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A Finite Volume Method Based on the Constrained Nonconforming Rotated Q_1-Constant Element for the Stokes Problem

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Abstract. We construct a finite volume element method based on the constrained nonconforming rotated Q_1-constant element $(CNRQ_1\text{-}P_0)$ for the Stokes problem. Two meshes are needed, which are the primal mesh and the dual mesh. We approximate the velocity by $CNRQ_1$ elements and the pressure by piecewise constants. The errors for the velocity in the H^1 norm and for the pressure in the L^2 norm are $\mathcal{O}(h)$ and the error for the velocity in the L^2 norm is $\mathcal{O}(h^2)$. Numerical experiments are presented to support our theoretical results.

AMS subject classifications: 65N08, 65N15, 65N30, 76D05

Key words: Stokes problem, finite volume method, constrained nonconforming rotated Q_1 element.

1 Introduction

Let Ω be a bounded, convex and open polygon of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following Stokes equations with the homogeneous Dirichlet boundary condition

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f},\qquad \text{in } \Omega, \tag{1.1a}$$
$$\text{div}\mathbf{u}=0,\qquad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u}=0,\qquad \text{on } \partial\Omega, \tag{1.1c}$$

where $\mathbf{u}=(u^1,u^2)$ represents the velocity vector, p is the pressure and $\mathbf{f}$ indicates a prescribed body force. Let $L^2_0(\Omega)$ be the set of all $L^2(\Omega)$ functions over Ω with zero integral mean and let

$$H^1_0(\Omega):=\big\{u\in H^1(\Omega): u=0 \text{ on } \partial\Omega\big\}.$$

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The variational formulation of (1.1a)-(1.1c) is: find a pair $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that (see [10])

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in H_0^1(\Omega)^2, \tag{1.2a}$$

$$b(\mathbf{u}, q) = 0, \qquad \forall q \in L_0^2(\Omega), \tag{1.2b}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \qquad b(\mathbf{v}, p) = -(p, \mathrm{div}\mathbf{v}).$$

Finite volume method (FVM) is an important numerical discretization technique for solving partial differential equations, especially for those arising from physical conservation laws including mass, momentum and energy. In general, FVM has both simplicity in implementation and local conservation property, so it has enjoyed great popularity in many fields, such as computational fluid dynamics, computational aerodynamics, petroleum engineering and so on. About some recent development of FVM, readers can refer to the monographs [6,7,9,13,16,19,20,27,28] for details.

In recent years, there have been a lot of studies on the mixed finite element methods and mixed finite volume element methods for the Stokes problem. Among these studies, some lower order quadrilateral finite elements seem to be more attractive, e.g., the conforming bilinear Q_1-P_0 element [7,17,24,26], the conforming bilinear Q_1-Q_1 element [1, 14] and nonconforming rotated Q_1-P_0 element [25] with some variants [4, 13]. All the approximation finite elements for the velocity need at least four degrees of freedom on each quadrilateral. In [23], Park and Sheen have introduced a P_1-nonconforming quadrilateral element which has only three degrees of freedom on each quadrilateral. Later, Man and Shi [20] proposed the P_1-nonconforming quadrilateral FVM for the elliptic problem by using a dual partition of overlapping type. Following the line of the finite element in [23], Hu and Shi [12] presented a constrained nonconforming rotated Q_1 ($CNRQ_1$) element and applied it to the second order elliptic problem. In [12], the authors also point out that the $CNRQ_1$ element and the P_1-nonconforming element are equivalent on a rectangle since the constraint and the continuity are the same, however, the two elements are different on a general quadrilateral. Afterwards, Hu, Man and Shi [11] and Mao and Chen [21] investigated and analyzed the $CNRQ_1$-P_0 finite element method for the Stokes problem on rectangular meshes. The application of the $CNRQ_1$ element to the nearly incompressible planar elasticity problem can be referred to [11,22]. Meanwhile, Mao and Chen [21] and Liu and Yan [18] discussed the supconvergence of the finite element scheme for the Stokes problem on rectangular meshes.

The purpose of this paper is to investigate a new mixed FVM for solving the Stokes problem on quadrilateral meshes. We will approximate the velocity by the $CNRQ_1$ element (see [12]) based on the primal quadrilateral partition, while the test function space of the velocity is the piecewise constant space associated to the nonoverlapping dual partition. Following the ideas presented in [11, 18, 21, 24], we employ a piecewise constant approximation for the pressure. We also analyze the stability of this new mixed FVM under certain reasonable assumptions and obtain the optimal convergence rate for the velocity in the broken H^1 seminorm and the pressure in the L^2 norm. In addition, an optimal convergence rate for the velocity in the L^2 norm is derived.

The organization of this paper is as follows. In Section 2, we investigate the $CNRQ_1$-P_0 finite volume element scheme for the Stokes problem on quadrilateral meshes. The stability of this FVM are discussed in Section 3. In Section 4, we prove the optimal convergence error estimates for this new method. Finally, in Section 5 numerical examples are presented on three types of quadrilateral meshes to confirm our theoretical results.

Throughout this paper, we denote C without subscripts as a generic positive constant which is not the same at different places and independent of the discretization parameters.

2 Finite volume element scheme

For a subdomain $T \subset \mathbb{R}^2$, $(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_{\partial T}$ denote the $L^2(T)$ and $L^2(\partial T)$ inner products, respectively. Let $\|\cdot\|_{s,T}$ and $|\cdot|_{s,T}, s \geq 0$ be the norm and the seminorm of the standard Sobolev space $H^s(T)$ or $H^s(T)^2$, respectively. The subscript T is omitted if $T = \Omega$.

Let $\mathcal{T}_h = \{K\}$, which is called primal partition, be a strictly convex and nonoverlapping quadrilateral partition of Ω. We denote the numbers of the elements and the interior nodes of $\mathcal{T}_h$ by N and N_i^v, respectively.

For a given $K \in \mathcal{T}_h$, its four vertices and midpoints of edges are denoted by $P_i = (x_i, y_i), 1 \leq i \leq 4$ and $M_i, 1 \leq i \leq 4$ in counterclockwise order (see Fig. 1), respectively and $V(K)$ denotes the set of four vertices of K. The point C_K denotes the joint of $\overline{M_1M_3}$ and $\overline{M_2M_4}$, which is the averaging center of K, then C_K becomes the midpoint of both $\overline{M_1M_3}$ and $\overline{M_2M_4}$. Set

$$m_1 = |\overline{M_1M_3}|, \quad m_2 = |\overline{M_2M_4}|, \quad \kappa = \frac{m_2}{m_1}, \quad \theta = \angle M_3C_KM_2.$$

Let $m(K)$ be the measure of K. And we shall give the following assumptions on $\mathcal{T}_h$.

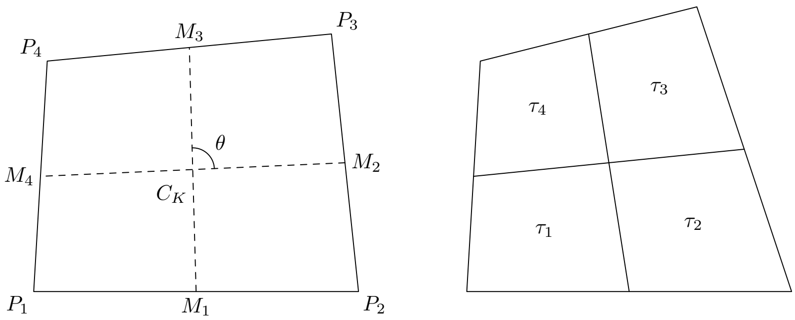

Figure 1: The element K and the element τ.

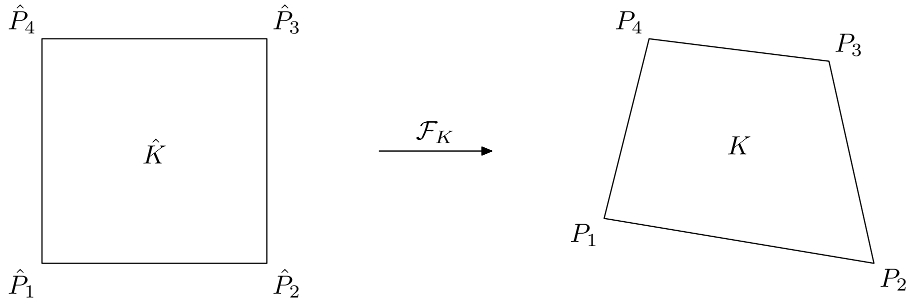

Figure 2: The bilinear mapping $\mathcal{F}_K : \hat{K} \to K$.

The primal partition $\mathcal{T}_h$ is supposed to be regular in the sense that there exist $\delta_2 > \delta_1 > 0$ and $0 < \sigma < 1$ independent of h such that for any $K \in \mathcal{T}_h$:

$$\delta_1 \le \kappa \le \delta_2, \tag{2.1a}$$

$$|\cos\theta_K| \le \sigma. \tag{2.1b}$$

And suppose $\mathcal{T}_h$ is quasi-uniform, i.e., there exists a positive constant C such that

$$Ch^2 \le m(K) \le h^2, \quad \forall K \in \mathcal{T}_h, \tag{2.2}$$

where h is the maximum meshsize of the partition.

We also assume each quadrilateral K in $\mathcal{T}_h$ satisfying quasi-parallel quadrilateral condition, i.e.,

$$|\overrightarrow{P_1P_4} + \overrightarrow{P_3P_2}| \le Ch^2. \tag{2.3}$$

Let $\hat{K} = [-1,1] \times [-1,1]$ in the $\xi\eta$-plane be a reference element with vertices $\hat{P}_1 = (-1,-1)$, $\hat{P}_2 = (1,-1)$, $\hat{P}_3 = (1,1)$, $\hat{P}_4 = (-1,1)$. Define the bilinear transformation $\mathcal{F}_K : \hat{K} \to K$:

$$\mathcal{F}_K : \begin{cases} x = c_0 + c_1\xi + c_2\eta + c_{12}\xi\eta, \\ y = d_0 + d_1\xi + d_2\eta + d_{12}\xi\eta, \end{cases}$$

where

$$\begin{pmatrix} c_0 & d_0 \\ c_1 & d_1 \\ c_2 & d_2 \\ c_{12} & d_{12} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}.$$

The Jacobian matrix of the bilinear transformation $\mathcal{F}_K$ can be expressed as

$$\mathcal{J}_K(\xi,\eta) = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} c_1 + c_{12}\eta & c_2 + c_{12}\xi \\ d_1 + d_{12}\eta & d_2 + d_{12}\xi \end{pmatrix},$$

with the determinant

$$J_K(\xi,\eta) = J_0 + J_1\xi + J_2\eta,$$

where

$$J_0 = c_1d_2 - c_2d_1 = m(K), \quad J_1 = c_1d_{12} - c_{12}d_1, \quad J_2 = c_{12}d_2 - c_2d_{12},$$

and its inverse is

$$\mathcal{J}_K^{-1}(\xi,\eta) = \begin{pmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\xi}{\partial y} \\ \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\eta}{\partial y} \end{pmatrix} = \frac{1}{J_K(\xi,\eta)} \begin{pmatrix} d_2 + d_{12}\xi & -c_2 - c_{12}\xi \\ -d_1 - d_{12}\eta & c_1 + c_{12}\eta \end{pmatrix}.$$

