Let $U_N=[u_0,\cdots,u_N]^T$ and $F_N=[f(x_0),\cdots,f(x_N)]^T$. The numerical scheme (2.6) leads to a system of equation of the form

$$U_N=F_N+AU_N, \tag{5.1}$$

where the entries of the matrix A is given by

$$a_{ij}=\left[\frac{T}{2}(1+x_i)\right]^{\frac{1}{2}}\sum_{k=0}^{N}\widetilde{K}(x_i,\tau_i(\theta_k))F_j(\tau_i(\theta_k)))w_k.$$

In our numerical tests, we only use the Chebyshev Gauss points

$$x_i=\cos\frac{(2i+1)\pi}{2N+2}$$

with the associated weights $w_i=\pi/(N+1)$. The results obtained using the Chebyshev Gauss-Radau collocation points or the Chebyshev Gauss-Lobatto collocation points are found of similar convergence behaviors.

Example 5.1. We consider the VIE of the form

$$y(t)=g(t)-\int_0^t(t-s)^{-\frac{1}{2}}y(s)ds,\quad 0\le t\le T, \tag{5.2}$$

with

$$g(t)=\pi\sin\frac{t}{2}\text{bessell}\left(0,\frac{t}{2}\right)+\frac{\sin t}{\sqrt{t}},$$

where bessell(ν,z) is the Bessell function defined by:

$$\text{bessell}(\nu,z)=\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-\frac{z^2}{4})^k}{k!\Gamma(\nu+k+1)}.$$

The above equation has the exact solution

$$y(t)=\frac{\sin t}{\sqrt{t}},\quad t\in[0,T].$$

Obviously, $y(t)$ possesses the property presented in the beginning of this paper, i.e., $y'(t)\sim t^{-1/2}$ near $t=0$. Table 1 and Fig. 1 present the errors of the computed solutions using the proposed Chebyshev collocation method. For this example, we choose $T=8$. It is seen clearly that the spectral rate of convergence is achieved.

It is pointed out that we need to evaluate $f(x)$ and $\tilde{g}(t)$, see, (1.4) and (1.8). The integral term in (1.4) can be approximated with spectral accuracy if we take $u\equiv 1$ in (2.2) and (2.4).

Table 1: Example 5.1: the L^∞ error and L^2_ω error for $\tilde{y}(t)$.

N	2	4	6	8
L^∞ Error	6.9287e-001	1.2720e-001	1.1819e-002	6.2614e-004
L^2_ω Error	7.7802e-001	1.6573e-001	1.6289e-002	9.0992e-004
N	10	12	14	16
L^∞ Error	2.2350e-005	5.5479e-007	1.0379e-008	1.4969e-010
L^2_ω Error	3.2840e-005	8.3039e-007	1.5549e-008	2.2448e-010

Table 2: Example 5.2: the L^∞ error and L^2_ω error for $\tilde{y}(t)$.

N	2	4	6	8
L^∞ Error	3.8007e-001	2.7672e-001	3.9601e-002	3.0376e-003
L^2_ω Error	4.5430e-001	3.2695e-002	1.6289e-002	2.0592e-003
N	10	12	14	16
L^∞ Error	1.4690e-004	9.9979e-006	1.8138e-007	3.5195e-009
L^2_ω Error	9.3713e-005	6.2508e-006	1.1140e-007	2.0263e-009

Our second example is concerned with a nonlinear Volterra integral equation with second kind.

Example 5.2. Consider the following the nonlinear Volterra integral equations of second kind with weakly singular kernels

$$y(t)=\int_0^t (t-s)^{-1/2}\Big(-y^2(s)\Big)ds+g(t), \quad 0\le t\le T, \tag{5.3}$$

with

$$g(t)=\frac{\pi}{2}\Big(1+\cos t \text{ besselj } (0,t)\Big)+\frac{\cos t}{t^{1/4}}, \quad t\in[0,T].$$

The above equation has the exact solution

$$y(t)=\frac{\cos t}{t^{1/4}}, \quad 0\le t\le T.$$

For this nonlinear problem, an iterated method is used to treat the nonlinearity, and the proposed spectral method is used to solve the resulting linear equations. It takes about 3 iterations in obtaining convergent results. Table 2 and Fig. 2 present the errors of computed solutions obtained by using the Chebyshev method at $T=8$. Again, exponential rate of convergence is observed for this example.

