INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 1-2, Pages 288–306

SUPERCONVERGENCE OF A QUADRATIC FINITE ELEMENT METHOD ON ADAPTIVELY REFINED ANISOTROPIC MESHES

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This paper is dedicated to the memory of Professor Benyu Guo

Abstract. We establish in this paper the supercloseness of the quadratic finite element solution of a two dimensional elliptic problem to the piecewise quadratic interpolation of its exact solution. The assumption is that the partition of the solution domain is quasi-uniform under a Riemannian metric and that each pair of the adjacent elements in the partition forms an approximate parallelogram. This result extends our previous one in [7] for the linear finite element approximations based on adaptively refined anisotropic meshes. It also generalizes the results by Huang and Xu in [13] for the supercloseness of the quadratic elements based on the mildly structured quasi-uniform meshes. A distinct feature of our analysis is that we transform the error estimates on each physical element to that on an equilateral standard element, and then focus on the algebraic properties of the Jacobians of the affine mappings from the standard element to the physical elements. We believe this idea is also useful for the superconvergence study of other types of elements on unstructured meshes.

Key words. Quadratic elements, superconvergence, anisotropic meshes.

1. Introduction

Superconvergence study in finite element approximations has been an area of research for several decades. Classical superconvergence analysis is mostly performed on approximations based on uniform meshes or structured meshes, since superconvergence is generally the result of cancellation of certain lower order terms in the discretization, which relies on the local symmetry of the partition of the solution domains, [2, 22, 26, 27]. There have been much recent developments in extending the study to the FEM based on general types of meshes, see, e.g., [3, 12, 15, 24, 25]. Bank and Xu [3] and Huang and Xu [13] introduced a number of basic identities which involves explicitly the geometric properties of the elements in the partition, and established the supercloseness of the linear and quadratic finite element solution of an two dimensional elliptic equation to the interpolation of its exact solution on general mildly structure quasi-uniform meshes.

For practical applications of the finite element method, the partition of the domain are often adaptively refined, and the meshes are no longer quasi-uniform or even shape regular. In this case, superconvergence is often still observed. Various error estimators have been designed based on such a property to guide the mesh refinement process, see, e.g., [10, 14, 16, 17, 20, 21]. Therefore, understanding of the superconvergence on adaptively refined unstructured meshes may offer useful insights to the practitioners of the finite element method. There have been some recent efforts in this area of study e.g., Wu and Zhang [23] established the superconvergence for linear finite element approximation of a singular perturbed problem based on a pre-defined graded mesh. However, the theoretical work in this area is still limited due to the technical complexity involved in deriving the expressions for discretization errors in general element geometries.

Received by the editors February 30, 2017 and, in revised form, May 12, 2017.
2000 *Mathematics Subject Classification.* 65N30, 65N15, 65N50.

By using the notion of quasi-uniform meshes under a Riemannian metric [9, 5, 6], we extended in [7] the superconvergence analysis for linear finite element approximations on mildly structure quasi-uniform meshes in [3] to certain adaptively refined anisotropic meshes. An innovation for the analysis in [7] is the development of the notion of approximate parallelogram for anisotropic meshes. Based on this superconvergence result, we established in [8] the effectiveness of several commonly used gradient recovery type error estimators for the FEM based on adaptively refined anisotropic meshes.

In this paper, we extend our analysis in [7] for linear FE approximations to a quadratic FE approximation on anisotropic meshes. We establish rigorously the supercloseness of the finite element solution of a two dimensional elliptic equation to the piecewise quadratic interpolation of the exact solution. This conclusion also generalizes the results by Huang and Xu [13] for the supercloseness of quadratic elements on the mildly structured quasi-uniform meshes As is documented in the study of a Laplace equation in [13], superconvergence analysis for quadratic elements on general meshes is much more complicated technically than for linear elements, and considering anisotropic features of the partition makes the study even more difficult. We simplify the task by performing the analysis on the standard element. More specifically, we select the equilateral triangle with vertices on the unit circle as the standard element, and transform the discretization errors on each physical element to the standard element. Then all the estimates involving the geometric properties of the physical elements become those involving the algebraic properties of the Jacobians of the affine mappings from the standard element to the physical elements. This approach was first used in our superconvergence study for linear elements in [7]. It made much easier technically the derivation of various error bounds needed for the analysis. We believe this idea is useful for the superconvergence analysis in other types of problems.

An outline of this paper is as follows: We describe in Section 2 the model problem, the anisotropic partitions of the solution domain, and the measure of the anisotropic features of the higher order derivatives of solutions. We list in Section 3 a number of basic lemmas and then establish the supercloseness of the quadratic finite element solution to the piecewise quadratic interpolation of the exact solution. We provide in Section 4 two numerical examples and finish the paper with some discussions in Section 5.

2. FE Approximation Based on Anisotropic Meshes

We consider the following homogeneous Dirichlet problem of a second order elliptic equation:

$$(1)\qquad \begin{cases} -\nabla \cdot (A\nabla u + \mathbf{b}\ u) + d\ u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

where A is symmetric positive definite (SPD) constant matrix, and $\mathbf{b}$, d, and f are suitably smooth functions. Furthermore, (1) is assumed to be strongly elliptic.

FE approximation: Let $\{\mathcal{T}_N\}$ be a family of triangulations of Ω satisfying the basic requirement that the intersection of the closures of any two elements is either the empty set, a point, or an entire edge. Here N stands for the total number of elements in $\mathcal{T}_N$. We use N, instead of the usual element diameter h to characterize the fineness of the partition, because in anisotropic meshes an element may have very different length scales in different directions. Define S_N be the space of continuous piecewise quadratic polynomials over partition $\mathcal{T}_N$, and $V_N =$

$S_N \cap H_0^1(\Omega)$. The finite element method for solving (1) is to find the approximate solution $u_N \in V_N$ satisfying

$$(2)\ a(u_N, v) = \int_\Omega [(A\nabla u_N)\cdot\nabla v + (\mathbf{b}\cdot\nabla v)u_N + d\ u_N v] = \int_\Omega fv,\ \forall v \in V_N.$$

In order to better describe and control the anisotropic mesh features, such as element sizes, aspect ratios, and alignment directions, we consider a class of meshes that are quasi-uniform under a given metric. Let M be a continuous Riemannian metric on $\bar{\Omega}$. For each element $\tau \in \mathcal{T}_N$, let M_τ be the average of M over τ. Its eigen-decomposition is of form

$$M_\tau = T_\tau \cdot \Lambda_\tau \cdot T_\tau', \tag{3}$$

where Λ_τ is diagonal and T_τ is orthonormal. Define

$$F_\tau = T_\tau \Lambda_\tau^{-\frac{1}{2}}. \tag{4}$$

We call a family of triangulations $\{\mathcal{T}_N\}$ quasi-uniform under metric M, if for all $\tau \in \mathcal{T}_N$, $\tilde{\tau} = F_\tau^{-1}\tau$ are shape regular and of about the same size, see [5, 6, 9]. Let J_τ be the Jacobian of the affine mapping from a standard element $\hat{\tau}$ to τ. Then $\{\mathcal{T}_N\}$ is quasi-uniform under metric M iff

$$\|F_\tau^{-1}J_\tau\| \simeq (\|J_\tau^{-1}F_\tau\|)^{-1} \simeq (C_M/N)^{1/2}, \qquad \forall \tau \in \mathcal{T}_N, \tag{5}$$

where $\simeq$ means the ratio of the two quantities involved are bounded from above and below by positive constants, and

$$C_M = \int_\Omega |\det(M)|^{1/2}. \tag{6}$$

In order to derive the convergence and superconvergence for the quadratic FE approximation on anisotropic meshes, we need certain quantities to characterize the anisotropic behavior of the third and fourth order derivatives of solution u.

Anisotropic measure of D^3u and D^4u: For any given point $\boldsymbol{x} \in \Omega$ and $m = 3$ or 4, we call a "suitable" 2×2 symmetric positive definite matrix Q_m an anisotropic measure of $D^m u$ at $\boldsymbol{x}$, if it satisfies

$$|(\mathbf{s}\cdot\nabla)^m u(\boldsymbol{x})| \le [\mathbf{s}\cdot Q_m(\boldsymbol{x})\mathbf{s}]^{m/2}, \qquad \forall \mathbf{s} \in \mathbf{R}^2. \tag{7}$$

Clearly, there are infinitely many SPD matrices satisfying the above inequality. For instance, Q_m can be chosen as $|D^m u|\ I$, where I is the identity matrix, and $|D^m u|$ is the maximum of all the m-th order directional derivatives at $\boldsymbol{x}$. However, such a Q_m reflects only the magnitude of $D^m u(\boldsymbol{x})$, and no information about the possible anisotropic behavior of $D^m u(\boldsymbol{x})$ is revealed. In order to include such information, Q_m should be chosen so that inequality (7) is satisfied as tight as possible, namely, Q_m should be selected as "small" as possible. One option is to choose such an anisotropic measure Q_m so that $\mathbf{s}\cdot Q_m\mathbf{s} = 1$ is the largest ellipse contained in $|(\mathbf{s}\cdot\nabla)^m u| \le 1$ on the $\mathbf{s}$-plane, see [6] for more details. Unfortunately, there is no explicit formula for Q_m in general. Furthermore, in some degenerated cases, such a largest ellipse can be infinite, and suitable regularization should be applied to define Q_m. We developed in [5, 6] a numerical algorithm to find an suitable Q_m approximately. More recently, Mirebeau [19] discussed the basic properties of Q_m for general m, and derived in particular an explicit formula in [18] for the "smallest" Q_3 measuring the anisotropic behaviors of third order derivatives D^3u. We shall use this formula for the anisotropic mesh generation in our numerical experiment in this paper.

Based on the anisotropic measure Q_m of the high order derivatives and its interplay with the metric M characterizing the anisotropic properties of the adaptive meshes, we can derive the error estimates for the piecewise polynomial interpolation based on anisotropic meshes, see [5, 6] for details. In particular, we state below an error estimate for the piecewise quadratic interpolation which will be needed in our analysis later.