In terms of the aforementioned mesh parameters, (2.3) implies (see [12])

$$|c_{12}| + |d_{12}| \le Ch^2, \tag{2.4a}$$

$$|J_1| + |J_2| \le Ch^3. \tag{2.4b}$$

For convenience, we give a brief introduction of the constrained nonconforming rotated Q_1 quadrilateral element proposed in [12]. For any edge $l \subset \partial K$, the edge functional i_h^l is defined as

$$i_h^l(v) = \frac{1}{|l|}\int_l v ds,$$

for any $v \in L^2(K)$. Let $P_1(\hat{K})$ be the linear function space on the reference element $\hat{K}$. Then the constrained nonconforming rotated Q_1 quadrilateral element space CNR^h and its homogenous space CNR_0^h read

$$CNR^h = \{v \in L^2(\Omega) : v|_K = \hat{v} \circ \mathcal{F}_K^{-1}, \hat{v} \in P_1(\hat{K}),\ v \text{ is continuous regarding } i_h^l,\ \forall K \in \mathcal{T}_h\},$$
$$CNR_0^h = \{v \in CNR^h : i_h^l(v) = 0,\ \forall l \subset \partial\Omega\}.$$

On the space CNR^h, we define the broken H^1 seminorm and the H^1 norm as follows,

$$|u_h|_{1,h} = \Big(\sum_{K\in\mathcal{T}_h} |u_h|_{1,K}^2\Big)^{\frac{1}{2}}, \quad \|u_h\|_{1,h} = \Big(\sum_{K\in\mathcal{T}_h} \|u_h\|_{1,K}^2\Big)^{\frac{1}{2}}, \quad \forall u_h \in CNR^h.$$

On the reference element $\hat{K}$, define

$$\hat{\phi}_1 = \frac{1}{4}(1-\xi-\eta), \qquad \hat{\phi}_2 = \frac{1}{4}(1+\xi-\eta),$$
$$\hat{\phi}_3 = \frac{1}{4}(1+\xi+\eta), \qquad \hat{\phi}_4 = \frac{1}{4}(1-\xi+\eta),$$

which are associated to vertices $\hat{P}_i$, $i = 1,2,3,4$ of $\hat{K}$, respectively. It has been proven in [12] and [23] that $\hat{\phi}_i$, $i = 1,2,3,4$ span $P_1(\hat{K})$. For each interior node P_j, let

$$E(j) = \cup_{i=1}^4 K_{i,j},$$

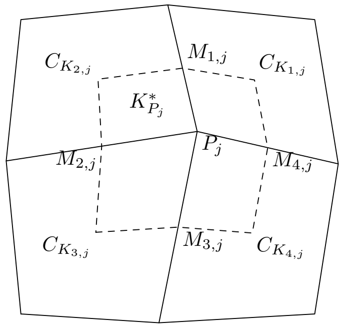

Figure 3: The dual element $K^*_{P_j}$.

where $K_{i,j}$ denotes the element with the node P_j as one of its vertices, we define

$$\phi_j(\mathbf{x}) = \begin{cases} \hat{\phi}_i(\mathcal{F}_K^{-1}(\mathbf{x})), & \mathbf{x} \in K \in E(j), \\ 0, & \mathbf{x} \in K \in \mathcal{T}_h \setminus E(j), \end{cases} \tag{2.5}$$

where the subscript i is determined by $P_j = P_{i,K} = \mathcal{F}_K(\hat{P}_i)$, with $P_{i,K}$, $i = 1,2,3,4$, four vertices of element K. It is easy to see that ϕ_j, $j = 1, \cdots, N_i^v$ are linearly independent, therefore $\{\phi_j\}_{j=1}^{N_i^v}$ is a basis of CNR_0^h. Then we introduce an interpolation operator $\pi_h : H^1(\Omega) \to CNR^h$ defined by

$$\pi_h v = \sum_{j=1}^{N_i^v} v(P_j)\phi_j, \quad v \in H^1(\Omega).$$

We also need to construct the dual partition $\mathcal{T}_h^*$ and the test function space. Divide each quadrilateral of the primal partition into four smaller quadrilaterals by connecting the opposite midpoints as shown in Fig. 1. The dual grid is defined as a union of polygons, each of which is made up of four smaller quadrilaterals. As shown in Fig. 3, the dual element associated to the node P_j is made up of the four small quadrilaterals which share the node P_j as a common base. Carrying out the construction for every interior node in the primal partition generates a dual partition for the domain. We denote the dual element based at P_j by the polygonal region $K^*_{P_j}$ with vertices $C_{K_{1,j}}, M_{1,j}, C_{K_{2,j}}, M_{2,j}, C_{K_{3,j}}, M_{3,j}, C_{K_{4,j}}, M_{4,j}$, and the dual partition by $\mathcal{T}_h^* = \{K^*_{P_j}, j = 1, \cdots, N_i^v\}$. Set

$$W_h = \Big\{\omega_h \in L^2(\Omega) : \omega_h|_{K^*_{i,j}} = \text{Const and } \omega_h|_{\partial\Omega} = 0,\ K^*_{i,j} = K_{i,j} \cap K^*_{P_j},\ K_{i,j} \in E(j)\Big\}.$$

Define an operator $\gamma_h : CNR_0^h \to W_h$ such that

$$\gamma_h v_h = \sum_{j=1}^{N_i^v} \sum_{i=1}^{4} \big(v_h|_{K_{i,j}}\big)(P_j)\chi_{i,j}, \quad \forall v_h \in CNR_0^h,$$

where $\chi_{i,j}$ is the characteristic function of $K^*_{i,j}$.

By virtue of this operator, the scalar test function space corresponding to $\mathcal{T}_h^*$ can be defined by

$$V_h := \mathrm{span}\Big\{\varphi_j : \varphi_j = \gamma_h \phi_j,\ j = 1, \cdots N_i^v\Big\},$$

where ϕ_j is given by (2.5).

Remark 2.1. Obviously, $V_h \subset W_h$. For each interior node P_j, $N(j)$ denotes the set of the adjacent nodes of P_j in $\mathcal{T}_h$ and $\overline{N}(j) = \{P : \ \overline{PP_j} \text{ is an interior edge in } \mathcal{T}_h\}$, then

$$\varphi_j = \begin{cases} \dfrac{3}{4}, & \text{in } K^*_{P_j}, \\ \dfrac{1}{4}, & \text{in } K^*_P \cap E(j), \text{ for } P \in \overline{N}(j), \\ -\dfrac{1}{4}, & \text{in } K^*_P \cap E(j), \text{ for } P \in N(j) \setminus \overline{N}(j), \\ 0, & \text{else.} \end{cases}$$

We now can propose the $CNRQ_1$-P_0 finite volume method for the Stokes problem now. Choose

$$\mathbf{U}_h = CNR_0^h \times CNR_0^h,$$

as the trial function space and

$$\mathbf{V}_h = V_h \times V_h,$$

as the test function space for the velocity. Set

$$\Pi_h = \pi_h \times \pi_h \quad \text{and} \quad \Gamma_h = \gamma_h \times \gamma_h.$$

Therefore we can easily check that the operators Π_h and Γ_h have the following approximation properties for any $K \in \mathcal{T}_h$ (see [2,8,12,15]):

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{m,K} \le Ch^{2-m}|\mathbf{u}|_{2,K}, \qquad \forall \mathbf{u} \in H^2(\Omega),\ m = 0,1, \tag{2.6a}$$

$$\|\mathbf{v}_h - \Gamma_h \mathbf{v}_h\|_{0,K} \le Ch|\mathbf{v}_h|_{1,K}, \qquad \forall \mathbf{v}_h \in \mathbf{U}_h. \tag{2.6b}$$

In fact, the estimate (2.6a) is the same as (3.7) in [12] when $\alpha = 1$. Write $P_j = (x_j, y_j)$, by Taylor's expansion we have

$$\begin{aligned} \|\mathbf{v}_h - \Gamma_h \mathbf{v}_h\|_{0,K^*_{i,j}} =& \|\mathbf{v}_h - \mathbf{v}_h|_{K_{i,j}}(P_j)\|_{0,K^*_{i,j}} \\ \le& \Big\| \frac{\partial \mathbf{v}_h(\xi)}{\partial x}(x - x_j)\Big\|_{0,K^*_{i,j}} + \Big\| \frac{\partial \mathbf{v}_h(\xi)}{\partial y}(y - y_j)\Big\|_{0,K^*_{i,j}} \\ \le& Ch|\mathbf{v}_h|_{1,K^*_{i,j}}. \end{aligned}$$

Then (2.6b) can be obtained.

1	1
1	1

$\hat{\mu}_1$

-1	1
-1	1

$\hat{\mu}_2$

1	1
-1	-1

$\hat{\mu}_3$

-1	1
1	-1

$\hat{\mu}_4$

Figure 4: Local basis functions of P_h' on $\hat{K}$.

For the pressure, we assume that the subdivision $\mathcal{T}_h$ is obtained from $\mathcal{T}_{2h} = \{\tau\}$ by dividing each element of $\mathcal{T}_{2h}$ into four smaller quadrilaterals through connecting the opposite midpoints. Let P_h' be a function space which consists of piecewise constant functions with respect to $\mathcal{T}_h$. The reference element $\hat{K}$ can be partitoned into subdomains

$$\hat{K}_{mn} = \big\{(\xi,\eta) \in \hat{K} : m-2 < \xi < m-1,\ n-2 < \eta < n-1,\ m,n = 1,2\big\},$$

and the local basis functions for P_h' on a 2×2 patch of $\hat{K}$ are defined by (see Fig. 4)

$$\hat{\mu_1}|_{\hat{K}_{mn}} = 1, \quad \hat{\mu_2}|_{\hat{K}_{mn}} = (-1)^n, \quad \hat{\mu_3}|_{\hat{K}_{mn}} = (-1)^m, \quad \hat{\mu_4}|_{\hat{K}_{mn}} = (-1)^{m+n}, \quad m,n = 1,2.$$

By [11,18,21,24], the finite element space for pressure has the form of

$$P_h = \Big\{p \in L_0^2(\Omega) : p|_\tau = \sum_{i=1}^{3} \lambda_i^\tau \hat{\mu}_i \circ \mathcal{F}_\tau^{-1}, \quad \forall \tau \in \mathcal{T}_{2h}\Big\}.$$