6 Summary

This work has been concerned with the error analysis for the Chebyshev-collocation spectral methods for the Volterra integral equations with a weakly singular kernel of the form

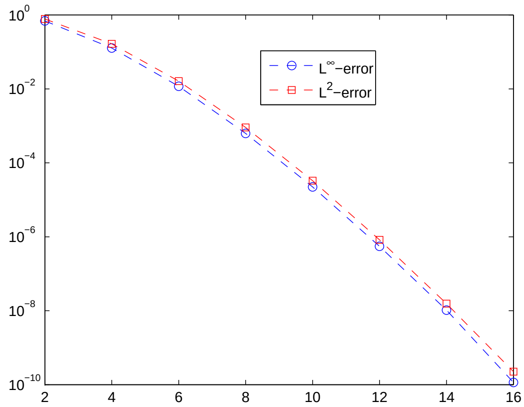

Figure 1: Example 5.1: L^∞ and L^2_ω errors versus the number of collocation points.

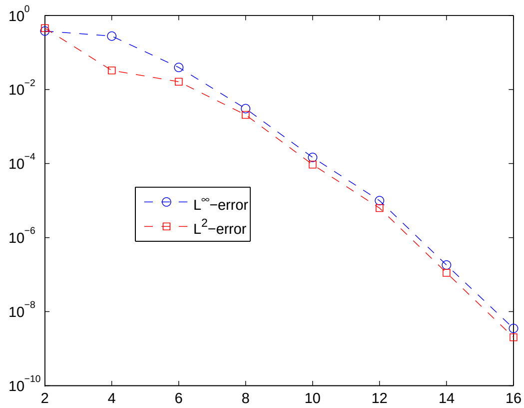

Figure 2: Example 5.2: L^∞ and L^2_ω errors versus the number of collocation points.

$(t-s)^{-\frac{1}{2}}$. The derivative $y'(t)$ of the solution of this equation behaves like $t^{-\frac{1}{2}}$ near the origin and this is expected to cause a loss in the global convergence order. To overcome this difficulty, the original equation was changed into a new Volterra integral equation which possesses better solution regularity, by applying some simple function and coordinate transformations. We also presented a discretization scheme for the resulting Volterra integral equation. It is demonstrated theoretically that both L^∞- and L^2-errors decay exponentially. These results were confirmed by numerical experiments.

In our future work, we will consider the Volterra integral equations of the second kind with a weakly singular kernel of the form $(t-s)^{-\mu}$, where $\mu\in(0,1)$. The Jacobi-collocation spectral analysis will be applied to obtain numerical solutions. Convergence analysis will be provided. We will also investigate the stability properties of these spectral approaches.

Acknowledgments

The authors would like to thank the referees for the helpful suggestions. This work is supported by National Science Foundation of China (Nos. 91430104, 11671157 and 11401347) and Lingnan Normal University Project (No. 2014YL1408).

References

[1] H. BRUNNER, *The numerical solutions of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comput., 45 (1985), pp. 417–437.

[2] H. BRUNNER, *Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels*, IMA J. Numer. Anal., 6 (1986), pp. 221–239.

[3] H. BRUNNER, Collocation Methods for Volterra Integral and Related Functional Equations Methods, Cambridge University Press 2004.

[4] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI AND T. A. ZANG, Spectral Methods Fundamentals in Single Domains, Springer-Verlag 2006.

[5] T. DIOGO, S. MCKEE AND T. TANG, *Collocation methods for second-kind Volterra integral equations with weakly singular kernels*, Proceedings of The Royal Society of Edinburgh, 124A (1994), pp. 199–210.

[6] A. GOGATISHVILL AND J. LANG, *The generalized hardy operator with kernel and variable integral limits in Banach function spaces*, J. Inequalities Appl., 4(1) (1999), pp. 1–16.