Theorem 2.1. *Suppose $\{\mathcal{T}_N\}$ is quasi-uniform under metric M, and u_q is the piecewise quadratic interpolation of function $u \in H^3(\Omega)$. Suppose the third order derivative D^3u satisfies assumption (7) about its anisotropic behavior. Then there exists a constant c independent of u and N such that*

$$\|u - u_q\|_{0,\Omega} \leq \; C_M^{3/2} N^{-3/2} \cdot \Big\{ \; c \int_\Omega \|F^T Q_3 F\|^3 \Big\}^{1/2}, \tag{8}$$

and

$$|u - u_q|_{1,\Omega} \leq \; C_M N^{-1} \cdot \Big\{ \; c \int_\Omega \|F^{-1}\|^2 \|F^T Q_3 F\|^3 \Big\}^{1/2}. \tag{9}$$

where F and C_M are determined by M as in (4) and (6).

Remark 2.1. *The above estimates hold for any quadratic interpolation as long as all the polynomials of degree ≤ 2 are invariant under the operation. In addition, by the ellipticity of the model problem, the above error estimate in H^1-seminorm is also true for the error $u - u_N$ of the quadratic finite element approximation u_N of (1).*

Remark 2.2. *Let $\lambda_1 \geq \lambda_2$ be the eigenvalues of $Q_3(\boldsymbol{x})$, and $\mathbf{v}_1, \mathbf{v}_2$ its corresponding eigenvectors. Then roughly speaking, λ_1 represents the largest third order directional derivative of u at $\boldsymbol{x}$ (along $\mathbf{v}_1$ direction), and λ_2 approximately the smallest third order directional derivative at x (along $\mathbf{v}_2$ direction). If we define a mesh metric $M_{3,1,2}$ as*

$$M_{3,1,2} \;=\; c(\lambda_1/\lambda_2)^{1/6} \cdot Q_3 \tag{10}$$

and the anisotropic mesh is quasi-uniform under $M_{3,1,2}$, then the error bound (9) for H^1-seminorm for the quadratic interpolation is minimized, see Theorem 2.1 in [6]*. In this case, we have*

$$|u - u_q|_{1,\Omega} \leq \; cN^{-1} \cdot \Big\{ \int_\Omega |\lambda_1|^{1/2} |\lambda_2|^{1/6} \; \Big\}^{3/2}. \tag{11}$$

On the other hand, if quasi-uniform meshes are used, then we would have

$$|u - u_q|_{1,\Omega} \leq \; cN^{-1} \cdot \Big\{ \int_\Omega |\lambda_1|^2 \; \Big\}^{1/2};$$

while if we are only allowed using isotropic adaptive refinements, then the best isotropic mesh metric to minimize the error bound in (9) would be $M = \lambda_1 I$, and the error bound for $\|u - u_N\|_{1,\Omega}$ becomes

$$|u - u_q|_{1,\Omega} \leq \; cN^{-1} \cdot \Big\{ \int_\Omega |\lambda_1|^{2/3} \; \Big\}^{3/2}.$$

Therefore, the improvements in error bounds brought by the anisotropic adaptivity is clearly manifested.

3. Supercloseness of FE solution

In this section we establish the supercloseness of the quadratic finite element solution to (1) to the quadratic interpolation of the exact solution. Since superconvergence is often the result of cancellation of certain lower order terms in the discretization, which relies on the local symmetry of the partition $\mathcal{T}_N$, we first recall the notion of $O(N^{-(1+\alpha)/2})$-approximate parallelograms introduced in [7].

Approximate parallelograms: Let the standard element $\hat{\tau}$ be an equilateral triangle with vertices on the unit circle. Let τ and τ' be a pair of elements sharing a common edge, and let J_τ and $J_{\tau'}$ be, respectively, the Jacobians of the affine mappings from $\hat{\tau}$ to τ and τ' that map a vertex of $\hat{\tau}$ to the opposite vertices of $\tau \cup \tau'$, cf. Figure 1. We call $\tau \cup \tau'$ forming an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, if

$$\|I + J_\tau^{-1} J_{\tau'}\| = O(N^{-\alpha/2}). \tag{12}$$

It can be shown that the above definition is independent of which of the two elements is taken as τ or τ'.

Note in the case that the affine mappings do not map the same vertex of $\hat{\tau}$ to the opposite vertices of $\tau \cup \tau'$, e.g., mapping $\mathcal{F}_\tau$ maps vertex $\hat{1}$ to vertex i in τ, but $\mathcal{F}_{\tau'}$ maps vertex $\hat{1}$ in $\hat{\tau}$ to vertex $i'-1$ in τ'. In this case, mapping $\mathcal{F}_{\tau'} \circ R_{120}$ maps vertex $\hat{1}$ into i' in τ', where R_{120} is the matrix for rotation by $120°$ counter-clock-wise, and condition (12) should be expressed as

$$\|I + J_\tau^{-1}(J_{\tau'} R_{120})\| = O(N^{-\alpha/2}). \tag{13}$$

When the partition $\{\mathcal{T}_N\}$ is quasi-uniform, the diameters of all elements in $\mathcal{T}_N$ are about $h = O(N^{-1/2})$, and the above definition is equivalent to the $O(h^{1+\alpha})$-approximate parallelograms for "mildly structured" and unstructured quasi-uniform meshes introduced in [3, 15, 24]. Indeed, anisotropic elements τ_1 and τ_2 form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, if they form an shape regular $O(N^{-(1+\alpha)/2})$ -approximate parallelogram under an affine mapping, see Lemma 2.2 in [7]. Also, if the two elements form an approximate parallelogram, then the ratio of their areas deviates from 1 by at most $O(N^{-\alpha/2})$, i.e.,

$$\frac{|\tau|}{|\tau'|} = 1 + O(N^{-\alpha/2}). \tag{14}$$

Now we introduce some basic notations and formulas about the piecewise linear, quadratic and cubic interpolations of a continuous function. Let $\tau = \Delta \boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3$ be an element in the partition. For $i = 1, 2, 3$, denote by $\mathbf{e}_i = \boldsymbol{x}_{i-1} - \boldsymbol{x}_{i+1}$. Define $\ell_i = |\mathbf{e}_i|$ be the length of edge $\mathbf{e}_i$, $\mathbf{t}_i = \mathbf{e}_i/\ell_i$ the unit tangent direction along $\mathbf{e}_i$, and $\mathbf{n}_i$ the unit outward normal on edge $\mathbf{e}_i$. See Figure 1. In addition, we define

$$\xi_i = \mathbf{n}_{i+1} \cdot A\mathbf{n}_{i-1}, \qquad \eta_i = \mathbf{n}_i \cdot A\mathbf{n}_i,$$

where A is the coefficient matrix in our model problem (1). All the indices regarding vertices and edges are considered congruent if they are equal modular 3. For simplifying notations, we write $\partial_{\mathbf{t}_i} = \frac{\partial}{\partial \mathbf{t}_i}$, $\partial_{\mathbf{t}_i^2} = \frac{\partial^2}{\partial \mathbf{t}_i^2}$, $\partial_{\mathbf{t}_{123}} = \frac{\partial^3}{\partial \mathbf{t}_1 \partial \mathbf{t}_2 \partial \mathbf{t}_3}$. We also denote by $\mathbf{P}_k$ the set of polynomials of degree $\leq k$.

For any element τ in the partition, we denote by ϕ_i the linear nodal basis function associated with vertex $\boldsymbol{x}_i$. It is elementary to verify that

$$\begin{cases} \nabla \phi_i = -\frac{\ell_i}{2|\tau|} \mathbf{n}_i; \\ \partial_{\mathbf{t}_i} \phi_i = 0, \quad \partial_{\mathbf{t}_i} \phi_{i+1} = -\frac{1}{\ell_i}, \quad \partial_{\mathbf{t}_i} \phi_{i-1} = \frac{1}{\ell_i}; \end{cases} \tag{15}$$

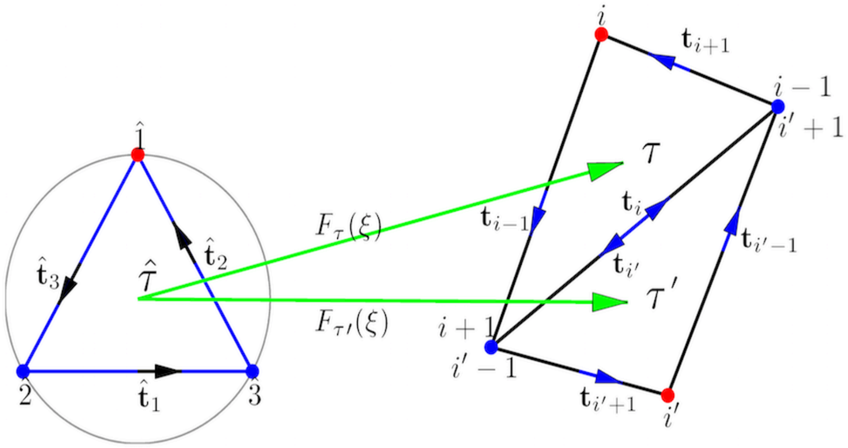

FIGURE 1. Standard element $\hat{\tau}$ with 3 vertices on the unit circle and a pair of adjacent elements τ and τ'.

Let u be a continuous function on Ω. We define $\Pi_\ell u|_\tau$ be the linear interpolation of $u|_\tau$ at the vertices of τ. Furthermore, we denote by $u_\ell : \Omega \to \mathbf{R}$, the piecewise linear interpolation of u over triangulation $\mathcal{T}_N$, i.e., $u_\ell|_\tau = \Pi_\ell u|_\tau$ for each $\tau \in \mathcal{T}_N$.