Next we define the interpolation operator $J_h : L_0^2(\Omega) \cap H^1(\Omega) \to P_h$ with respect to τ:

$$J_h p|_{\tau_i} = \begin{cases} p_i^\tau - \dfrac{1}{4}\alpha_\tau, & i = 1,4, \\ p_i^\tau + \dfrac{1}{4}\alpha_\tau, & i = 2,3, \end{cases}$$

where

$$\alpha_\tau = p_1^\tau - p_2^\tau + p_3^\tau - p_4^\tau, \qquad p_i^\tau = \frac{1}{m(\tau_i)} \int_{\tau_i} p dxdy,$$

and τ_i, $i = 1,2,3,4$, are smaller elements in τ (see Fig. 1). A direct calculation shows that

$$J_h p|_\tau = \frac{1}{4}\Big[\Big(\sum_{i=1}^{4} p_i^\tau\Big)\mu_1^\tau + (-p_1^\tau + p_2^\tau + p_3^\tau - p_4^\tau)\mu_2^\tau + (-p_1^\tau - p_2^\tau + p_3^\tau + p_4^\tau)\mu_3^\tau\Big],$$

which implies that $J_h p \in P_h$ for $p \in L_0^2(\Omega)$, where $\mu_i^\tau = \hat{\mu}_i \circ \mathcal{F}_\tau^{-1}$, $i = 1,2,3,4$. It is easy to check that (see [2,8])

$$\|p - J_h p\|_0 \le Ch|p|_1, \quad \forall p \in L_0^2(\Omega) \cap H^1(\Omega). \tag{2.7}$$

The FVM for the Stokes problem (1.1a)-(1.1c) investigated in this paper is: find a pair $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times P_h$, such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.8a}$$

$$b'_h(\mathbf{u}_h, q_h) = 0, \qquad \forall q_h \in P_h, \tag{2.8b}$$

where

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = -\sum_{j=1}^{N_i^v} \sum_{i=1}^{4} \big(\mathbf{v}_h|_{K_{i,j}}\big)(P_j) \cdot \int_{\partial K^*_{P_j} \cap K_{i,j}} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds, \tag{2.9a}$$

$$b_h(\mathbf{v}_h, p_h) = \sum_{j=1}^{N_i^v} \sum_{i=1}^{4} \big(\mathbf{v}_h|_{K_{i,j}}\big)(P_j) \cdot \int_{\partial K^*_{P_j} \cap K_{i,j}} p_h \mathbf{n} ds, \tag{2.9b}$$

$$b'_h(\mathbf{u}_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K (\mathrm{div} \mathbf{u}_h) q_h dxdy. \tag{2.9c}$$

3 Stability

In this section, we will show that the bilinear form $a_h(\cdot,\cdot)$ is coercive and the bilinear form $b_h(\cdot,\cdot)$ satisfies the inf-sup condition.

Lemma 3.1. *It holds that*

$$b_h(\Gamma_h \mathbf{v}_h, q_h) = b'_h(\mathbf{v}_h, q_h), \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad q_h \in P_h. \tag{3.1}$$

Proof. As $P_h \subset P'_h$, we only have to prove that

$$b_h(\Gamma_h \mathbf{v}_h, q_h) = b'_h(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{U}_h \times P'_h. \tag{3.2}$$

Set $\mathbf{v}_h = (v_h^1, v_h^2)$. Using the notations in Fig. 1, the right-hand side of (2.9b) can be rewritten as

$$\sum_{j=1}^{N_i^v} \sum_{i=1}^{4} \big(\mathbf{v}_h|_{K_{i,j}}\big)(P_j) \cdot \int_{\partial K^*_{P_j} \cap K_{i,j}} q_h \mathbf{n} ds = \sum_{K \in \mathcal{T}_h} b_{h,K}(\Gamma_h \mathbf{v}_h, q_h), \tag{3.3}$$

where

$$\begin{aligned}
b_{h,K}(\Gamma_h \mathbf{v}_h, q_h) =& \sum_{P \in V(K)} \big(\mathbf{v}_h|_K\big)(P) \cdot \int_{\partial K^*_P \cap K} q_h \mathbf{n} ds \\
=& \sum_{P \in V(K)} (v_h^1|_K)(P) \int_{\partial K^*_P \cap K} q_h dy - (v_h^2|_K)(P) \int_{\partial K^*_P \cap K} q_h dx \\
=& q_h|_K \big\{ -d_2(v_h^1|_K(P_2) - v_h^1|_K(P_1)) - d_2(v_h^1|_K(P_3) - v_h^1|_K(P_4)) \\
& + d_1(v_h^1|_K(P_4) - v_h^1|_K(P_1)) + d_1(v_h^1|_K(P_3) - v_h^1|_K(P_2)) \\
& + c_2(v_h^2|_K(P_2) - v_h^2|_K(P_1)) + c_2(v_h^2|_K(P_3) - v_h^2|_K(P_4)) \\
& - c_1(v_h^2|_K(P_4) - v_h^2|_K(P_1)) - c_1(v_h^2|_K(P_3) - v_h^2|_K(P_2)) \big\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned} b_h'(\mathbf{v}_h,q_h) &= \sum_{K\in\mathcal{T}_h}\int_K \mathrm{div}\mathbf{v}_h q_h dxdy = \sum_{K\in\mathcal{T}_h} q_h|_K \int_K \Big(\frac{\partial v_h^1}{\partial x}+\frac{\partial v_h^2}{\partial y}\Big)dxdy \\ &= \sum_{K\in\mathcal{T}_h} 4q_h|_K\Big(d_2\frac{\partial v_h^1}{\partial \xi}\Big|_K - d_1\frac{\partial v_h^1}{\partial \eta}\Big|_K - c_2\frac{\partial v_h^2}{\partial \xi}\Big|_K + c_1\frac{\partial v_h^2}{\partial \eta}\Big|_K\Big). \end{aligned}$$

With

$$\begin{aligned} \frac{\partial v_h^i}{\partial \xi}\Big|_K &= \frac{1}{4}\big(v_h^i|_K(P_2)+v_h^i|_K(P_3)-v_h^i|_K(P_4)-v_h^i|_K(P_1)\big), \\ \frac{\partial v_h^i}{\partial \eta}\Big|_K &= \frac{1}{4}\big(v_h^i|_K(P_3)+v_h^i|_K(P_4)-v_h^i|_K(P_1)-v_h^i|_K(P_2)\big), \end{aligned}$$

where $i=1,2$, we can obtain (3.2) by summing over K. $\square$

Now we define a discrete norm on $\mathbf{U}_h$:

$$|||\mathbf{u}_h|||=\Big(\sum_{K\in\mathcal{T}_h}|||\mathbf{u}_h|||_K^2\Big)^{\frac{1}{2}}, \tag{3.4}$$

where

$$|||\mathbf{u}_h|||_K^2 = \|\mathbf{u}_h(M_2)-\mathbf{u}_h(M_4)\|^2+\|\mathbf{u}_h(M_3)-\mathbf{u}_h(M_1)\|^2,$$

and $\|\cdot\|=(\cdot,\cdot)^{1/2}$ is the Euclidian norm of vectors.

Lemma 3.2. *Assume that the partition $\mathcal{T}_h$ satisfies* (2.2), *then for any $\mathbf{u}_h\in\mathbf{U}_h$, there exist positive constants β_1 and β_2 independent of h such that*

$$\beta_1|||\mathbf{u}_h|||\le|\mathbf{u}_h|_{1,h}\le\beta_2|||\mathbf{u}_h|||. \tag{3.5}$$

Proof. Set $\mathbf{u}_h=(u_h^1,u_h^2)$, then we only have to show the equivalence of $|u_h^1|_{1,K}$ and $|||u_h^1|||_K$, where

$$|||u_h^1|||_K=\big[u_h^1(M_2)-u_h^1(M_4)\big]^2+\big[u_h^1(M_3)-u_h^1(M_1)\big]^2.$$

Write

$$\nabla u_h^1=\Big(\frac{\partial u_h^1}{\partial x},\frac{\partial u_h^1}{\partial y}\Big)^T \quad\text{and}\quad \hat{\nabla}u_h^1=\Big(\frac{\partial u_h^1}{\partial \xi},\frac{\partial u_h^1}{\partial \eta}\Big)^T.$$

According to [8,16]

$$\|\nabla u_h^1\|^2\le\|\mathcal{J}_K^{-1}(\xi,\eta)\|_F^2\|\hat{\nabla}u_h^1\|^2,\qquad \|\hat{\nabla}u_h^1\|^2\le\|\mathcal{J}_K(\xi,\eta)\|_F^2\|\nabla u_h^1\|^2,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. So

$$|u_h^1|_{1,K}^2=\int_K\|\nabla u_h^1\|^2dxdy\le\int_{\hat{K}}\|\mathcal{J}_K^{-1}(\xi,\eta)\|_F^2 J_K(\xi,\eta)\|\hat{\nabla}u_h^1\|^2d\xi d\eta, \tag{3.6}$$

and

$$\int_{\hat{K}} \|\hat{\nabla} u_h^1\|^2 d\xi d\eta \leq \int_{\hat{K}} \|\mathcal{J}_K(\xi,\eta)\|_F^2 \|\nabla u_h^1\|^2 d\xi d\eta \leq \int_K \frac{\|\mathcal{J}_K(\xi,\eta)\|_F^2}{J_K(\xi,\eta)} \|\nabla u_h^1\|^2 dxdy. \quad (3.7)$$

Because of $\|\mathcal{J}_K(\xi,\eta)\|_F \leq Ch$ and (2.2), we obtain

$$\|\mathcal{J}_K^{-1}(\xi,\eta)\|_F^2 J_K(\xi,\eta) = \frac{\|\mathcal{J}_K(\xi,\eta)\|_F^2}{J_K(\xi,\eta)} \leq C.$$

This inequality together with (3.6) and (3.7) leads to

$$\frac{1}{C}|u_h^1|_{1,K}^2 \leq \int_{\hat{K}} \|\hat{\nabla} u_h^1\|^2 d\xi d\eta \leq C|u_h^1|_{1,K}^2. \quad (3.8)$$

Through direct calculation, we have

$$\int_{\hat{K}} \|\hat{\nabla} u_h^1\|^2 d\xi d\eta = \big[u_h^1(M_2) - u_h^1(M_4)\big]^2 + \big[u_h^1(M_3) - u_h^1(M_1)\big]^2.$$

Therefore $|||u_h^1|||_K$ and $|u_h^1|_{1,K}$ are equivalent by (3.8). □

Lemma 3.3. *Suppose that the partition $\mathcal{T}_h$ satisfies the conditions* (2.1)-(2.3)*, then for sufficiently small h, there exists a constant $C_0 > 0$, such that*

$$a_h(\mathbf{u}_h, \Gamma_h \mathbf{u}_h) \geq C_0 |\mathbf{u}_h|_{1,h}^2, \quad \forall \mathbf{u}_h \in \mathbf{U}_h. \quad (3.9)$$

Proof. Using the notations in Fig. 1, we rearrange the line integrals of the right-hand side of (2.9a) to get

$$a_h(\mathbf{u}_h, \Gamma_h \mathbf{u}_h) = \sum_{K \in \mathcal{T}_h} I_K(\mathbf{u}_h, \Gamma_h \mathbf{u}_h),$$

where

$$\begin{aligned} I_K(\mathbf{u}_h, \Gamma_h \mathbf{u}_h) = -\Big\{ & (\mathbf{u}_h|_K(P_1) - \mathbf{u}_h|_K(P_2)) \cdot \int_{\partial K_{P_1}^* \cap \partial K_{P_2}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds \\ & + (\mathbf{u}_h|_K(P_4) - \mathbf{u}_h|_K(P_3)) \cdot \int_{\partial K_{P_4}^* \cap \partial K_{P_3}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds \\ & + (\mathbf{u}_h|_K(P_4) - \mathbf{u}_h|_K(P_1)) \cdot \int_{\partial K_{P_4}^* \cap \partial K_{P_1}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds \\ & + (\mathbf{u}_h|_K(P_3) - \mathbf{u}_h|_K(P_2)) \cdot \int_{\partial K_{P_3}^* \cap \partial K_{P_2}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} ds \Big\}. \end{aligned}$$