[7] I. G. GRAHAM AND I. H. SLOAN, *Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in* $\mathbb{R}^3$, Numer. Math., 92 (2002), pp. 289–323.

[8] Q. HU, *Stieltjes derivatives and polynomial spline collocation for Volterra integro-differential equations with singularities*, SIAM J. Numer. Anal., 33 (1996), pp. 208–220.

[9] A. KUFNER AND L. E. PERSSON, Weighted Inequalities of Hardy Type, World Scientific, New York, 2003.

[10] CH. LUBICH, *Fractional linear multi-step methods for Abel-Volterra integral equations of the second kind,* Math. Comput., 45 (1985), pp. 463–469.

[11] G. MASTROIANNI AND D. OCCORSIO, *Optimal systems of nodes for Lagrange interpolation on bounded intervals: A survey*, J. Comput. Appl. Math., 134 (2001), pp. 325–341.

[12] P. NEVAI, *Mean convergence of Lagrange interpolation III*, Trans. Amer. Math. Soc., 282 (1984), pp. 669–698.

[13] D. L. RAGOZIN, *Polynomial approximation on compact manifolds and homogeneous spaces*, Trans. Amer. Math. Soc., 150 (1970), pp. 41–53.

[14] D. L. RAGOZIN, *Constructive polynomial approximation on spheres and projective spaces*, Trans. Amer. Math. Soc., 162 (1971), pp. 157–170.

[15] H. J.J. TE RIELE, *Collocation methods for weakly singular second-kind Volterra integral equations with non-smooth solution*, IMA J. Numer. Anal., 2 (1982), pp. 437–449.

[16] S. G. SAMKO AND R. P. CARDOSO, *Sonine integral equations of the first kind in* $L_p(0,b)$, Fract. Calc. Appl. Anal., 6 (2003), pp. 235–258.

[17] J. SHEN AND T. TANG, Spectral and High-Order Methods with Applications, Science Press, Beijing, 2006.

[18] T. TANG, *Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations*, Numer. Math., 61 (1992), pp. 373–382.

[19] T. TANG, *A note on collocation methods for Volterra integro-differential equations with weakly singular kernels*, IMA J. Numer. Anal., 13 (1993), pp. 93–99.

[20] D. WILLETT, *A linear generalization of Gronwall's inequality*, Proceedings of the American Mathematical Society, 16 (1965), pp. 774–778.

[21] Y. CHEN, X. LI AND T. TANG, *Convergence analysis of the Jacobi spectral-collocation methods for weakly singular Volterra integral equation with smooth solution*, J. Comput. Appl. Math., 233 (2009), pp. 938–950.

[22] Y. CHEN, X. LI AND T. TANG, *A note on Jacobi spectral-collocation methods for weakly singular Volterra integral equations with smooth solutions*, J. Comput. Math., 31(1) (2013), pp. 47–56.

[23] Y. CHEN AND T. TANG, *Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equation with aweakly singular kernel*, Math. Comput., 79 (2010), pp. 147–167.

[24] YANPING CHEN AND ZHENDONG GU, *Legendre spectral-collocation method for Volterra integral differential equations with non-vanishing delay*, Commun. Appl. Math. Comput. Sci., 8(1) (2013), pp. 67–98.

[25] ZHENDONG GU AND YANPING CHEN, *Chebyshev spectral-collocation method for Volterra integral equations*, Contemp. Math., 586 (2013), pp. 163–170.

[26] Y. YANG, Y. CHEN, Y. HUANG AND W. YANG, *Convergence analysis of Legendre-collocation methods for nonlinear volterra type integro equations*, Adv. Appl. Math. Mech., 7(1) (2015), pp. 74–88.

[27] YIN YANG, YANPING CHEN AND YUNQING HUANG, *Spectral-collocation method for fractional Fredholm integro-differential equations*, J. Korean Math. Soc., 51(1) (2014), pp. 203–224.

[28] X. SHI AND Y. CHEN, *Spectral-collocation method for Volterra delay integro-differential equations with weakly singular kernels,* Adv. Appl. Math. Mech., 8(4) (2016), pp. 648–669.