Define u_q be the piecewise quadratic projection-interpolation of u over $\mathcal{T}_N$ as follows: on each $\tau \in \mathcal{T}_N$,

$$u_q|_\tau = \Pi_q u|_\tau = \Pi_\ell u|_\tau + \sum_{i=1}^{3} \alpha_i \phi_{i+1}\phi_{i-1}, \tag{16}$$

where constants $\alpha_i, 1 \le i \le 3$, are determined by the following condition:

$$\int_{e_i} (u - u_q) = 0, \qquad \forall i = 1, 2, 3. \tag{17}$$

Since $\Pi_q u$ on an element edge is determined uniquely by u on the edge, thus $\Pi_q u$ is continuous over Ω. Furthermore, if $u \in \mathbf{P}_2$, then we have by Lemma 2.2 in [13] that

$$\alpha_i = -\frac{\ell_i^2}{2} \partial_{\mathbf{t}_i^2} u. \tag{18}$$

Define u_c be the piecewise cubic projection-interpolation of u over $\mathcal{T}_N$ as follows: on each $\tau \in \mathcal{T}_N$,

$$u_c|_\tau = \Pi_c\, u|_\tau = \Pi_q u|_\tau + \sum_{i=0}^{3} \beta_i \psi_i, \tag{19}$$

where

$$\psi_0 = \phi_1\phi_2\phi_3,$$

and

$$\psi_i = \phi_{i+1}\phi_{i-1}(\phi_{i+1} - \phi_{i-1}), \quad 1 \le i \le 3.$$

The constants β_i, $0 \le i \le 3$, are determined by

$$\begin{cases} \text{(i):} & \int_\tau (u - u_c) = 0; \\ \text{(ii):} & \int_{e_i} (u - u_c) p_1 = 0, \qquad \forall p_1 \in \mathbf{P}_1, \qquad i = 1, 2, 3. \end{cases} \tag{20}$$

Since $\Pi_c u$ on an element edge is determined uniquely by u on the edge, thus u_c is continuous over Ω. Furthermore, if $u \in \mathbf{P}_3$, then we have from Lemma 2.3 in [13] that

$$\left\{ \begin{array}{l} \beta_0 = \frac{1}{4}\ell_1\ell_2\ell_3\partial_{\mathbf{t}_{123}}u + \frac{1}{6}\sum_{k=1}^{3}\ell_k^2\ell_{k+1}\partial_{\mathbf{t}_k^2\mathbf{t}_{k+1}}u; \\ \beta_i = \frac{1}{12}\ell_i^3\partial_{\mathbf{t}_i^3}u, \qquad 1 \le i \le 3. \end{array} \right. \tag{21}$$

Lemma 3.1. *For any $v \in \mathbf{P}_2$, we have*

$$\nabla\cdot(A\nabla v) = -\frac{1}{4|\tau|^2}\sum_{i=1}^{3}\ell_i^2\ell_{i+1}\ell_{i-1}\xi_i\cdot\partial_{\mathbf{t}_i^2}v. \tag{22}$$

Proof: Since $v \in \mathbf{P}_2$ and A is a constant matrix, we have

$$\begin{aligned} \nabla\cdot(A\nabla v) &= \sum_{i=1}^{3}\ \alpha_i\ \nabla\cdot A\nabla(\phi_{i+1}\phi_{i-1}) \\ &= \sum_{i=1}^{3}\ 2\,\alpha_i\cdot\nabla\phi_{i+1}\cdot A\nabla\phi_{i-1}. \end{aligned}$$

Then (22) follows directly from applying (15) and (18) to the right hand side of the above equation. ∎

Lemma 3.2. *For any $v \in \mathbf{P}_2$, we have for $i = 1,2,3$, that*

$$\begin{aligned} \partial_{\mathbf{t}_i}(\mathbf{n}_i\cdot A\nabla v) = \ & \frac{1}{4|\tau|}\ \{\ \ell_i(\ell_{i+1}\xi_{i-1} - \ell_{i-1}\xi_{i+1})\partial_{\mathbf{t}_i^2}v \\ & +\eta_i(\ell_{i+1}^2\partial_{\mathbf{t}_{i+1}^2}v - \ell_{i-1}^2\partial_{\mathbf{t}_{i-1}^2}v)\ \}. \end{aligned} \tag{23}$$

Proof: Since $v \in \mathbf{P}_2$ and A is constant, we have

$$\begin{aligned} \partial_{\mathbf{t}_i}(\mathbf{n}_i\cdot A\nabla v) &= \sum_{k=1}^{3}\alpha_k\partial_{\mathbf{t}_i}[\mathbf{n}_i\cdot A(\phi_{k+1}\nabla\phi_{k-1})] \\ &= \sum_{k=1}^{3}\alpha_k\mathbf{n}_i\cdot A(\nabla\phi_{k+1}\partial_{\mathbf{t}_i}\phi_{k-1} + \nabla\phi_{k-1}\partial_{\mathbf{t}_i}\phi_{k+1}). \end{aligned} \tag{24}$$

By using (15), we have for $k = i$,

$$\begin{aligned} & \mathbf{n}_i\cdot A(\nabla\phi_{i+1}\partial_{\mathbf{t}_i}\phi_{i-1} + \nabla\phi_{i-1}\partial_{\mathbf{t}_i}\phi_{i+1}) \\ = \ & \frac{1}{2|\tau|\,\ell_i}\ [-\ell_{i+1}\,\mathbf{n}_i\cdot A\mathbf{n}_{i+1} + \ell_{i-1}\,\mathbf{n}_i\cdot A\mathbf{n}_{i-1}] \\ = \ & \frac{1}{2|\tau|\,\ell_i}\ [-\ell_{i+1}\xi_{i-1} + \ell_{i-1}\xi_{i+1}]; \end{aligned}$$

for $k = i+1$,

$$\mathbf{n}_i\cdot A(\nabla\phi_{i+2}\partial_{\mathbf{t}_i}\phi_i + \nabla\phi_i\partial_{\mathbf{t}_i}\phi_{i+2}) = -\frac{1}{2|\tau|}\mathbf{n}_i\cdot A\mathbf{n}_i = -\frac{1}{2|\tau|}\eta_i;$$

and for $k = i-1$,

$$\mathbf{n}_i\cdot A(\nabla\phi_i\partial_{\mathbf{t}_i}\phi_{i-2} + \nabla\phi_{i-2}\partial_{\mathbf{t}_i}\phi_i) = \frac{1}{2|\tau|}\mathbf{n}_i\cdot A\mathbf{n}_i = \frac{1}{2|\tau|}\eta_i;$$

Hence (23) follows readily from the application of (18) and the above three equalities to (24). ∎

Lemma 3.3. *For any $u \in \mathbf{P}_3$, let $u_q = \Pi_q u$ be the piecewise quadratic interpolation of u defined in (16). Then for any $v \in \mathbf{P}_2$,*

$$\int_{\partial\tau}(u - u_q)(\mathbf{n}\cdot A\nabla v) = -\frac{1}{720}\ \sum_{i=1}^{3}\ \ell_i^5\ \partial_{\mathbf{t}_i^3}u\ \partial_{\mathbf{t}_i}(\mathbf{n}_i\cdot A\nabla v). \tag{25}$$

Proof: Clearly,

$$\int_{\partial\tau}(u-u_q)(\mathbf{n}\cdot A\nabla v)=\sum_{i=1}^{3}\int_{\mathbf{e}_i}(u-u_q)(\mathbf{n}\cdot A\nabla v).$$

Note that for $u\in\mathbf{P}_3$, we have $u-u_q=\beta_i\psi_i$ on edge $\mathbf{e}_i$ with β_i given in (21). In addition,

$$\psi_i=\phi_{i+1}\phi_{i-1}(\phi_{i+1}-\phi_{i-1})=\frac{\ell_i}{2}\partial_{\mathbf{t}_i}(\phi_{i+1}^2\phi_{i-1}^2).$$

Thus, by using integration by parts,

$$\begin{aligned}\textstyle\int_{\mathbf{e}_i}(u-u_q)(\mathbf{n}\cdot A\nabla v) &= \textstyle\int_{\mathbf{e}_i}\beta_i\psi_i(\mathbf{n}\cdot A\nabla v)\\ &= \tfrac{1}{2}\,\ell_i\,\beta_i\textstyle\int_{\mathbf{e}_i}\partial_{\mathbf{t}_i}(\phi_{i+1}^2\phi_{i-1}^2)(\mathbf{n}\cdot A\nabla v)\\ &= -\tfrac{1}{2}\,\ell_i\,\beta_i\textstyle\int_{\mathbf{e}_i}(\phi_{i+1}^2\phi_{i-1}^2)\partial_{\mathbf{t}_i}(\mathbf{n}\cdot A\nabla v).\end{aligned}$$

By the facts that $\partial_{\mathbf{t}_i}(\mathbf{n}\cdot A\nabla v)$ is a constant since $v\in\mathbf{P}_2$, and that

$$\int_{\mathbf{e}_i}(\phi_{i+1}^2\phi_{i-1}^2)=\frac{\ell_i}{30},$$

we have

$$\int_{\mathbf{e}_i}(u-u_q)(\mathbf{n}\cdot A\nabla v)=-\frac{1}{60}\,\ell_i^2\beta_i\,\partial_{\mathbf{t}_i}(\mathbf{n}_i\cdot A\nabla v).$$

Finally, (25) follows from inserting β_i described by (21) into the above equation. ∎

Lemma 3.4. *For any $u\in\mathbf{P}_3$, let $u_q=\Pi_q u$ be the piecewise quadratic interpolation of u defined in (16). Then for any $v\in\mathbf{P}_2$,*

$$\begin{aligned}&\textstyle\int_\tau A\nabla(u-u_q)\cdot\nabla v\\ =\;&\tfrac{1}{2880|\tau|}\;\sum_{i=1}^{3}\;\partial_{\mathbf{t}_i^2}v\cdot\ell_i^2\;\{\ell_{i+1}\ell_{i-1}\xi_i\,[\,3\,\ell_1\ell_2\ell_3\partial_{\mathbf{t}_{123}}u\\ &+2\sum_{k=1}^{3}\;\ell_k^2\ell_{k+1}\partial_{\mathbf{t}_k^2\mathbf{t}_{k+1}}u\,]-\ell_i^4(\ell_{i+1}\xi_{i-1}-\ell_{i-1}\xi_{i+1})\partial_{\mathbf{t}_i^3}u\\ &+(\ell_{i-1}^5\eta_{i-1}\partial_{\mathbf{t}_{i-1}^3}u-\ell_{i+1}^5\eta_{i+1}\partial_{\mathbf{t}_{i+1}^3}u)\;\}.\end{aligned}\tag{26}$$

Proof: Using integration by parts, we have

$$\begin{aligned}&\textstyle\int_\tau A\nabla(u-u_q)\cdot\nabla v\\ =\;&-\textstyle\int_\tau(u-u_q)\nabla\cdot(A\nabla v)+\int_{\partial\tau}(u-u_q)(\mathbf{n}\cdot A\nabla v)\\ =\;&-\nabla\cdot(A\nabla v)\textstyle\int_\tau(u-u_q)+\int_{\partial\tau}(u-u_q)(\mathbf{n}\cdot A\nabla v).\end{aligned}\tag{27}$$

Since $u\in\mathbf{P}_3$, we have from (19) that

$$\int_\tau(u-u_q)=\sum_{i=0}^{4}\int_\tau\beta_i\psi_i=\beta_0\int_\tau\psi_0=\frac{|\tau|}{60}\cdot\beta_0,$$

where we have used the facts that $\int_\tau\psi_0=\frac{|\tau|}{60}$, and that $\int_\tau\psi_i=0$ for all $i=1,2,3$.