Noticing that

$$\begin{aligned} &\mathbf{u}_h|_K(P_1) - \mathbf{u}_h|_K(P_2) = \mathbf{u}_h|_K(P_4) - \mathbf{u}_h|_K(P_3) = \mathbf{u}_h(M_4) - \mathbf{u}_h(M_2), \\ &\mathbf{u}_h|_K(P_4) - \mathbf{u}_h|_K(P_1) = \mathbf{u}_h|_K(P_3) - \mathbf{u}_h|_K(P_2) = \mathbf{u}_h(M_3) - \mathbf{u}_h(M_1), \end{aligned}$$

so

$$I_K(\mathbf{u}_h,\Gamma_h\mathbf{u}_h)=(\mathbf{u}_h(M_2)-\mathbf{u}_h(M_4))\cdot\int_{M_1}^{M_3}\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}ds-(\mathbf{u}_h(M_3)-\mathbf{u}_h(M_1))\cdot\int_{M_4}^{M_2}\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}ds.$$

It is easy to see by the geometry transformation and chain rule that on $\overline{M_1M_3}$ $(\xi=0)$

$$\begin{aligned}
&\int_{M_1}^{M_3}\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}ds=\int_{M_1}^{M_3}\frac{\partial\mathbf{u}_h}{\partial x}dy-\frac{\partial\mathbf{u}_h}{\partial y}dx\\
=&\int_{-1}^{1}\Big(\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial\eta}{\partial x}\Big)\Big(\frac{\partial y}{\partial\xi}d\xi+\frac{\partial y}{\partial\eta}d\eta\Big)-\Big(\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial\eta}{\partial y}\Big)\Big(\frac{\partial x}{\partial\xi}d\xi+\frac{\partial x}{\partial\eta}d\eta\Big)\\
=&\int_{-1}^{1}\frac{1}{J(0,\eta)}\Big[(d_2^2+c_2^2)\frac{\partial\mathbf{u}_h}{\partial\xi}+(-d_1d_2-c_1c_2-(d_{12}d_2+c_{12}c_2)\eta)\frac{\partial\mathbf{u}_h}{\partial\eta}\Big]d\eta,
\end{aligned}$$

and on $\overline{M_4M_2}$ $(\eta=0)$

$$\begin{aligned}
&\int_{M_4}^{M_2}\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}ds=\int_{M_4}^{M_2}\frac{\partial\mathbf{u}_h}{\partial x}dy-\frac{\partial\mathbf{u}_h}{\partial y}dx\\
=&\int_{-1}^{1}\Big(\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial\eta}{\partial x}\Big)\Big(\frac{\partial y}{\partial\xi}d\xi+\frac{\partial y}{\partial\eta}d\eta\Big)-\Big(\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial\eta}{\partial y}\Big)\Big(\frac{\partial x}{\partial\xi}d\xi+\frac{\partial x}{\partial\eta}d\eta\Big)\\
=&\int_{-1}^{1}\frac{1}{J(\xi,0)}\Big[(d_1d_2+c_1c_2+(d_{12}d_2+c_{12}c_2)\xi)\frac{\partial\mathbf{u}_h}{\partial\xi}+(-d_1^2-c_1^2)\frac{\partial\mathbf{u}_h}{\partial\eta}\Big]d\xi.
\end{aligned}$$

With

$$\frac{\partial\mathbf{u}_h}{\partial\xi}=\frac{1}{2}(\mathbf{u}_h(M_2)-\mathbf{u}_h(M_4))\quad\text{and}\quad\frac{\partial\mathbf{u}_h}{\partial\eta}=\frac{1}{2}(\mathbf{u}_h(M_3)-\mathbf{u}_h(M_1)),$$

we can rewrite the above bilinear form with quadric form

$$I_K(\mathbf{u}_h,\Gamma_h\mathbf{u}_h)=\alpha W\alpha^T=\frac{1}{2}\alpha(W+W^T)\alpha^T,$$

where $\alpha=(\mathbf{z}^1,\mathbf{z}^2)$ is a composite line vector with

$$\mathbf{z}^i=\big(u_h^i(M_2)-u_h^i(M_4),u_h^i(M_3)-u_h^i(M_1)\big),\quad i=1,2,$$

and the partition matrix is

$$W=\begin{pmatrix}A&0\\0&A\end{pmatrix}.$$

The entries of the matrix $A_{2\times2}$ are specified as follows

$$\begin{aligned}
&(A)_{1,1}=|M_1M_3|^2\int_I\frac{1}{J_K(0,\eta)}d\eta,\qquad(A)_{2,2}=|M_4M_2|^2\int_I\frac{1}{J_K(\xi,0)}d\xi,\\
&(A)_{1,2}=-\overrightarrow{M_1M_3}\cdot\overrightarrow{M_2M_4}\int_I\frac{1}{J_K(0,\eta)}d\eta-\frac{(\overrightarrow{P_2P_1}+\overrightarrow{P_4P_3})\cdot\overrightarrow{M_1M_3}}{2}\int_I\frac{\eta}{J_K(0,\eta)}d\eta,\\
&(A)_{2,1}=-\overrightarrow{M_1M_3}\cdot\overrightarrow{M_2M_4}\int_I\frac{1}{J_K(\xi,0)}d\xi-\frac{(\overrightarrow{P_2P_1}+\overrightarrow{P_4P_3})\cdot\overrightarrow{M_1M_3}}{2}\int_I\frac{\xi}{J_K(\xi,0)}d\xi.
\end{aligned}$$

Since each quadrilateral K in $\mathcal{T}_h$ satisfies quasi-parallel quadrilateral condition, we assume that K is a parallelogram first, then define

$$\tilde{A} = \frac{1}{2}(A + A^T),$$

as follows:

$$\begin{aligned}\tilde{A} =& \frac{4}{m(K)} \begin{pmatrix} |M_1M_3|^2 & -\overrightarrow{M_1M_3} \cdot \overrightarrow{M_4M_2} \\ -\overrightarrow{M_1M_3} \cdot \overrightarrow{M_4M_2} & |M_4M_2|^2 \end{pmatrix} \\ =& \frac{4m_1^2}{m(K)} \begin{pmatrix} 1 & -\kappa\cos\theta \\ -\kappa\cos\theta & \kappa^2 \end{pmatrix} = \frac{4m_1^2}{m(K)} \tilde{A}_\kappa.\end{aligned}$$

Obviously the matrix $\tilde{A}_\kappa$ is positive definite with its minimum eigenvalue $\lambda_\kappa > 0$. Using the regularity conditions (2.1) and noticing that $m(K) = m_1 m_2 \sin\theta_K$, we obtain

$$\lambda_{\min}(\tilde{A}) \geq \lambda_\kappa \frac{4m_1^2}{m(K)} \geq \lambda_\kappa \frac{4}{\kappa \sin\theta_K} \geq C\lambda_\kappa,$$

where $\lambda_{\min}(\tilde{A})$ denotes the minimum eigenvalue of $\tilde{A}$.

Next we should consider the difference between the matrix $\tilde{A}$ on a parallelogram and the matrix $(A + A^T)/2$ on an almost parallelogram. Set

$$D = \frac{1}{2}(A + A^T) - \tilde{A}.$$

Under the condition (2.3), we see from [16, 28] that

$$\lim_{h\to 0} \frac{|J_K(\xi,\eta)|}{m(K)} = 1,$$

which verifies that when h is small enough,

$$|(D)_{i,j}| \leq C\frac{h^3}{m(K)}, \quad 1 \leq i,j \leq 2.$$

Since the partition is quasi-uniform, then

$$\lambda_{\max}(D) \leq \|D\|_\infty \leq 2|(D)_{ij}| \leq Ch,$$

where $\lambda_{\max}(D)$ denotes the maximum eigenvalue of the matrix D.

Combining the results above with the definition (3.4), for sufficiently small h we have

$$I_K(\mathbf{u}_h, \Gamma_h \mathbf{u}_h) \geq C(C\lambda_\kappa - Ch)|||\mathbf{u}_h|||_K^2 \geq C|||\mathbf{u}_h|||_K^2.$$

By (3.5), summing over all quadrilaterals yields (3.9). $\square$

Lemma 3.4. *The pair $(\mathbf{U}_h,P_h)$ satisfies a uniform inf-sup condition, i.e., there holds*

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{U}_h}\frac{b_h(\Gamma_h\mathbf{v},q_h)}{|\mathbf{v}|_{1,h}}\geq C\|q_h\|_0,\quad \forall q_h\in P_h. \tag{3.10}$$

Proof. Let V_{bh} be the standard conforming bilinear finite element space and $\mathbf{V}_{bh}=V_{bh}\times V_{bh}$. For any $\mathbf{u}\in\mathbf{V}_{bh}$, we can see from the definition of Π_h that

$$\int_K \mathrm{div}(\mathbf{u}-\mathbf{\Pi}_h\mathbf{u})dxdy=0.$$

Then taking $\bar{\mathbf{v}}=\mathbf{\Pi}_h\mathbf{u}$, we have

$$(\mathrm{div}\mathbf{u},q_h)_K=(\mathrm{div}\bar{\mathbf{v}},q_h)_K,\quad (\mathrm{div}\mathbf{u},q_h)_\Omega=(\mathrm{div}\bar{\mathbf{v}},q_h)_\Omega,\quad \forall q_h\in P_h. \tag{3.11}$$

Therefore the pair $(\mathbf{U}_h,P_h)$ shares the same stable property of $(\mathbf{V}_{bh},P_h)$.

By the references [3,10,24], the pair $(\mathbf{V}_{bh},P_h)$ satisfies

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_{bh}}\frac{b'_h(\mathbf{v},q_h)}{|\mathbf{v}|_1}\geq C\|q_h\|_0,\quad \forall q_h\in P_h. \tag{3.12}$$

We can state an equivalent formulation of (3.12) as in [21]: for any $q_h\in P_h$, there is a function $\mathbf{u}\in\mathbf{V}_{bh}$ such that

$$(\mathrm{div}\mathbf{u},q_h)=\|q_h\|_0^2\quad\text{and}\quad |\mathbf{u}|_1\leq C\|q_h\|_0.$$

Then noticing

$$|\bar{\mathbf{v}}|_{1,h}\leq c|\mathbf{u}|_1,$$

we can complete the proof of the assertion by virtue of (3.1) and (3.11). ☐

Remark 3.1. Thanks to Lemma 3.1, (3.9) and (3.10) are equivalent to (see [3,10,29])

$$\begin{aligned}&\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,q_h)\in\mathbf{U}_h\times P_h}\frac{a_h(\mathbf{w}_h,\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,\chi_h)+b'_h(\mathbf{w}_h,q_h)}{|\mathbf{v}_h|_{1,h}+\|q_h\|_0}\\ \geq& C(|\mathbf{w}_h|_{1,h}+\|\chi_h\|_0),\quad \forall(\mathbf{w}_h,\chi_h)\in\mathbf{U}_h\times P_h.\end{aligned} \tag{3.13}$$

4 Error estimates

In this section, we will present error estimates for the finite volume element scheme (2.8a)-(2.8b). First we will derive two lemmas which are important to our error estimates.