Putting β_0 described in (21) into the right hand side of (27) and using Lemma 3.1, we have

$$\begin{aligned}&\textstyle\int_\tau A\nabla(u-u_q)\cdot\nabla v\\ =\;&\tfrac{1}{2880|\tau|}\;(\sum_{i=1}^{3}\ell_i^2\ell_{i+1}\ell_{i-1}\xi_i\;\partial_{\mathbf{t}_i^2}v)\cdot[\,3\,\ell_1\ell_2\ell_3\partial_{\mathbf{t}_{123}}u+2\sum_{k=1}^{3}\ell_k^2\ell_{k+1}\partial_{\mathbf{t}_k^2\mathbf{t}_{k+1}}u]\\ &-\tfrac{1}{2880|\tau|}\;\sum_{i=1}^{3}\;\ell_i^5\,\partial_{\mathbf{t}_i^3}u\,\{\ell_i(\ell_{i+1}\xi_{i-1}-\ell_{i-1}\xi_{i+1})\partial_{\mathbf{t}_i^2}v\\ &+\eta_i(\ell_{i+1}^2\partial_{\mathbf{t}_{i+1}^2}v-\ell_{i-1}^2\partial_{\mathbf{t}_{i-1}^2}v)\},\end{aligned}$$

which leads to (26) after regrouping the terms. □

Next we state two lemmas about the partial cancellation of directional derivatives on a pair of adjacent elements τ and τ' when they form an approximate parallelogram. First we are concerned with this cancellation between two $\xi_i = \mathbf{n}_{i+1} \cdot A\mathbf{n}_{i-1}$ and between two $\eta_i = \mathbf{n}_i \cdot A\mathbf{n}_i$ after affine transforms. Let $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ be the affine mappings from $\hat{\tau}$ to τ and τ', resp.. J_τ and $J_{\tau'}$ are their Jacobians. For any function u, let $\hat{u}_\tau = u \circ \mathcal{F}_\tau$ and $\hat{u}_{\tau'} = u \circ \mathcal{F}_{\tau'}$. Note that $\hat{\nabla}\hat{u}_\tau = J_\tau \nabla u$ on τ and $\hat{\nabla}\hat{u}_{\tau'} = J_{\tau'}\nabla u$ on τ'. Let

$$\hat{A}_\tau = J_\tau^{-1} A J_\tau^{-T}, \qquad \hat{A}_{\tau'} = J_{\tau'}^{-1} A J_{\tau'}^{-T}.$$

Also, let i and i' be the indices for the opposite vertices in $\tau \cup \tau'$. Define

$$\hat{\xi}_i = \hat{\mathbf{n}}_{i+1} \cdot \hat{A}_\tau \hat{\mathbf{n}}_{i-1}, \qquad \hat{\xi}_{i'} = \hat{\mathbf{n}}_{i'+1} \hat{A}_{\tau'} \hat{\mathbf{n}}_{i'-1},$$

and

$$\hat{\eta}_i = \hat{\mathbf{n}}_i \cdot \hat{A}_\tau \hat{\mathbf{n}}_i, \qquad \hat{\eta}_{i'} = \hat{\mathbf{n}}_{i'} \hat{A}_{\tau'} \hat{\mathbf{n}}_{i'}.$$

Lemma 3.5. *Suppose τ and τ' are a pair of adjacent elements in the partition, and they form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Then for any $1 \le i \le 3$, we have*

$$|\hat{\xi}_i - \hat{\xi}_{i'}| + |\hat{\eta}_i - \hat{\eta}_{i'}| \le cN^{-\alpha/2}(\|J_\tau^{-1}\|^2 + \|J_{\tau'}^{-1}\|^2). \tag{28}$$

Proof: First we assume that $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ map the same vertex of $\hat{\tau}$ into opposite vertices in $\tau \cup \tau'$. In this case, we have $F_\tau^{-1}\mathbf{t}_i = F_{\tau'}^{-1}\mathbf{t}_i'$ and $\hat{\mathbf{n}}_i = \hat{\mathbf{n}}_i'$. Therefore

$$\begin{aligned}
|\hat{\xi}_i - \hat{\xi}_{i'}| &= |\hat{\mathbf{n}}_{i+1} \cdot \hat{A}_\tau \hat{\mathbf{n}}_{i-1} - \hat{\mathbf{n}}_{i'+1} \cdot \hat{A}_{\tau'} \hat{\mathbf{n}}_{i'-1}| \\
&\le |\hat{\mathbf{n}}_{i+1}| \cdot \|\hat{A}_\tau - \hat{A}_{\tau'}\| \cdot |\hat{\mathbf{n}}_{i-1}| \\
&= \|J_\tau^{-1} A J_\tau^{-T} - J_{\tau'}^{-1} A J_{\tau'}^{-T}\| \\
&\le \|(J_\tau^{-1} + J_{\tau'}^{-1}) A J_\tau^{-T}\| + \|J_{\tau'}^{-1} A (J_\tau^{-T} + J_{\tau'}^{-T})\| \\
&\le \|I + J_\tau^{-1} J_{\tau'}\| \cdot \|J_{\tau'}^{-1} A J_\tau^{-T}\| + \|J_{\tau'}^{-1} A J_{\tau'}^{-T}\| \cdot \|I + J_\tau^{-1} J_{\tau'}\| \\
&\le cN^{-\alpha/2}(\|J_\tau^{-1}\|^2 + \|J_{\tau'}^{-1}\|^2).
\end{aligned}$$

The estimate for $\hat{\eta}_i - \hat{\eta}_{i'}$ is similar.

In the cases where $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ do not map the same vertex of $\hat{\tau}$ into opposite vertices in $\tau \cup \tau'$. For instance, mapping $\mathcal{F}_{\tau'}$ maps vertex $\hat{1}$ to vertex i' in τ', but mapping $\mathcal{F}_\tau$ maps vertex $\hat{1}$ to vertex $i-1$ in τ. In this case, $\hat{\mathbf{n}}_i = R_{120}\hat{\mathbf{n}}_{i'}$, and mapping $\tilde{\mathcal{F}}_\tau = \mathcal{F}_\tau \circ R_{120}$ maps vertex $\hat{1}$ into i in τ, cf. Figure 1. Therefore,

$$\begin{aligned}
\hat{\xi}_i &= \hat{\mathbf{n}}_{i+1} \cdot \hat{A}_\tau \hat{\mathbf{n}}_{i-1} \\
&= R_{120}\hat{\mathbf{n}}_{i'+1} \cdot \hat{A}_\tau \; R_{120}\hat{\mathbf{n}}_{i'-1} \\
&= \hat{\mathbf{n}}_{i'+1} \cdot R_{120}^T \hat{A}_\tau \; R_{120}\hat{\mathbf{n}}_{i'-1} \\
&= \hat{\mathbf{n}}_{i'+1} \cdot R_{120}^T J_\tau^{-1} A J_\tau^{-T} R_{120} \; \hat{\mathbf{n}}_{i'-1} \\
&= \hat{\mathbf{n}}_{i'+1} \cdot (J_\tau R_{120})^{-1} A (J_\tau R_{120})^{-T} \; \hat{\mathbf{n}}_{i'-1}
\end{aligned}$$

Then we may proceed in the same way as for the first case, and prove the estimates for $|\hat{\xi}_i - \hat{\xi}_{i'}|$ by noting (13). □

Lemma 3.6. *Suppose τ and τ' are a pair of adjacent elements in the partition, and they form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ are the affine mappings from standard element $\hat{\tau}$ to τ and τ', resp.. Let $\hat{u}_\tau = u|_\tau \circ \mathcal{F}_\tau$ and*

$\hat{u}_{\tau'} = u|_{\tau'} \circ \mathcal{F}_{\tau'}$. *Then we have for any* $1 \le i, j, k \le 3$ *that*

$$
\begin{aligned}
&\left|\int_{\hat{\tau}} \left(\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'}\right) d\hat{\tau}\ \right| \\
(29) \qquad \le\ & cN^{-\alpha/2}\left\{\tfrac{1}{|\tau|}\int_\tau \left\|J_\tau^T Q_3 J_\tau\right\|^3 + \tfrac{1}{|\tau'|}\int_{\tau'} \left\|J_{\tau'}^T Q_3 J_{\tau'}\right\|^3\right\}^{1/2} \\
&+ c\left\{\tfrac{1}{|\tau|}\int_\tau \left\|J_\tau^T Q_4 J_\tau\right\|^4 + \tfrac{1}{|\tau'|}\int_{\tau'} \left\|J_{\tau'}^T Q_4 J_{\tau'}\right\|^4\right\}^{1/2},
\end{aligned}
$$

where i', j', k' *are the indices for the vertices opposite to* i, j, k *in* $\tau \cup \tau'$*, while* J_τ *and* $J_{\tau'}$ *are the Jacobians of* $\mathcal{F}_\tau$ *and* $\mathcal{F}_{\tau'}$*, resp..* Q_3 *and* Q_4 *are the anisotropic measures of* D^3u *and* D^4u *defined in (7).*

Proof: First we assume that the affine mappings $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ map the same vertex on $\hat{\tau}$ into opposite vertices on $\tau \cup \tau'$, otherwise, either $\mathcal{F}_\tau$ or $\mathcal{F}_{\tau'}$ is the composition of such a mapping with a $\pm 120°$ rotation, and the estimates can be dealt with similarly. Furthermore, for simplifying notations, we assume both $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ map $\hat{\mathbf{e}}_1$ to $\tau \cap \tau' = \mathbf{e}_i = -\mathbf{e}_{i'}$.

To begin with, note that u_τ and $u_{\tau'}$ only share the common values on edge $\mathbf{e}_i$, we define the averages of $\hat{u}_\tau$ and $\hat{u}_{\tau'}$ on $\hat{\mathbf{e}}_1$ as follows:

$$
\begin{aligned}
\mathcal{G}(\hat{u}_\tau) &= \tfrac{1}{\hat{\tau}} \cdot \tfrac{1}{|\hat{\mathbf{e}}_1|} \cdot \int_{\hat{\mathbf{e}}_1} \partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau = \tfrac{4}{9}\int_{\hat{\mathbf{e}}_1} \partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau, \\
\mathcal{G}(\hat{u}_{\tau'}) &= \tfrac{1}{\hat{\tau}} \cdot \tfrac{1}{|\hat{\mathbf{e}}_1|} \cdot \int_{\hat{\mathbf{e}}_1} \partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'} = \tfrac{4}{9}\int_{\hat{\mathbf{e}}_1} \partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|\int_{\hat{\tau}} \left(\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'}\right) d\hat{\tau}\ \right| \\
(30) \qquad \le\ & \left|\int_{\hat{\tau}} \left(\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau - \mathcal{G}(\hat{u}_\tau)\ \right)\right| + \left|\int_{\hat{\tau}'} \left(\partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'} - \mathcal{G}(\hat{u}_{\tau'})\ \right)\right| \\
&+ \left|\mathcal{G}(\hat{u}_\tau) + \mathcal{G}(\hat{u}_{\tau'})\right|.
\end{aligned}
$$

We first estimate the first term on the right hand side of the above inequality. Note that $\left|\int_{\hat{\tau}} (\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau - \mathcal{G}(\hat{u}_\tau)\ )\right|$ is invariant if $\hat{u}_\tau$ is replaced by $\hat{u}_\tau + p_3$ for any $p_3 \in \mathbf{P}_3$, and that $H^4(\hat{\tau}) \hookrightarrow W^{3,1}(\hat{\tau})$, we have

$$
\begin{aligned}
\left|\int_{\hat{\tau}} (\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau - \mathcal{G}(\hat{u}_\tau)\ )\right| &\le c \inf_{p_3} \|\hat{u}_\tau + p_3\|_{H^4(\hat{\tau})} \\
&\le c\, |\hat{u}_\tau|_{H^4(\hat{\tau})} \\
(31) \qquad &\le c \left\{\int_{\hat{\tau}} |\hat{D}^4 \hat{\tau}|^2 d\hat{\tau}\right\}^{1/2} \\
&\le c \left\{\int_{\hat{\tau}} \left\|J_\tau^T Q_4 J_\tau\right\|^4 d\hat{\tau}\right\}^{1/2} \\
&\le c\, |\tau|^{-1/2} \left\{\int_\tau \left\|J_\tau^T Q_4 J_\tau\right\|^4 d\tau\right\}^{1/2}.
\end{aligned}
$$

The second term on the right hand side of (30) can be estimated similarly.

To bound the last term in (30), we consider first the case $i = 1, j = 2, k = 3$, and thus $\partial_{\hat{\mathbf{t}}_{ijk}} = \partial_{\hat{\mathbf{t}}_{123}}$. Since $D^3 u_\tau = D^3 u_{\tau'}$ on $\tau \cap \tau'$, we have on $\hat{\mathbf{e}}_1$ that

$$
\partial_{\hat{\mathbf{t}}_{ijk}} \hat{u}_\tau = \partial_{\hat{\mathbf{t}}_{123}} \hat{u}_\tau = (\hat{\mathbf{t}}_1 \cdot \hat{\nabla})(\hat{\mathbf{t}}_2 \cdot \hat{\nabla})(\hat{\mathbf{t}}_3 \cdot \hat{\nabla}) \hat{u}_\tau = (J_\tau \hat{\mathbf{t}}_1 \cdot \nabla)(J_\tau \hat{\mathbf{t}}_2 \cdot \nabla)(J_\tau \hat{\mathbf{t}}_3 \cdot \nabla) u,
$$

and similarly

$$
\partial_{\hat{\mathbf{t}}_{i'j'k'}} \hat{u}_{\tau'} = \partial_{\hat{\mathbf{t}}_{123}} \hat{u}_{\tau'} = (J_{\tau'} \hat{\mathbf{t}}_1 \cdot \nabla)(J_{\tau'} \hat{\mathbf{t}}_2 \cdot \nabla)(J_{\tau'} \hat{\mathbf{t}}_3 \cdot \nabla) u.
$$

Recall that $\hat{\nabla} \hat{u}_\tau = J_\tau^T \nabla u_\tau$ and $\hat{\nabla} \hat{u}_{\tau'} = J_{\tau'}^T \nabla u_{\tau'}$, and use the fact that

$$
J_\tau \hat{\mathbf{t}}_1 + J_{\tau'} \hat{\mathbf{t}}_1 = \frac{1}{\sqrt{3}}(\ \mathbf{e}_i + \mathbf{e}_{i'}\ ) = \mathbf{0},
$$

we have on $\hat{\mathbf{e}}_1$ that

$$
\begin{array}{rl}
 & \left|\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}\right| \\
= & \left|(J_\tau\hat{\mathbf{t}}_1\cdot\nabla)\ [\ (J_\tau\hat{\mathbf{t}}_2\cdot\nabla)(J_\tau\hat{\mathbf{t}}_3\cdot\nabla)u - (J_{\tau'}\hat{\mathbf{t}}_2\cdot\nabla)(J_{\tau'}\hat{\mathbf{t}}_3\cdot\nabla)u\ ]\right| \\
\leq & \left|(J_\tau\hat{\mathbf{t}}_1\cdot\nabla)(J_\tau\hat{\mathbf{t}}_2\cdot\nabla)[(J_\tau + J_{\tau'})\hat{\mathbf{t}}_3\cdot\nabla]u\ \right| \\
 & +\left|(J_\tau\hat{\mathbf{t}}_1\cdot\nabla)[(J_\tau + J_{\tau'})\hat{\mathbf{t}}_2\cdot\nabla)](J_{\tau'}\hat{\mathbf{t}}_3\cdot\nabla)u\ ]\right| \\
\leq & \left|(\hat{\mathbf{t}}_1\cdot\hat{\nabla})(\hat{\mathbf{t}}_2\cdot\hat{\nabla})[(I + J_\tau^{-1}J_{\tau'})\hat{\mathbf{t}}_3\cdot\hat{\nabla}]\hat{u}_\tau\ \right| \\
 & +\left|(\hat{\mathbf{t}}_1\cdot\hat{\nabla})((I + J_{\tau'}^{-1}J_\tau)\hat{\mathbf{t}}_2\cdot\hat{\nabla})(\hat{\mathbf{t}}_3\cdot\hat{\nabla})\hat{u}_{\tau'}\ \right| \\
\leq & |\hat{\mathbf{t}}_1|\ |\hat{\mathbf{t}}_2|\ |(I + J_\tau^{-1}J_{\tau'})\hat{\mathbf{t}}_3|\cdot|\hat{D}^3\hat{u}_\tau| + |\hat{\mathbf{t}}_1|\ |(I + J_{\tau'}^{-1}J_\tau)\hat{\mathbf{t}}_2|\ |\hat{\mathbf{t}}_3|\cdot|\hat{D}^3\hat{u}_{\tau'}| \\
\leq & cN^{-\alpha/2}(|\hat{D}^3\hat{u}_\tau| + |\hat{D}^3\hat{u}_{\tau'}|).
\end{array}
\tag{32}
$$

Hence we have from the embedding $H^1(\hat{\tau}) \hookrightarrow L^1(\hat{\mathbf{e}}_1)$ that

$$
\begin{array}{rl}
 & \left|\mathcal{G}(\hat{u}_\tau) + \mathcal{G}(\hat{u}_{\tau'})\right| \\
\leq & \frac{4}{9}\int_{\hat{\mathbf{e}}_1}|\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}| \\
\leq & cN^{-\alpha/2}\cdot\int_{\hat{\mathbf{e}}_1}(|\hat{D}^3\hat{u}_\tau| + |\hat{D}^3\hat{u}_{\tau'}|) \\
\leq & cN^{-\alpha/2}\cdot\{\int_{\hat{\tau}}(|\hat{D}^3\hat{u}_\tau|^2 + |\hat{D}^4\hat{u}_\tau|^2 + |\hat{D}^3\hat{u}_{\tau'}|^2 + |\hat{D}^4\hat{u}_{\tau'}|^2)d\hat{\tau}\}^{1/2} \\
\leq & cN^{-\alpha/2}\cdot\{\int_{\hat{\tau}}(\|J_\tau^TQ_3J_\tau\|^3 + \|J_\tau^TQ_4J_\tau\|^4 \\
 & +\|J_{\tau'}^TQ_3J_{\tau'}\|^3 + \|J_{\tau'}^TQ_4J_{\tau'}\|^4)d\hat{\tau}\}^{1/2} \\
\leq & cN^{-\alpha/2}\cdot\{\frac{1}{|\tau|}\int_\tau(\|J_\tau^TQ_3J_\tau\|^3 + \|J_\tau^TQ_4J_\tau\|^4)d\tau \\
 & +\frac{1}{|\tau'|}\int_{\tau'}(\|J_{\tau'}^TQ_3J_{\tau'}\|^3 + \|J_{\tau'}^TQ_4J_{\tau'}\|^4)d\tau'\ \}^{1/2}
\end{array}
\tag{33}
$$

Putting estimates (31) and (33) into (30), we have the estimate (29).