Lemma 4.1. *Suppose $(\mathbf{u},p)\in(H_0^1(\Omega)\cap H^2(\Omega))^2\times(L_0^2(\Omega)\cap H^1(\Omega))$. Then*

$$|a_h(\mathbf{u}-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h)|\leq Ch|\mathbf{u}|_2|\mathbf{v}_h|_{1,h},\qquad \forall\mathbf{v}_h\in\mathbf{U}_h, \tag{4.1a}$$

$$|b_h(\Gamma_h\mathbf{v}_h,p-J_hp)|\leq Ch|p|_1|\mathbf{v}_h|_{1,h},\qquad \forall\mathbf{v}_h\in\mathbf{U}_h, \tag{4.1b}$$

$$|b'_h(\mathbf{u}-\Pi_h\mathbf{u},q_h)|\leq Ch|\mathbf{u}|_2|\|q_h\|_0,\qquad \forall q_h\in P_h. \tag{4.1c}$$

Proof. With the notations in Fig. 1, by (2.9a) we have

$$a_h(\mathbf{u}-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h} I_K(\mathbf{u}-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h),$$

where

$$\begin{aligned}
&I_K(\mathbf{u}-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h)\\
=&\frac{1}{2}(\mathbf{v}_h|_K(P_2)+\mathbf{v}_h|_K(P_3)-\mathbf{v}_h|_K(P_1)-\mathbf{v}_h|_K(P_4))\cdot\int_{\overline{M_1M_3}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds\\
&-\frac{1}{2}(\mathbf{v}_h|_K(P_3)+\mathbf{v}_h|_K(P_4)-\mathbf{v}_h|_K(P_1)-\mathbf{v}_h|_K(P_2))\cdot\int_{\overline{M_4M_2}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds\\
=&(\mathbf{v}_h(M_2)-\mathbf{v}_h(M_4))\cdot\int_{\overline{M_1M_3}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds\\
&-(\mathbf{v}_h(M_3)-\mathbf{v}_h(M_1))\cdot\int_{\overline{M_4M_2}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds.
\end{aligned}$$

It follows from the definitions (3.4) and (3.5) that

$$\|\mathbf{v}_h(M_2)-\mathbf{v}_h(M_4)\|\le|\mathbf{v}_h|_{1,K},\qquad\|\mathbf{v}_h(M_3)-\mathbf{v}_h(M_1)\|\le|\mathbf{v}_h|_{1,K}.\tag{4.2}$$

Using the Cauchy-Schwarz inequality and the trace theorem, by (2.6a) we can show that

$$\begin{aligned}
&\Big\|\int_{\overline{M_1M_3}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds\Big\|\le Ch^{\frac{1}{2}}\big(|\mathbf{u}-\Pi_h\mathbf{u}|_{H^1(\overline{M_1M_3})}\big)\\
\le&Ch^{\frac{1}{2}}\big(|\mathbf{u}-\Pi_h\mathbf{u}|_{1,K}^{\frac{1}{2}}|\mathbf{u}-\Pi_h\mathbf{u}|_{2,K}^{\frac{1}{2}}\big)\\
\le&Ch|\mathbf{u}|_{2,K}.
\end{aligned}\tag{4.3}$$

Similarly,

$$\Big\|\int_{\overline{M_4M_2}}\frac{\partial(\mathbf{u}-\Pi_h\mathbf{u})}{\partial\mathbf{n}}ds\Big\|\le Ch|\mathbf{u}|_{2,K}.$$

Therefore,

$$|I_K(\mathbf{u}-\Pi_h\mathbf{u},\mathbf{v}_h)|\le Ch|\mathbf{u}|_{2,K}|\mathbf{v}_h|_{1,K},$$

which leads to (4.1a) by summing over K. In the same way, (3.3) implies that

$$\begin{aligned}
b_{h,K}(\Gamma_h\mathbf{v}_h,p-J_hp)=&-(\mathbf{v}_h(M_2)-\mathbf{v}_h(M_4))\cdot\int_{\overline{M_1M_3}}(p-J_hp)\mathbf{n}ds\\
&+(\mathbf{v}_h(M_3)-\mathbf{v}_h(M_1))\cdot\int_{\overline{M_4M_2}}(p-J_hp)\mathbf{n}ds.
\end{aligned}$$

And using similar approach in (4.3) we can show

$$\Big\| \int_{\overline{M_1M_3}} (p - J_h p)\mathbf{n}ds \Big\| \le Ch|p|_{1,K}, \qquad \Big\| \int_{\overline{M_4M_2}} (p - J_h p)\mathbf{n}ds \Big\| \le Ch|p|_{1,K}.$$

Owing to (4.2) and (2.7), there holds

$$|b_{h,K}(\Gamma_h \mathbf{v}_h, p - J_h p)| \le Ch|p|_{1,K}|\mathbf{v}_h|_{1,K}.$$

Then (4.1b) follows. At last, the Cauchy-Schwarz inequality and the interpolation estimate (2.6a) lead to

$$\begin{aligned}|b_h'(\mathbf{u} - \Pi_h\mathbf{u}, q_h)| &= \Big|\sum_{K\in\mathcal{T}_h} (\mathrm{div}(\mathbf{u} - \Pi_h\mathbf{u}), q_h)_K\Big| \\ &\le \sum_{K\in\mathcal{T}_h} |\mathbf{u} - \Pi_h\mathbf{u}|_{1,K}\|q_h\|_{0,K} \le Ch|\mathbf{u}|_2\|q_h\|_0.\end{aligned}$$

So, the lemma is proved. □

Lemma 4.2. *Under the hypothesis of Lemma 4.1, we have*

$$|\hat{a}_h(\mathbf{u}, \Gamma_h\mathbf{v}_h)| \le Ch|\mathbf{u}|_2|\mathbf{v}_h|_{1,h}, \qquad \forall \mathbf{v}_h \in \mathbf{U}_h, \tag{4.4a}$$

$$|\hat{b}_h(\Gamma_h\mathbf{v}_h, p)| \le Ch|\mathbf{v}_h|_{1,h}|p|_1, \qquad \forall \mathbf{v}_h \in \mathbf{U}_h, \tag{4.4b}$$

where

$$\hat{a}_h(\mathbf{u}, \Gamma_h\mathbf{v}_h) = -\sum_{K\in\mathcal{T}_h}\sum_{P\in V(K)} (v_h|_K)(P) \cdot \int_{\partial K\cap K_P^*} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} ds, \tag{4.5a}$$

$$\hat{b}_h(\Gamma_h\mathbf{v}_h, p) = \sum_{K\in\mathcal{T}_h}\sum_{P\in V(K)} (v_h|_K)(P) \cdot \int_{\partial K\cap K_P^*} p\mathbf{n}ds. \tag{4.5b}$$

Proof. Rearrange the line integrals of the right-hand side of (4.5a) to give

$$\hat{a}_h(\mathbf{u}, \Gamma_h\mathbf{v}_h) = -\sum_{e\in\mathscr{E}} \hat{I}_e(\mathbf{u}, \Gamma_h\mathbf{v}_h),$$

where $\mathscr{E}$ is the set of the interior edges in $\mathcal{T}_h$.

We only have to discuss the left situation as shown in Fig. 5. The right situation in Fig. 5 is similar. Let P_1, P_2 be the two nodes of the interior edge e, M_{12} be the midpoint of e and K_e^1, K_e^2 be the two primal cells which share the common edge e, then

$$\begin{aligned}\hat{I}_e(\mathbf{u}, \mathbf{v}_h) =&\Big((\mathbf{v}_h|_{K_e^1})(P_1) - (\mathbf{v}_h|_{K_e^2})(P_1)\Big) \cdot \int_{P_1}^{M_{12}} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} ds + \Big((\mathbf{v}_h|_{K_e^1})(P_2) - (\mathbf{v}_h|_{K_e^2})(P_2)\Big) \cdot \int_{M_{12}}^{P_2} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} ds \\ =&\Big(\mathbf{v}_h(M_1) - \mathbf{v}_h(M_2) + \mathbf{v}_h(M_4) - \mathbf{v}_h(M_3)\Big) \cdot \Big(\int_{P_1}^{M_{12}} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} ds - \int_{M_{12}}^{P_2} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} ds\Big),\end{aligned}$$

where M_{2i-1} and M_{2i} are the midpoints of adjacent edges of e in K_e^i, $i=1,2$. Using Lemma 3.2, we get

$$\big\|\mathbf{v}_h(M^1_{K_e^i})-\mathbf{v}_h(M^2_{K_e^i})\big\|\leq|\mathbf{v}_h|_{1,K_e^i}\leq|\mathbf{v}_h|_{1,h},\quad i=1,2. \tag{4.6}$$

By Lemma 4.5.3 in [2] and mean value theorems of integral and differential, we obtain

$$\Big\|\int_{P_1}^{M_{12}}\frac{\partial\mathbf{u}}{\partial\mathbf{n}}ds-\int_{M_{12}}^{P_2}\frac{\partial\mathbf{u}}{\partial\mathbf{n}}ds\Big\|\leq Ch^2|U|_{2,\infty}\leq Ch|U|_2,$$

which together with (4.6) implies (4.4a). A similar argument yields (4.4b). □

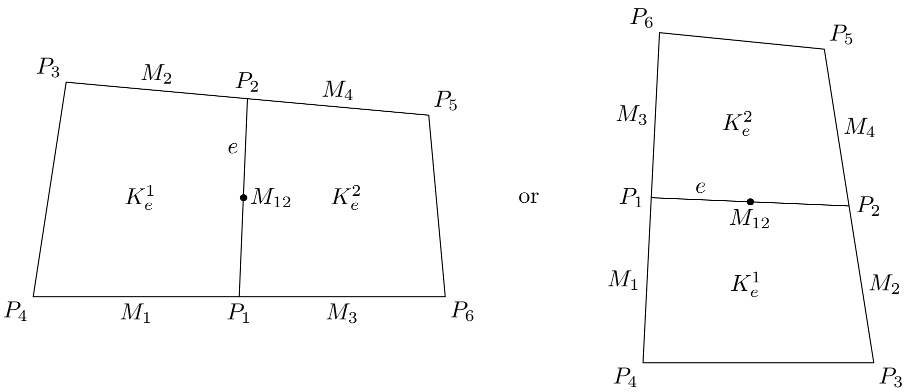

Figure 5: The elements K_e^1 and K_e^2.

From Lemmas 4.1 and 4.2, we can prove the following error estimate result.