For other cases of indices i, j, k, if one of them corresponds to $\hat{\mathbf{t}}_1$, then a proceedure similar to (32) and (33) also leads to estimate (29). If none of the indices corresponds to $\hat{\mathbf{t}}_1$, we only have to split $\partial_{\hat{\mathbf{t}}_{ijk}} + \partial_{\hat{\mathbf{t}}_{i'j'k'}}$ into $[\partial_{\hat{\mathbf{t}}_{ijk}} + \partial_{\hat{\mathbf{t}}_{i'jk}}] - [\partial_{\hat{\mathbf{t}}_{i'jk}} + \partial_{\hat{\mathbf{t}}_{i'j'k}}] + [\partial_{\hat{\mathbf{t}}_{i'j'k}} + \partial_{\hat{\mathbf{t}}_{i'j'k'}}]$, then proceed similarly as in (32) to reach (29). $\square$

Theorem 3.1. *Let u be the exact solution of model problem (1) and u_q its piecewise quadratic interpolation. Let u_N be the quadratic finite element solution of (1) based on partition $\mathcal{T}_N$, and $J : \Omega \to \mathbf{R}^{2\times2}$ is defined on each element τ by $J|_\tau = J_\tau$, the Jacobian of the affine mapping from standard element $\hat{\tau}$ to τ. Let Q_3 and Q_4 be the anisotropic measure of D^3u and D^4u satisfying (7), and assume each pair of adjacent elements in $\mathcal{T}_N$ forms an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Then we have*

$$
\|u_N - u_q\|_{1,\Omega} \ \leq\ c\left\{\ \int_\Omega\ [\ (1 + N^{-\alpha/2}\|J\|^2\|J^{-1}\|^4)\cdot\|J^TQ_3J\|^3 + \|J\|^2\|J^{-1}\|^4\cdot\|J^TQ_4J\|^4\ ]\ \right\}^{1/2}
\tag{34}
$$

Furthermore, if the partition $\mathcal{T}_N$ is quasi-uniform under metric M, then we have

$$
\|u_N - u_q\|_{1,\Omega} \leq C_MN^{-1}\Big\{c\int_\Omega[(C_MN^{-1} + N^{-\alpha}\|F\|^2\|F^{-1}\|^4)\cdot\|F^TQ_3F\|^3 + C_MN^{-1}\|F\|^2\|F^{-1}\|^4\cdot\|F^TQ_4F\|^4\ \Big\}^{1/2}.
\tag{35}
$$

where F and C_M are determined by metric M as in (4) and (6).

Proof: By the ellipticity of $a(\cdot,\cdot)$ and the orthogonality of the finite element solution u_N to S_N,

$$\|u_N - u_q\|_{1,\Omega} \ \le\ c \sup_{v\in S_N} \frac{|a(u_N-u_q,v)|}{\|v\|_{1,\Omega}} \ =\ c \sup_{v\in S_N} \frac{|a(u-u_q,v)|}{\|v\|_{1,\Omega}}.$$

To estimate $a(u-u_q,v)$, let u_c be the piecewise cubic interpolation of u defined by (19). We have for any $v \in S_N$ that

$$\begin{aligned}(A\nabla(u-u_c),\nabla v) &= \sum_\tau \int_\tau A\nabla(u-u_c)\cdot\nabla v\\ &= \sum_\tau \{-\int_\tau (u-u_c)\,\nabla\cdot(A\nabla v) \ +\ \int_{\partial\tau} (u-u_c)(\mathbf{n}\cdot A\nabla v)\ \}\\ &= 0,\end{aligned}$$

where we have used the fact that $\nabla\cdot(A\nabla v)$ is constant and $\mathbf{n}\cdot A\nabla v$ is linear on each element τ, together with condition (20) in the definition of u_c. Therefore,

$$a(u-u_q,v) \ =\ (A\nabla(u_c-u_q),\nabla v) + (u-u_q,\ \mathbf{b}\cdot\nabla v + d\ v). \tag{36}$$

For the lower order term on the right hand side of the above equation, we have from Theorem 2.1 that

$$\begin{aligned}|\ (u-u_q,\ \mathbf{b}\cdot\nabla v + d\ v)\ | &\le\ \ \|u-u_q\|\cdot\|v\|_{1,\Omega}\\ &\le\ \ c\left\{\ \int_\Omega \|J^TQ_3J\|^3\ \right\}^{1/2}\cdot\|v\|_{1,\Omega}.\end{aligned}\tag{37}$$

Now we focus on the estimate for $(A\nabla(u_c-u_q),\nabla v)$ in (36).

We transform all the integrals on physical elements τ to those on standard element $\hat\tau$. Denote by $\mathcal{F}_\tau$ the affine mapping from $\hat\tau$ to τ, and J_τ its Jacobian. Define $\hat u_q = u_q\circ F_\tau$, and $\hat u_c = u_c\circ F_\tau$. It is easy to see that [13]

$$\hat u_q = \hat\Pi_q\hat u, \qquad \text{and} \qquad \hat u_c = \hat\Pi_c\hat u.$$

Noting that for standard element $\hat\tau$, we have $|\hat\tau| = \frac{3\sqrt3}{4}$ and $\hat\ell_1 = \hat\ell_2 = \hat\ell_3 = \sqrt3$. Thus by applying Lemma 3.4 to integrals on $\hat\tau$, we have

$$\begin{aligned}(A\nabla(u_c-u_q),\nabla v) =&\ \ \sum_\tau \int_\tau \nabla(u_c-u_q)\cdot A\nabla v\\ =&\ \ \tfrac{1}{|\hat\tau|}\sum_\tau |\tau|\int_{\hat\tau}\hat\nabla(\hat u_c-\hat u_q)\cdot\hat A\nabla v\\ =&\ \ \tfrac{1}{80|\hat\tau|}\sum_\tau |\tau|\,\partial_{\hat{\mathbf{t}}_i^2}\hat v\ \{\ \hat\xi_i\,(\,3\,\partial_{\hat{\mathbf{t}}_{123}}\hat u_\tau + 2\sum_{k=1}^3 \partial_{\hat{\mathbf{t}}_k^2\hat{\mathbf{t}}_{k+1}}\hat u_\tau\,)\\ &+(\hat\xi_{i+1}-\hat\xi_{i-1})\partial_{\hat{\mathbf{t}}_i^3}\hat u_\tau\ -\ \hat\eta_{i-1}\partial_{\hat{\mathbf{t}}_{i-1}^3}\hat u_\tau\ +\ \hat\eta_{i+1}\partial_{\hat{\mathbf{t}}_{i+1}^3}\hat u_\tau\ \}\\ =&\ \ \tfrac{1}{80|\hat\tau|^2}\sum_\tau\{\ |\tau|\sum_{i=1}^3 B_i(\hat\Pi_c\hat u_\tau)\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v\ \},\end{aligned}\tag{38}$$

where

$$\begin{aligned}B_i(w) =\int_{\hat\tau}\Big[&\hat\xi_i\,(\,3\,\partial_{\hat{\mathbf{t}}_{123}}w + 2\sum_{k=1}^3\partial_{\hat{\mathbf{t}}_k^2\hat{\mathbf{t}}_{k+1}}w\,)\\ &+(\hat\xi_{i+1}-\hat\xi_{i-1})\partial_{\hat{\mathbf{t}}_i^3}w - \hat\eta_{i-1}\partial_{\hat{\mathbf{t}}_{i-1}^3}w + \hat\eta_{i+1}\partial_{\hat{\mathbf{t}}_{i+1}^3}w\ \Big]\,d\hat\tau.\end{aligned}$$

Regroupping the terms in the sum-up on the right hand side of (38) so that the two terms involving the same $\partial_{\hat{\mathbf{t}}_i^2}\hat{v}$ are combined, we have

$$\begin{array}{rl} & a(u_c-u_q,v) \\ = & \frac{1}{80|\hat\tau|^2}\sum\limits_{\mathbf{e}_i=\tau\cap\tau'}\left[\,|\tau|\,B_i(\hat\Pi_c\hat u_\tau)+|\tau'|\,B_{i'}(\hat\Pi_c\hat\tau')\,\right]\;\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v \\ = & \frac{1}{80|\hat\tau|^2}\sum\limits_{\mathbf{e}_i=\tau\cap\tau'}\{\;\left[\,|\tau|\,B_i(\hat\Pi_c\hat u_\tau-\hat u_\tau)+|\tau'|\,B_{i'}(\hat\Pi_c\hat\tau'-\hat u_{\tau'})\,\right]\;\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v \\ & +\left[\,|\tau|\,B_i(\hat u_\tau)+|\tau'|\,B_{i'}(\hat\tau')\,\right]\;\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v\;\}. \end{array} \tag{39}$$

First, we estimate $B_i(\hat\Pi_c\hat u_\tau-\hat u_\tau)$ in the first term on the right hand side of (39). Note that for any $1\le i,j,k\le 3$, we have $\partial_{\hat{\mathbf{t}}_{ijk}}(\hat\Pi_c\hat u_\tau-\hat u_\tau)$ is invariant if $\hat u_\tau$ is replaced by $\hat u_\tau+p_3$ for any $p_3\in\mathbf{P}_3$. Thus

$$\begin{array}{rl} \int_{\hat\tau}|\partial_{\hat{\mathbf{t}}_{ijk}}(\hat\Pi_c\hat u_\tau-\hat u_\tau)|d\hat\tau & \le\; c\inf\limits_{p_3}\|\hat u_\tau+p_3\|_{H^4(\hat\tau)}\le c|\hat u_\tau|_{H^4(\hat\tau)} \\ & \le\; c\left\{\int_{\hat\tau}|\hat D^4\hat u_\tau|^2d\hat\tau\right\}^{1/2} \\ & \le\; c\left\{\int_{\hat\tau}\|J_\tau^TQ_4J_\tau\|^4d\hat\tau\right\}^{1/2} \\ & =c\,|\tau|^{-1/2}\left\{\int_\tau\|J_\tau^TQ_4J_\tau\|^4d\tau\right\}^{1/2}. \end{array} \tag{40}$$

In addition,

$$|\hat\xi_i|=|\hat{\mathbf{n}}_{i+1}\cdot\hat A\hat{\mathbf{n}}_{i-1}|\le\|\hat A\|=\|J_\tau^{-1}AJ_\tau^{-T}\|\le c\,\|J_\tau^{-1}\|^2,$$

and

$$|\hat\eta_i|=|\hat{\mathbf{n}}_i\cdot\hat A\hat{\mathbf{n}}_i|\le c\,\|J_\tau^{-1}\|^2.$$

Hence we have

$$|B_i(\hat\Pi_c\hat u_\tau-\hat u_\tau)|\le c|\tau|^{-1/2}\|J_\tau^{-1}\|^2\{\int_\tau\|J_\tau^TQ_4J_\tau\|^4d\tau\}^{1/2}. \tag{41}$$

We may estimate $|B_{i'}(\hat\Pi_c\hat u_{\tau'}-\hat u_{\tau'})|$ in a similar way. In addition, since $v\in\mathbf{P}_2$, which is a finite dimensional subspace, we have

$$\begin{array}{rl} |\partial_{\hat{\mathbf{t}}_i^2}\hat v| & =|\hat\tau|^{-1/2}\{\int_{\hat\tau}|\partial_{\hat{\mathbf{t}}_i^2}\hat v|^2d\hat\tau\}^{1/2} \\ & \le\; c\;\{\int_{\hat\tau}|\hat D^2\hat v|^2d\hat\tau\}^{1/2} \\ & \le\; c\;\{\int_{\hat\tau}|\hat D\hat v|^2d\hat\tau\}^{1/2} \\ & \le c\;|\tau|^{-1/2}\{\int_\tau\|J_\tau\|^2\,|Dv|^2d\tau\}^{1/2}. \end{array} \tag{42}$$

Thus we have

$$\begin{array}{l} \frac{1}{80|\hat\tau|^2}\sum\limits_{\mathbf{e}_i=\tau\cap\tau'}\{\;\left[\,|\tau|\,B_i(\hat\Pi_c\hat u_\tau-\hat u_\tau)+|\tau'|\,B_{i'}(\hat\Pi_c\hat\tau'-\hat u_{\tau'})\,\right]\;\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v \\ \le c\sum\limits_\tau\{\int_\tau\|J_\tau^{-1}\|^4\|J_\tau^TQ_4J_\tau\|^4d\tau\}^{1/2}\cdot\{\int_\tau\|J_\tau\|^2\,|Dv|^2d\tau\}^{1/2} \\ \le c\,\{\sum\limits_\tau\int_\tau\|J_\tau\|^2J_\tau^{-1}\|^4\|J_\tau^TQ_4J_\tau\|^4d\tau\}^{1/2}\cdot\;\{\sum\limits_\tau\int_\tau|Dv|^2d\tau\}^{1/2} \\ \le c\{\int_\Omega\|J\|^2\|J^{-1}\|^4\|J^TQ_4J\|^4d\tau\}^{1/2}\cdot|v|_{1,\Omega}. \end{array} \tag{43}$$

Now we deal with the last part on the right hand side of (39). We split it as follows

$$\begin{array}{rl} & \frac{1}{80|\hat\tau|^2}\sum\limits_{\mathbf{e}_i=\tau\cap\tau'}\left[\,|\tau|\,B_i(\hat u_\tau)+|\tau'|\,B_{i'}(\hat\tau')\,\right]\;\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v\;\} \\ = & \frac{1}{80|\hat\tau|^2}\sum\limits_{\mathbf{e}_i=\tau\cap\tau'}\{\,(|\tau|-|\tau'|)B_i(\hat u_\tau)\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v \\ & +|\tau'|(B_i(\hat u_\tau)+B_{i'}(\hat\tau'))\cdot\partial_{\hat{\mathbf{t}}_i^2}\hat v\;\}. \end{array} \tag{44}$$

For the first term on the right hand side, by using the fact that $\tau \cup \tau'$ forms an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, we have

$$\Big|\, |\tau| - |\tau'| \,\Big| \;=\; |\tau| \Big|\, 1 - \frac{|\tau'|}{|\tau|} \,\Big| \;\leq\; cN^{-\alpha/2}|\tau|.$$

Note that $B_i(\hat{u}_\tau)$ involves only ξ_k, η_k, $1 \leq k \leq 3$, and the third order derivatives of $\hat{u}_\tau$. They can be bounded by $\|J_\tau^{-1}\|^2$ and $|\hat{D}^3\hat{u}_\tau|$ respectively. Thus

$$\begin{aligned} |B_i(\hat{u}_\tau)| \;&\leq\; c\textstyle\int_{\hat\tau} \|J_\tau^{-1}\|^2 \cdot |\hat{D}^3\hat{u}_\tau|\, d\hat\tau \\ &\leq\; c\textstyle\int_{\hat\tau} \|J_\tau^{-1}\|^2 \cdot \|J_\tau^T Q_3 J_\tau\|^{3/2}\, d\hat\tau \\ &\leq\; c\,|\tau|^{-1/2}\{\ \textstyle\int_\tau \|J_\tau^{-1}\|^4 \cdot \|J_\tau^T Q_3 J_\tau\|^3\, d\tau\ \}^{1/2}. \end{aligned}$$

Putting the above two inequalities together, we have

$$\begin{aligned} &\tfrac{1}{80|\hat\tau|^2} \sum_{\mathbf{e}_i = \tau\cap\tau'} \Big|\, (|\tau| - |\tau'|)B_i(\hat{u}_\tau)\cdot \partial_{\hat{\mathbf{t}}_i^2}\hat{v}\,\Big| \\ (45)\quad &\leq cN^{-\alpha/2}\sum_\tau \{\textstyle\int_\tau \|J_\tau\|^2\|J_\tau^{-1}\|^4\|J^TQ_4J\|^4 d\tau\,\}^{1/2}\cdot\ \{\sum_\tau \int_\tau |Dv|^2 d\tau\,\}^{1/2} \\ &\leq c\{\textstyle\int_\Omega \|J\|^2\|J^{-1}\|^4(N^{-\alpha}\|J^TQ_3J\|^3 + \|J^TQ_4J\|^4)d\tau\,\}^{1/2}\cdot |v|_{1,\Omega} \end{aligned}$$

For the second part on the right hand side of (44), we note that $B_i(\hat{u}_\tau)$ is composed of terms like $\hat{\xi}_l\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau$ with varrious l, i, j, k, and $B_{i'}\hat{u}_{\tau'}$ the terms like $\hat{\xi}_{l'}\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}$ with the corresponding indexes l', i', j', k'. Hence all the terms in $B_i(\hat{u}_\tau) + B_{i'}(\hat{u}_{\tau'})$ are in the form of

$$\hat{\xi}_l\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \hat{\xi}_{l'}\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}.$$

We may estimate each of them in the following manners:

$$|\hat{\xi}_l\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \hat{\xi}_{l'}\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}| \leq |\hat{\xi}_l - \hat{\xi}_{l'}|\cdot\int_{\hat\tau}|\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau| + |\hat{\xi}_{l'}|\cdot\Big|\int_{\hat\tau}[\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}]\Big|,$$

where by (28) the first term on the right hand side can be bounded by

$$\begin{aligned} &|\hat{\xi}_l - \hat{\xi}_{l'}|\cdot\textstyle\int_{\hat\tau}|\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau| \\ \leq\;& cN^{-\alpha/2}(\|J_\tau^{-1}\|^2 + \|J_{\tau'}^{-1}\|^2)\{\textstyle\int_{\hat\tau}|\hat{D}^3\hat{u}_\tau|^2\}^{1/2} \\ \leq\;& c|\tau|^{-1/2}N^{-\alpha/2}(\|J_\tau^{-1}\|^2 + \|J_{\tau'}^{-1}\|^2)\{\textstyle\int_\tau\|J_\tau\|^2\|J^TQ_3J\|^3\}^{1/2}; \end{aligned}$$

while by Lemma 3.6, the second term can be bounded by

$$\begin{aligned} &|\hat{\xi}_{l'}|\cdot|\textstyle\int_{\hat\tau}\partial_{\hat{\mathbf{t}}_{ijk}}\hat{u}_\tau + \partial_{\hat{\mathbf{t}}_{i'j'k'}}\hat{u}_{\tau'}| \\ \leq\;& c|\tau|^{-1/2}N^{-\alpha/2}\|J_{\tau'}^{-1}\|^2\{\textstyle\int_\tau[\|J^TQ_3J\|^3 + \|J^TQ_4J\|^4] \\ &+ \textstyle\int_{\tau'}[\|J_{\tau'}^TQ_3J_{\tau'}\|^3 + \|J_{\tau'}^TQ_4J_{\tau'}\|^4]\}^{1/2}. \end{aligned}$$

Combining the above estimates and the bound for $\partial_{\hat{\mathbf{t}}_i^2}\hat{v}$ in (42), we end up with

$$\begin{aligned} &\tfrac{1}{80|\hat\tau|^2}\sum_{\mathbf{e}_i=\tau\cap\tau'}|\tau'|\cdot|B_i(\hat{u}_\tau) + B_{i'}(\hat\tau')\,|\cdot|\partial_{\hat{\mathbf{t}}_i^2}\hat{v}| \\ \leq\;& c\sum_\tau \|J_\tau\|\cdot\|J_\tau^{-1}\|^2\{\textstyle\int_\tau (N^{-\alpha}\|J_\tau^TQ_3J_\tau\|^3 + \|J_\tau^TQ_4J_\tau\|^4)d\tau\}^{1/2} \\ (46)\quad &\cdot\{\sum_\tau\textstyle\int_\tau|Dv|^2d\tau\}^{1/2} \\ \leq\;& c\{\textstyle\int_\Omega\|J\|^2\|J^{-1}\|^4(N^{-\alpha}\|J^TQ_3J\|^3 + \|J^TQ_4J\|^4)d\tau\,\}^{1/2}\cdot|v|_{1,\Omega}. \end{aligned}$$

Finally, (34) follows from a combination of estimates (37), (39), (43), (45), and (46).

The second inequality (35) of this theorem follows easily from applying $J_\tau \simeq (C_M/N)^{1/2} F_\tau$ for all $\tau \in \mathcal{T}_N$ in (34) when the partition is quasi-uniform under metric M. $\square$

4. Numerical Results

We present in this section some numerical results for the H^1-error $|u - u_N|_{1,\Omega}$ of the quadratic finite element solution u_N and the error $|u_N - u_q|_{1,\Omega}$ between u_N and the quadratic interpolation of solution u. We choose in the model problem (1) the domain $\Omega = [0,1]^2$, and coefficients $A = I, \mathbf{b} = 0, d = 1$. The right hand side function f and the Dirichlet boundary condition are selected so that equation (1) admits the following exact solutions:

Example (1): $\quad u(x,y) = \tanh[-100(\sqrt{x^2+y^2} - 0.3)];$

Example (2): $\quad u(x,y) = [\cosh(Kx) + \cosh(Ky)]/[2\cosh(K)], \quad \text{with } K = 100.$

These two examples were used in [7] to demonstrate the convergence and superconvergence of the linear finite element approximation based on unstructured anisotropic adaptive meshes. The first example involves a solutions with steep front inside the domain, and the second one near the boundary. These examples were used also for the numerical study of some a-posteriori error estimators of linear elements based on anisotropic meshes, see, e.g., [14, 20, 25].