Theorem 4.1. *Let the pair* $(\mathbf{u},p)\in(H_0^1(\Omega)\cap H^2(\Omega))^2\times(L_0^2(\Omega)\cap H^1(\Omega))$ *be the solution of* (1.1a)-(1.1c) *and* $(\mathbf{u}_h,p_h)\in\mathbf{U}_h\times P_h$ *be the solution of* (2.8a)-(2.8b). *If the conditions* (2.1)-(2.3) *hold, there exists a positive constant C independent of h such that*

$$|\mathbf{u}-\mathbf{u}_h|_{1,h}+\|p-p_h\|_0\leq Ch(\|\mathbf{u}\|_2+\|p\|_1). \tag{4.7}$$

Proof. Replacing $\mathbf{w}_h$ by $\mathbf{u}_h-\Pi_h\mathbf{u}$ and χ_h by p_h-J_hp in (3.13), respectively, we have

$$\begin{aligned}&C(|\mathbf{u}_h-\Pi_h\mathbf{u}|_{1,h}+\|p_h-J_hp\|_0)\\ \leq&\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,q_h)\in\mathbf{U}_h\times P_h}\frac{a_h(\mathbf{u}_h-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,p_h-J_hp)+b_h'(\mathbf{u}_h-\Pi_h\mathbf{u},q_h)}{|\mathbf{v}_h|_{1,h}+\|q_h\|_0}.\end{aligned} \tag{4.8}$$

For $(\mathbf{v}_h,q_h)\in\mathbf{U}_h\times P_h$, from (2.8a)-(2.8b), there holds

$$a_h(\mathbf{u}_h,\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,p_h)+b_h'(\mathbf{u}_h,q_h)=(\mathbf{f},\Gamma_h\mathbf{v}_h). \tag{4.9}$$

Multiplying (1.1a) by $\Gamma_h\mathbf{v}_h$ and (1.1b) by q_h, respectively, we can show that

$$a_h(\mathbf{u},\Gamma_h\mathbf{v}_h)+\hat{a}_h(\mathbf{u},\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,p)+\hat{b}_h(\Gamma_h\mathbf{v}_h,p)+b_h'(\mathbf{u},q_h)=(\mathbf{f},\Gamma_h\mathbf{v}_h). \tag{4.10}$$

Subtracting (4.10) from (4.9) gives

$$a_h(\mathbf{u}_h-\mathbf{u},\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,p_h-p)+b'_h(\mathbf{u}_h-\mathbf{u},q_h)+\hat{a}_h(\mathbf{u},\Gamma_h\mathbf{v}_h)+\hat{b}_h(\Gamma_h\mathbf{v}_h,p)=0. \tag{4.11}$$

Combine (4.8) with (4.11) to get

$$\begin{aligned}
&C(|\mathbf{u}_h-\Pi_h\mathbf{u}|_{1,h}+\|p_h-J_hp\|_0)\\
\le&\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,q_h)\in\mathbf{U}_h\times P_h}\frac{a_h(\mathbf{u}-\Pi_h\mathbf{u},\Gamma_h\mathbf{v}_h)+b_h(\Gamma_h\mathbf{v}_h,p-J_hp)+b'_h(\mathbf{u}-\Pi_h\mathbf{u},q_h)}{|\mathbf{v}_h|_{1,h}+\|q_h\|_0}\\
&+\frac{\hat{a}_h(\mathbf{u},\Gamma_h\mathbf{v}_h)+\hat{b}_h(\Gamma_h\mathbf{v}_h,p)}{|\mathbf{v}_h|_{1,h}+\|q_h\|_0}.
\end{aligned}$$

Using Lemmas 4.1 and 4.2 gives

$$|\mathbf{u}_h-\Pi_h\mathbf{u}|_{1,h}+\|p_h-J_hp\|_0\le Ch\Big(|\mathbf{u}|_2+|p|_1\Big).$$

Finally, an application of the triangle inequality and the interpolation estimates imply

$$\begin{aligned}
&|\mathbf{u}-\mathbf{u}_h|_{1,h}+\|p-p_h\|_0\\
\le&|\mathbf{u}-\Pi_h\mathbf{u}|_{1,h}+|\Pi_h\mathbf{u}-\mathbf{u}_h|_{1,h}+\|p-J_hp\|_0+\|J_hp-p_h\|_0\\
\le&Ch\Big(\|\mathbf{u}\|_2+\|p\|_1\Big).
\end{aligned}$$

Therefore, the proof is completed. □

To establish the error estimate in the L^2 norm, we consider the following dual problem,

$$\begin{aligned}
&-\Delta\mathbf{\Phi}+\nabla\Psi=\mathbf{u}-\mathbf{u_h}, &&\text{in }\ \Omega, &(4.12a)\\
&\text{div}\mathbf{\Phi}=0, &&\text{in }\ \Omega, &(4.12b)\\
&\mathbf{\Phi}=0, &&\text{on }\ \partial\Omega. &(4.12c)
\end{aligned}$$

Assume that the problem (4.12a)-(4.12c) is regular, i.e.,

$$\|\mathbf{\Phi}\|_2+\|\Psi\|_1\le C\|\mathbf{u}-\mathbf{u}_h\|_0. \tag{4.13}$$

Lemma 4.3. *Suppose* (2.3) *is satisfied, then for any* $\mathbf{v}_h\in\mathbf{U}_h$, *we have*

$$\Big\|\int_K(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy\Big\|\le Ch^3|\mathbf{v}_h|_{1,K}. \tag{4.14}$$

Proof. Exact computations show that (see Fig. 1)

$$\int_K(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy=\frac{1}{3}J_1[\mathbf{v}_h(M_4)-\mathbf{v}_h(M_2)]+\frac{1}{3}J_2[\mathbf{v}_h(M_1)-\mathbf{v}_h(M_3)],$$

which together with (2.4b) and (3.4) yields (4.14). □

Lemma 4.4. *For any* $\mathbf{w}\in H_0^1(\Omega)^2\cup\mathbf{U}_h$, *there hold*

$$\Big|\sum_{K\in\mathcal{T}_h}\langle\frac{\partial\mathbf{v}}{\partial\mathbf{n}},\mathbf{w}\rangle_{\partial K}\Big|\le Ch\|\mathbf{v}\|_2|\mathbf{w}|_{1,h},\quad \forall\mathbf{v}\in(H_0^1(\Omega)\cap H^2(\Omega))^2, \tag{4.15a}$$

$$\Big|\sum_{K\in\mathcal{T}_h}\langle q,\mathbf{w}\cdot\mathbf{n}\rangle_{\partial K}\Big|\le Ch\|q\|_1|\mathbf{w}|_{1,h},\quad \forall q\in H^1(\Omega). \tag{4.15b}$$

Proof. Similar to [4] and [5], (4.15a)-(4.15b) can follow from the approximation properties, the Cauchy-Schwartz inequality and a standard trace theorem. □

Lemma 4.5. *Suppose* (2.3) *is satisfied, then for any* $\mathbf{f}\in H^1(\Omega)^2$, *there exists a positive constant* C *independent of* h, *such that*

$$|(\mathbf{f},\Pi_h\mathbf{\Phi}-\Gamma_h\Pi_h\mathbf{\Phi})|\le Ch^2\|\mathbf{f}\|_1\|\mathbf{\Phi}\|_2,\quad \forall\mathbf{\Phi}\in H_0^1(\Omega)^2. \tag{4.16}$$

Proof. The operator π_0 is defined by

$$\pi_0\mathbf{f}|_K:=\frac{1}{m(K)}\int_K\mathbf{f}(x,y)dxdy,\quad \forall K\in\mathcal{T}_h,$$

which gives

$$\|\mathbf{f}-\pi_0\mathbf{f}\|_0\le Ch\|\mathbf{f}\|_1\quad\text{and}\quad\|\pi_0\mathbf{f}|_K\|\le Ch^{-1}\|\mathbf{f}\|_{0,K}.$$

Employing (2.6b), we have

$$\begin{aligned}
&|(\mathbf{f},\Pi_h\mathbf{\Phi}-\Gamma_h\mathbf{\Pi}_h\mathbf{\Phi})|\\
\le&|(\mathbf{f}-\pi_0\mathbf{f},\Pi_h\mathbf{\Phi}-\Gamma_h\mathbf{\Pi}_h\mathbf{\Phi})|+|(\pi_0\mathbf{f},\Pi_h\mathbf{\Phi}-\Gamma_h\mathbf{\Pi}_h\mathbf{\Phi})|\\
\le&\|\mathbf{f}-\pi_0\mathbf{f}\|_0\|\Pi_h\mathbf{\Phi}-\Gamma_h\mathbf{\Pi}_h\mathbf{\Phi}\|_0+\sum_{K\in\mathcal{T}_h}\|\pi_0\mathbf{f}|_K\|\Big\|\int_K(\Pi_h\mathbf{\Phi}-\Gamma_h\mathbf{\Pi}_h\mathbf{\Phi})dxdy\Big\|\\
\le&Ch^2(\|\mathbf{f}\|_1|\Pi_h\mathbf{\Phi}|_{1,h}+\|\mathbf{f}\|_0|\Pi_h\mathbf{\Phi}|_{1,h})\\
\le&Ch^2\|\mathbf{f}\|_1\|\mathbf{\Phi}\|_2.
\end{aligned}$$

Thus, the proof is completed. □

Define the following parameters:

$$\begin{aligned}
A_{11}&=\int_{-1}^0\frac{1}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}\ln\frac{J_0-J_2}{J_0-J_2-J_1}, & A_{12}&=\int_{-1}^0\frac{1}{J_K(\xi,1)}d\xi=\frac{1}{J_1}\ln\frac{J_0+J_2}{J_0+J_2-J_1},\\
A_{21}&=\int_0^1\frac{1}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}\ln\frac{J_0-J_2+J_1}{J_0-J_2}, & A_{22}&=\int_0^1\frac{1}{J_K(\xi,1)}d\xi=\frac{1}{J_1}\ln\frac{J_0+J_2+J_1}{J_0+J_2},\\
B_{11}&=\int_{-1}^0\frac{\xi}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}(1-(J_0-J_2)A_{11}), & B_{12}&=\int_{-1}^0\frac{\xi}{J_K(\xi,1)}d\xi=\frac{1}{J_1}(1-(J_0+J_2)A_{12}),\\
B_{21}&=\int_0^1\frac{\xi}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}(1-(J_0-J_2)A_{21}), & B_{22}&=\int_0^1\frac{\xi}{J_K(\xi,1)}d\xi=\frac{1}{J_1}(1-(J_0+J_2)A_{22}),\\
C_{11}&=\int_{-1}^0\frac{\xi^2}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}(-\frac{1}{2}-(J_0-J_2)B_{11}), & C_{12}&=\int_{-1}^0\frac{\xi^2}{J_K(\xi,1)}d\xi=\frac{1}{J_1}(-\frac{1}{2}-(J_0+J_2)B_{12}),\\
C_{21}&=\int_0^1\frac{\xi^2}{J_K(\xi,-1)}d\xi=\frac{1}{J_1}(\frac{1}{2}+(J_0-J_2)B_{21}), & C_{22}&=\int_0^1\frac{\xi^2}{J_K(\xi,1)}d\xi=\frac{1}{J_1}(\frac{1}{2}+(J_0+J_2)B_{22}).
\end{aligned}$$