In order to generate the anisotropic meshes suitable for the quadratic approximation of the solution u, we need an accurate measure of the anisotropic behavior of $D^3 u$. We use here the exact formula developed by Mirebeau [18] for the "smallest possible" Q_3. More specifically, at any given point (x,y), let

$$c_0 = \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x,y), \quad c_1 = \frac{1}{2}\frac{\partial^3 u}{\partial x^2 \partial y}(x,y), \quad c_2 = \frac{1}{2}\frac{\partial^3 u}{\partial x \partial y^2}(x,y), \quad c_3 = \frac{1}{6}\frac{\partial^3 u}{\partial y^3}(x,y).$$

Define

$$p_3(t) = c_0 t^3 + c_1 t^2 + c_2 t + c_3$$

and its discriminant

$$disc(p_3) = (c_1 c_2)^2 - 4(c_0 c_2^3 + c_1^3 c_3) - 27(c_0 c_3)^2 + 18 c_0 c_1 c_2 c_3.$$

Define

$$Q_3 = (\Phi^{-1})^T \Phi^{-1}.$$

where Φ is selected according to $disc(p_3)$ as follows:

(i) When $disc(p_3) > 0$, p_3 has three distinct real roots $r_1 < r_2 < r_3$. In this case,

$$\Phi = \frac{c_0}{\sqrt[3]{2\, disc(p_3)}} \cdot \begin{bmatrix} r_1(r_2+r_3) - 2r_2 r_3, & \sqrt{3}\, r_1(r_2 - r_3) \\ 2r_1 - (r_2+r_3), & \sqrt{3}\,(r_2 - r_3) \end{bmatrix};$$

(ii) When $disc(p_3) < 0$, p_3 has one real root r_1 and two complex roots $r_2 = \bar{r}_3$. Suppose $\mathrm{Im}(r_2) > 0$. Then we choose

$$\Phi = \frac{c_0}{\sqrt{2}\,\sqrt[3]{disc(p_3)}} \cdot \begin{bmatrix} r_1(r_2+r_3) - 2r_2 r_3, & \sqrt{3}\,\mathrm{i}\; r_1(r_2 - r_3) \\ 2r_1 - (r_2+r_3), & \sqrt{3}\,\mathrm{i}\;(r_2 - r_3) \end{bmatrix}.$$

In the degenerate case $c_0 = 0$, we may apply a rotation of the coordinates to convert $D^3 u$ into the case with $c_0 \neq 0$ and then use the inverse of the rotation afterwards to get the anisotropic measure Q_3. Also for the degenerate cases when p_3 has repeated real roots, we may use small perturbation to convert them into the cases of distinct real roots, and regularize the resulting Q_3 by putting a "floor" on its eigenvalues to avoid being close to singular.

Once the anisotropic measure Q_3 is determined, we compute the metric $M_{3,1,2}$ according to (10), which minimizes the upper bound of the quadratic interpolation errors in H^1-seminorm. Metric $M_{3,1,2}$ with various c is then supplied to the bi-dimensional anisotropic mesh generator (`bamg`) [11] to generate the adaptive anisotropic meshes with desired number of elements for our computations. We display in Figure 2 and 3 typical meshes resulted from this procedure for the two examples in this section.

We list in Table 1 the H^1-error $|u-u_N|_{1,\Omega}$ of the quadratic finite element solution u_N and the error $|u_N - u_q|_{1,\Omega}$ between u_N and the quadratic interpolation of the solution u based on the above generated anisotropic adaptive meshes. To determine the rate of convergence, we use linear least square fitting on the logarithms of the errors as well as the number of elements. It turns out that for Example (1), the convergence rates (with respect to N^{-1}) are approximate 1.0713 and 1.0875 for $|u - u_N|_{1,\Omega}$ and $|u_N - u_q|_{1,\Omega}$, respectively; while for Example (2) these rates are 1.1919 and 1.2421, respectively. Clearly second order of convergence (corresponding to $O(N^{-1})$) for the quadratic element solutions are achieved in both examples, and the magnitude of the error $|u_N - u_q|_{1,\Omega}$ is about $\frac{1}{4}$ and $\frac{1}{10}$ of the corresponding error $|u - u_N|_{1,\Omega}$ for example (1) and (2) respectively. However, the convergence of $|u_N - u_q|_{1,\Omega}$ is only slightly better than quadratic. More precisely, they are approximately of order $O(N^{-(1+\alpha/2)})$ with $\alpha = 0.1750$ and 0.4842 in example (1) and (2), resp.. The relative small values of α in the improvement of the rate of convergence for $|u_N - u_q|_{1,\Omega}$ is mainly due to the fact that in unstructured adaptive mesh refinement the mesh adaptation to metrics or solutions is the major goal, and the closeness of each element pair to a parallelogram, which often conflicts with this goal due to geometric constraints, is only secondary. Similar results were reported before for the linear element approximations on unstructured meshes, too, see [3] for the case of unstructured shape regular meshes and [7] for the case of unstructured anisotropic meshes. As the meshes get finer and finer, it is expected to have more and more pairs of elements become closer and closer to parallelograms, and indeed this is observed in the cases of linear elements in [7]. However, it seems that superconvergence for quadratic approximations is even more sensitive to the closeness of the element pairs to parallelograms. Our computation here for the two examples is still not well within regime of substantial order improvement yet.

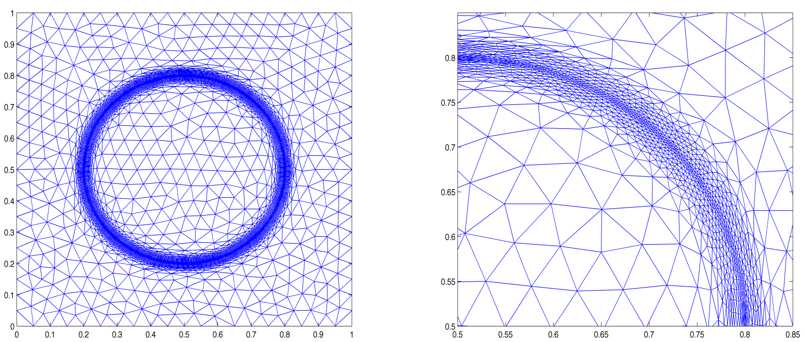

FIGURE 2. Example (1): Adaptive mesh generated by `bamg` based on metric $M_{3,1,2}$ and its close-up look. Total number of elements $N = 8548$.

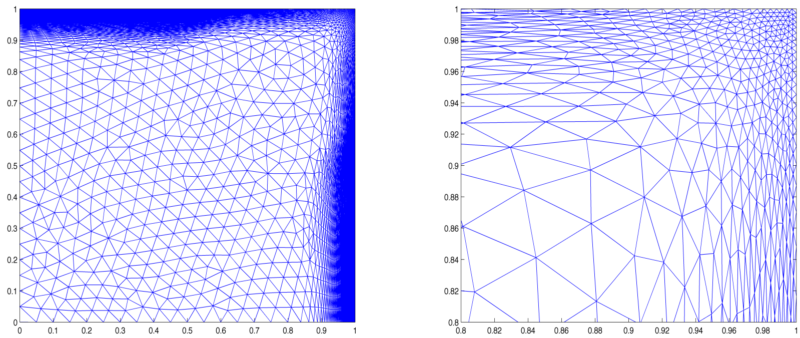

FIGURE 3. Example (2): Adaptive mesh generated by `bamg` based on metric $M_{3,1,2}$ and its close-up look. Total number of elements $N = 8452$.

TABLE 1. Errors of the quadratic element solution to (1) and errors between the FE solution and the quadratic interpolation of the exact solution.

Example (1)			Example (2)		
N	$\|u - u_N\|_{1,\Omega}$	$\|u_N - u_q\|_{1,\Omega}$	N	$\|u - u_N\|_{1,\Omega}$	$\|u_N - u_q\|_{1,\Omega}$
2278	4.70495e-01	1.08851e-01	2674	4.12428e-02	4.59787e-03
4146	1.97713e-01	4.34775e-02	4336	1.69532e-02	2.14723e-03
8548	8.26101e-02	1.73568e-02	8452	6.71769e-03	8.18282e-04
16548	4.15003e-02	8.67958e-03	16565	3.07508e-03	3.77876e-04
32014	2.18243e-02	4.74147e-03	32166	1.49450e-03	1.19576e-04
62318	1.16236e-02	2.54865e-03	65568	6.93893e-04	7.67566e-05
124668	5.98247e-03	1.23921e-03	130805	3.65857e-04	3.78686e-05
order in N	-1.0713	-1.0875	order in N	-1.1919	-1.2421

5. Discussions

For the quadratic finite element approximation of elliptic equations based on adaptively refined anisotropic meshes in two dimensions, we proved the supercloseness of the finite element solutions to the quadratic interpolation of the exact solutions in energy norm. Our basic assumptions are that the partition is quasi-uniform under a Riemannian metric and that each pair of adjacent elements in the meshes forms an approximate parallelogram. This result extends our earlier one for the linear finite element approximation on anisotropic meshes [7]. It can also considered as a generalization of the results in [13] for quadratic elements based on mildly structured shape regular meshes.

It is noted that for finite element approximation based on unstructured meshes, though the finite element solution converges to the quadratic interpolation of the exact solution faster than it does to the exact solution its self, the improvement in the rate of convergence depends on the closeness of each element pair to a parallelogram. In particular, this dependence seems to be more critical in quadratic approximations than in linear approximations. Unfortunately, making element pairs close to parallelograms is subject to the geometric constraints and also often conflicts with the goal of mesh adaptation. Therefore, inevitablely there are more or less certain percentage of element pairs that do not form approximate parallelograms to higher degrees. One issue of practical interests is to understand how much more smaller the error $u_N - u_q$ is compared to the error $u - u_N$ in such situations. Another issue under our current consideration is rigorous proof of the effectiveness of the gradient recovery techniques for quadratic finite elements on adaptively refined unstructured anisotropic meshes.

Acknowledgments

This work was supported in part by the National Science Foundation of China grant 11571224.

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