According to (2.4b) and using Taylor expansion of $\ln(1+x)$, $(x\to 0)$, one can get

$$A_{11}-A_{21}=\mathcal{O}(h^{-1}),\quad A_{12}-A_{22}=\mathcal{O}(h^{-1}),\quad A_{11}-A_{21}-A_{12}+A_{22}=\mathcal{O}(1), \tag{4.17a}$$
$$B_{11}+B_{21}=\mathcal{O}(h^{-1}),\quad B_{12}+B_{22}=\mathcal{O}(h^{-1}),\quad B_{11}+B_{21}-B_{12}-B_{22}=\mathcal{O}(1), \tag{4.17b}$$
$$B_{11}-B_{21}=\mathcal{O}(h^{-2}),\quad B_{12}-B_{22}=\mathcal{O}(h^{-2}),\quad B_{11}-B_{21}-B_{12}+B_{22}=\mathcal{O}(h^{-1}), \tag{4.17c}$$
$$C_{11}+C_{21}=\mathcal{O}(h^{-2}),\quad C_{12}+C_{22}=\mathcal{O}(h^{-2}),\quad C_{11}+C_{21}-C_{12}-C_{22}=\mathcal{O}(h^{-1}). \tag{4.17d}$$

Lemma 4.6. *Suppose* (2.3) *holds, there exists a positive constant C independent of h, such that for any $\mathbf{u}_h,\mathbf{v}_h\in\mathbf{U}_h$,*

$$|a_h(\mathbf{u_h},\Gamma_h\mathbf{v}_h)-a(\mathbf{u_h},\mathbf{v}_h)|\le Ch^2|\mathbf{u}_h|_{1,h}|\mathbf{v}_h|_{1,h}. \tag{4.18}$$

Proof. Making use of integral by parts, we have (see [6])

$$a_h(\mathbf{u_h},\Gamma_h\mathbf{v}_h)-a(\mathbf{u_h},\mathbf{v}_h):=E_1+E_2,$$

where

$$\begin{aligned}
E_1&=\sum_{K\in\mathcal{T}_h}\int_K\Delta\mathbf{u}_h\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy=\sum_{K\in\mathcal{T}_h}\int_K\Big(\frac{\partial^2\mathbf{u}_h}{\partial x^2}+\frac{\partial^2\mathbf{u}_h}{\partial y^2}\Big)\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy\\
&=\sum_{K\in\mathcal{T}_h}\Big\{\int_K\frac{\partial^2\mathbf{u}_h}{\partial x^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy+\int_K\frac{\partial^2\mathbf{u}_h}{\partial y^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy\Big\},\\
E_2&=-\int_{\partial K}\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)ds\\
&=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\Big[-\frac{\partial\mathbf{u}_h}{\partial x}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dy+\frac{\partial\mathbf{u}_h}{\partial y}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dx\Big].
\end{aligned}$$

Using the chain rule of differentiation, we get

$$\begin{aligned}
E_1=&\sum_{K\in\mathcal{T}_h}\int_K\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy+\sum_{K\in\mathcal{T}_h}\int_K\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy\\
&+\sum_{K\in\mathcal{T}_h}\int_K\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial^2\xi}{\partial y^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy+\sum_{K\in\mathcal{T}_h}\int_K\frac{\partial\mathbf{u}_h}{\partial\eta}\frac{\partial^2\eta}{\partial y^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)dxdy\\
:=&I_1+I_2+I_3+I_4.
\end{aligned}$$

We shall only estimate the first term of E_1, the estimate of the rest is analogous. Set

$$\overline{\frac{\partial^2\xi}{\partial x^2}}=\frac{\partial^2\xi}{\partial x^2}\Big|_{(\xi=0,\eta=0)}.$$

It can be verified that

$$\begin{aligned}
I_1&=\sum_{K\in\mathcal{T}_h}\int_{\hat K}\frac{\partial\mathbf{u}_h}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}\cdot(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)J_Kd\xi d\eta\\
&=\sum_{K\in\mathcal{T}_h}\frac{\partial\mathbf{u}_h}{\partial\xi}\cdot\Big\{\int_{\hat K}\Big(\frac{\partial^2\xi}{\partial x^2}-\overline{\frac{\partial^2\xi}{\partial x^2}}\Big)(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)J_Kd\xi d\eta\ +\overline{\frac{\partial^2\xi}{\partial x^2}}(\mathbf{v}_h-\Gamma_h\mathbf{v}_h)J_Kd\xi d\eta\Big\}.
\end{aligned}$$

By the definitions (3.4) and (3.5), we obtain

$$\Big\|\frac{\partial \mathbf{u}_h}{\partial \xi}\Big|_K\Big\| = \frac{1}{2}\|\mathbf{u}_h(M_2) - \mathbf{u}_h(M_4)\| \le C|\mathbf{u}_h|_{1,K}. \tag{4.19}$$

Notice that

$$\Big|\frac{\partial^2 \xi}{\partial x^2}\Big|_{\infty,K} \le Ch^{-1} \quad \text{and} \quad \Big|\Big(\frac{\partial^2 \xi}{\partial x^2} - \overline{\frac{\partial^2 \xi}{\partial x^2}}\Big)J_K\Big|_{\infty,K} \le Ch^2,$$

from Proposition 2 and Proposition 7 in [19], where $|\cdot|_{\infty,K}$ is the norm of the space $L^\infty(K)$. And use the Cauchy-Schwarz inequality and (4.14) to give

$$\begin{aligned}
&\Big|\sum_{K\in\mathcal{T}_h}\frac{\partial \mathbf{u}_h}{\partial \xi}\cdot\int_{\hat{K}}\Big(\frac{\partial^2 \xi}{\partial x^2} - \overline{\frac{\partial^2 \xi}{\partial x^2}}\Big)(\mathbf{v}_h - \Gamma_h\mathbf{v}_h)J_K d\xi d\eta\Big| \\
\le & Ch^2\sum_{K\in\mathcal{T}_h}|\mathbf{u}_h|_{1,K}\|\mathbf{v}_h - \Gamma_h\mathbf{v}_h\|_{0,\hat{K}} \le Ch^2|\mathbf{u}_h|_{1,h}|\mathbf{v}_h|_{1,h}, \\
&\Big|\sum_{K\in\mathcal{T}_h}\frac{\partial \mathbf{u}_h}{\partial \xi}\cdot\int_{\hat{K}}\overline{\frac{\partial^2 \xi}{\partial x^2}}(\mathbf{v}_h - \Gamma_h\mathbf{v}_h)J_K d\xi d\eta\Big| \\
\le & Ch^{-1}\sum_{K\in\mathcal{T}_h}|\mathbf{u}_h|_{1,K}\Big|\int_{\hat{K}}(\mathbf{v}_h - \Gamma_h\mathbf{v}_h)J_K d\xi d\eta\Big| \le Ch^2|\mathbf{u}_h|_{1,h}|\mathbf{v}_h|_{1,h}.
\end{aligned}$$

Hence, we get

$$|E_1| \le Ch^2|\mathbf{u}_h|_{1,h}|\mathbf{v}_h|_{1,h}.$$

To estimate E_2, set

$$E_{\partial K,ij} = \int_{\overline{P_iP_j}}\Big[-\frac{\partial \mathbf{u}_h}{\partial x}\cdot(\mathbf{v}_h - \Gamma_h\mathbf{v}_h)dy + \frac{\partial \mathbf{u}_h}{\partial y}\cdot(\mathbf{v}_h - \Gamma_h\mathbf{v}_h)dx\Big].$$

It can be verified that

$$E_2 = \sum_{K\in\mathcal{T}_h}(E_{\partial K,12} + E_{\partial K,23} + E_{\partial K,34} + E_{\partial K,41}).$$

We consider the sum of integrals on one pair of opposite edges: $E_{\partial K,12} + E_{\partial K,34}$. Write

$$\begin{aligned}
&e_1 = d_1d_2 + c_1c_2, \quad e_2 = d_1d_{12} + c_1c_{12}, \quad e_3 = d_2d_{12} + c_2c_{12}, \\
&e_4 = d_{12}^2 + c_{12}^2, \qquad e_5 = d_1^2 + c_1^2.
\end{aligned}$$

Then

$$\begin{aligned}
E_{\partial K,12} + E_{\partial K,34} =&\frac{\partial \mathbf{u}_h}{\partial \xi}\Big|_K\cdot[e_1(A_{11} - A_{21} - A_{12} + A_{22} + B_{11} + B_{21} - B_{12} - B_{22}) \\
&- e_3(A_{11} - A_{21} + A_{12} - A_{22} + B_{11} + B_{21} + B_{12} + B_{22}) \\
&+ e_2(B_{11} - B_{21} - B_{12} + B_{22} + C_{11} + C_{21} - C_{12} - C_{22}) \\
&- e_4(B_{11} - B_{21} + B_{12} - B_{22} + C_{11} + C_{21} + C_{12} + C_{22})]\frac{\partial \mathbf{v}_h}{\partial \xi}\Big|_K \\
&+ \frac{\partial \mathbf{u}_h}{\partial \eta}\Big|_K\cdot[2e_2(A_{11} - A_{21} + A_{12} - A_{22} + B_{11} + B_{21} + B_{12} + B_{22}) \\
&- (e_5 + e_4)(A_{11} - A_{21} - A_{12} + A_{22} + B_{11} + B_{21} - B_{12} - B_{22})]\frac{\partial \mathbf{v}_h}{\partial \xi}\Big|_K,
\end{aligned}$$

which, combined with (4.17a)-(4.17d) and (4.19), implies

$$|E_{\partial K,12} + E_{\partial K,34}| \le Ch^2 |\mathbf{u}_h|_{1,K} |\mathbf{v}_h|_{1,K}.$$

Similarly, we can show

$$|E_{\partial K,23} + E_{\partial K,41}| \le Ch^2 |\mathbf{u}_h|_{1,K} |\mathbf{v}_h|_{1,K}.$$

Gather over all elements to yield

$$|E_2| \le Ch^2 |\mathbf{u}_h|_{1,h} |\mathbf{v}_h|_{1,h}.$$

Thus, the desired result is obtained. □

From Lemmas 4.4-4.6, the result of the L^2-error estimate for the velocity can be proved as follows.

Theorem 4.2. *Let* $(\mathbf{u}, p) \in (H_0^1(\Omega) \cap H^2(\Omega))^2 \times (L_0^2(\Omega) \cap H^1(\Omega))$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times P_h$ *be the solutions of* (1.1a)-(1.1c) *and* (2.8a)-(2.8b)*, respectively. If* $\mathbf{f} \in H^1(\Omega)^2$*, then*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^2 \Big(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{f}\|_1 \Big). \tag{4.20}$$

Proof. Multiplying (4.12a) and (4.12b) by $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$, respectively, integrating by parts and using Green's formula on each element, we can get

$$\begin{aligned} \|\mathbf{u} - \mathbf{u_h}\|_0^2 =& a(\mathbf{u} - \mathbf{u_h}, \mathbf{\Phi}) + b(\mathbf{u} - \mathbf{u_h}, \Psi) + b(\mathbf{\Phi}, p - p_h) \\ & - \sum_{K \in \mathcal{T}_h} \Big\langle \frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}}, \mathbf{u} - \mathbf{u_h} \Big\rangle_{\partial K} + \sum_{K \in \mathcal{T}_h} \langle \Psi, (\mathbf{u} - \mathbf{u_h}) \cdot \mathbf{n} \rangle_{\partial K}. \end{aligned} \tag{4.21}$$

Replacing $(\mathbf{v_h}, q_h)$ by $(\Gamma_h \Pi_h \mathbf{\Phi}, J_h \Psi)$ in (2.8a)-(2.8b) yields

$$a_h(\mathbf{u}_h, \Gamma_h \Pi_h \mathbf{\Phi}) + b_h(\Gamma_h \Pi_h \mathbf{\Phi}, p_h) + b_h'(\mathbf{u}_h, J_h \Psi) = (\mathbf{f}, \Gamma_h \Pi_h \mathbf{\Phi}). \tag{4.22}$$

Multiply (1.1a) and (1.1b) by $\Pi_h \mathbf{\Phi}$ and $J_h \Psi$ with Green's formula on each element to show that

$$\begin{aligned} a(\mathbf{u}, \Pi_h \mathbf{\Phi}) + b(\Pi_h \mathbf{\Phi}, p) + b(\mathbf{u}, J_h \Psi) - \sum_{K \in \mathcal{T}_h} \Big\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}}, \Pi_h \mathbf{\Phi} \Big\rangle_{\partial K} & \\ + \sum_{K \in \mathcal{T}_h} \langle p, \Pi_h \mathbf{\Phi} \cdot \mathbf{n} \rangle_{\partial K} = (\mathbf{f}, \Pi_h \mathbf{\Phi}). & \end{aligned} \tag{4.23}$$

Note that

$$\sum_{K \in \mathcal{T}_h} \Big\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}}, \mathbf{\Phi} \Big\rangle_{\partial K} = 0, \qquad \sum_{K \in \mathcal{T}_h} \langle p, \mathbf{\Phi} \cdot \mathbf{n} \rangle_{\partial K} = 0. \tag{4.24}$$

Applying (4.22)-(4.24), (4.21) can be rewritten as

$$\begin{aligned}\|\mathbf{u}-\mathbf{u_h}\|_0^2 =&a(\mathbf{u}-\mathbf{u_h},\mathbf{\Phi}-\Pi_h\mathbf{\Phi})+b(\mathbf{u}-\mathbf{u_h},\Psi-J_h\Psi)+b(\mathbf{\Phi}-\Pi_h\mathbf{\Phi},p-p_h)\\&+\sum_{K\in\mathcal{T}_h}\Big\langle\frac{\partial\mathbf{u}}{\partial\mathbf{n}},\mathbf{\Phi}-\Pi_h\mathbf{\Phi}\Big\rangle_{\partial K}-\sum_{K\in\mathcal{T}_h}\langle p,(\mathbf{\Phi}-\Pi_h\mathbf{\Phi})\cdot\mathbf{n}\rangle_{\partial K}\\&-\sum_{K\in\mathcal{T}_h}\Big\langle\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}},\mathbf{u}-\mathbf{u_h}\Big\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\langle\Psi,(\mathbf{u}-\mathbf{u_h})\cdot\mathbf{n}\rangle_{\partial K}\\&-a(\mathbf{u_h},\Pi_h\mathbf{\Phi})+a_h(\mathbf{u_h},\Gamma_h\Pi_h\mathbf{\Phi})+(\mathbf{f},\Pi_h\mathbf{\Phi}-\Gamma_h\Pi_h\mathbf{\Phi}).\end{aligned}\tag{4.25}$$

Since the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, with (2.6a), (2.7) and (4.7) we can deduce that

$$\begin{aligned}&|a(\mathbf{u}-\mathbf{u_h},\mathbf{\Phi}-\Pi_h\mathbf{\Phi})+b(\mathbf{u}-\mathbf{u_h},\Psi-J_h\Psi)+b(\mathbf{\Phi}-\Pi_h\mathbf{\Phi},p-p_h)|\\\le&C\Big(|\mathbf{u}-\mathbf{u_h}|_{1,h}+\|p-p_h\|_0\Big)\Big(|\mathbf{\Phi}-\Pi_h\mathbf{\Phi}|_{1,h}+\|\Psi-J_h\Psi\|_0\Big)\\\le&Ch^2\Big(\|\mathbf{u}\|_2+\|p\|_1\Big)\Big(\|\mathbf{\Phi}\|_2+\|\Psi\|_1\Big).\end{aligned}$$

It can also be seen from (4.15a)-(4.15b) and (4.7) that

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{T}_h}\Big\langle\frac{\partial\mathbf{u}}{\partial\mathbf{n}},\mathbf{\Phi}-\Pi_h\mathbf{\Phi}\Big\rangle_{\partial K}-\sum_{K\in\mathcal{T}_h}\langle p,\mathbf{\Phi}-\Pi_h\mathbf{\Phi}\cdot\mathbf{n}\rangle_{\partial K}\Big|\\\le&Ch(\|\mathbf{u}\|_2+\|p\|_1)|\mathbf{\Phi}-\Pi_h\mathbf{\Phi}|_{1,h}\le Ch^2(\|\mathbf{u}\|_2+\|p\|_1)\|\mathbf{\Phi}\|_2,\\&\Big|\sum_{K\in\mathcal{T}_h}\Big\langle\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}},\mathbf{u}-\mathbf{u_h}\Big\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\langle\Psi,(\mathbf{u}-\mathbf{u_h})\cdot\mathbf{n}\rangle_{\partial K}\Big|\\\le&Ch(\|\mathbf{\Phi}\|_2+\|\Psi\|_1)|\mathbf{u}-\mathbf{u}_h|_{1,h}\le Ch^2(\|\mathbf{u}\|_2+\|p\|_1)(\|\mathbf{\Phi}\|_2+\|\Psi\|_1).\end{aligned}$$

Combining the above three inequalities with (4.16) and (4.18), we complete the proof under the assumption (4.13). □

5 Numerical examples

In this section, we present some numerical results to confirm the theoretical error estimates obtained in this paper. We consider numerical experiments of Stokes equations (1.1a)-(1.1c) with $\Omega=[0,1]\times[0,1]$, where

$$u^1=\frac{1}{\pi}\sin^2(\pi x)\sin(2\pi y),\tag{5.1a}$$

$$u^2=-\frac{1}{\pi}\sin(2\pi x)\sin^2(\pi y),\tag{5.1b}$$

$$p=\cos(\pi x)\cos(\pi y).\tag{5.1c}$$

Three kinds of quadrilateral meshes are used in our experiments. The first kind is the usual square mesh, the second one is a trapezoidal mesh and the third one is a

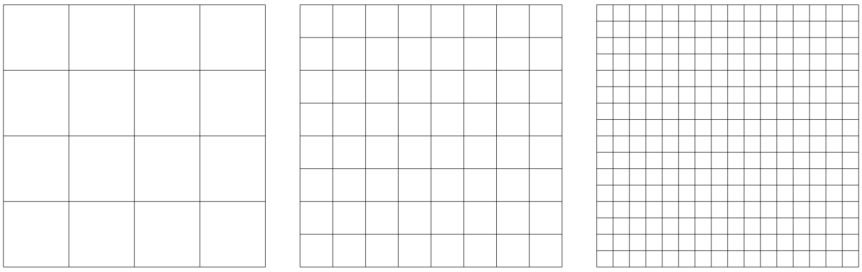

Figure 6: Square mesh.

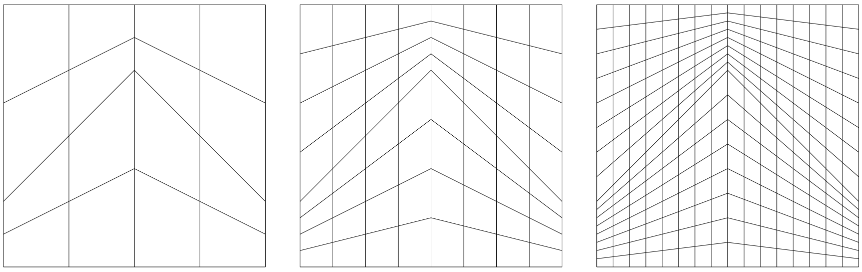

Figure 7: Trapezoidal mesh.

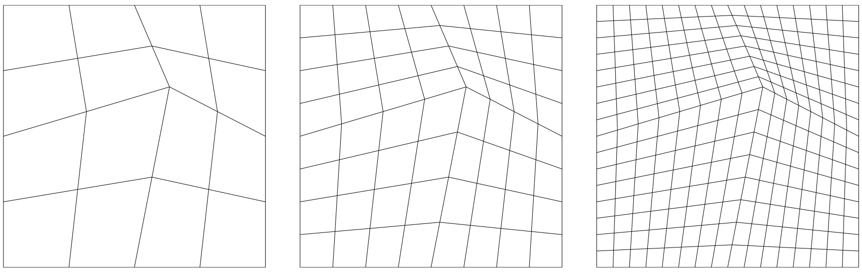

Figure 8: Random mesh.

random mesh. The way of refining meshes is symmetric refinement of quadrilaterals via bisection on edges. The meshes used in our experiments illustrated by Fig. 6-Fig. 8 and the corresponding numerical results are listed in Table 1-Table 3, from which we can see that the optimal convergence rate for the velocity in the broken H^1 seminorm and the pressure in the L^2 norm are of first order and the optimal convergence rate for the velocity in the L^2 norm is of second order.

Table 1: Numerical results on square mesh.

1/h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_0$	order
4	0.033059		0.80097		0.27969	
8	0.0081705	2.0166	0.39738	1.0112	0.10426	1.4237
16	0.0020693	1.9813	0.19703	1.0121	0.043597	1.2578
32	0.00051851	1.9967	0.098263	1.0037	0.020497	1.0888
64	0.00012969	1.9993	0.049098	1.001	0.010078	1.0243
128	3.2427e-005	1.9998	0.024545	1.0002	0.0050172	1.0062

Table 2: Numerical results on trapezoidal mesh.

1/h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_0$	order
4	0.056164		0.91456		0.69364	
8	0.012644	2.1512	0.47338	0.95006	0.19862	1.8042
16	0.0030822	2.0364	0.23264	1.0249	0.063431	1.6468
32	0.00076453	2.0113	0.11567	1.0081	0.026124	1.2798
64	0.00019069	2.0034	0.057749	1.0022	0.012296	1.0872
128	4.7643e-005	2.0009	0.028864	1.0005	0.0060496	1.0233

Table 3: Numerical results on random mesh.

1/h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|p-p_h\|_0$	order
4	0.042335		0.87868		0.27727	
8	0.010205	2.0526	0.43856	1.0026	0.13623	1.0252
16	0.0025288	2.0127	0.21666	1.0174	0.049859	1.4501
32	0.00063072	2.0034	0.10791	1.0056	0.022227	1.1655
64	0.00015757	2.001	0.0539	1.0015	0.010756	1.0472
128	3.9384e-005	2.0003	0.026943	1.0004	0.0053324	1.0122

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