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Elastic Waves in Generalized Thermo-Piezoelectric Transversely Isotropic Circular Bar Immersed in Fluid

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Abstract. In this paper, a mathematical model is developed to study the wave propagation in an infinite, homogeneous, transversely isotropic thermo-piezoelectric solid bar of circular cross-sections immersed in inviscid fluid. The present study is based on the use of the three-dimensional theory of elasticity. Three displacement potential functions are introduced to uncouple the equations of motion and the heat and electric conductions. The frequency equations are obtained for longitudinal and flexural modes of vibration and are studied based on Lord-Shulman, Green-Lindsay and Classical theory theories of thermo elasticity. The frequency equations of the coupled system consisting of cylinder and fluid are developed under the assumption of perfect-slip boundary conditions at the fluid-solid interfaces, which are obtained for longitudinal and flexural modes of vibration and are studied numerically for PZT-4 material bar immersed in fluid. The computed non-dimensional frequencies are compared with Lord-Shulman, Green-Lindsay and Classical theory theories of thermo elasticity for longitudinal and flexural modes of vibrations. The dispersion curves are drawn for longitudinal and flexural modes of vibrations. Moreover, the dispersion of specific loss and damping factors are also analyzed for longitudinal and flexural modes of vibrations.

AMS subject classifications: 43.40.Ey, 43.88.Fx, 43.40.Cw, 43.20.Bi

Key words: Piezoelectric cylinders/plates, thermo-elastic, thermal cylinder immersed in fluid, solid- fluid interaction, transversely isotropic cylinder.

1 Introduction

The piezoelectric materials have been used in numerous fields taking advantage of the flexible characteristics of these polymers. Some of the applications of these polymers include Audio device-microphones, high frequency speakers, tone generators and acoustic modems; Pressure switches–position switches, accelerometers, impact detectors, flow

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meters and load cells; Actuators-electronic fans and high shutters. Since piezoelectric polymers allow their use in a multitude of compositions and geometrical shapes for a large variety of applications from transducers in acoustics, ultrasonic's and hydrophone applications to resonators in band pass filters, power supplies, delay lines, medical scans and some industrial non-destructive testing instruments.

The piezoelectricity was discovered by the brothers Curie in 1880 (Curie and Curie, 1880). The piezoelectric materials generally are physically strong and chemically inert, and they are relatively inexpensive to manufacture. The composition, shape and dimension of piezoelectric ceramic elements can be tailored to meet the requirements of a specific purpose. Ceramics manufactured from formulations of lead zirconate/ lead titanate exhibit greater sensitivity and higher operating temperatures, relative to ceramics of other compositions and the materials PZT-4 are most widely used piezoelectric ceramics.

The coupling between the thermal/electric/elastic fields in piezo electric materials provides a mechanism for sensing thermo mechanical disturbances from measurements of induced electric potentials, and for altering structural responses via applied electric fields. One of the applications of the piezo thermoelastic material is to detect the responses of a structure by measuring the electric charge, sensing or to reduce excessive responses by applying additional electric forces or thermal forces actuating. If sensing and actuating can be integrated smartly, a so-called intelligent structure can be designed. The piezoelectric materials are also often used as resonators whose frequencies need to be precisely controlled. The coupling between the thermoelastic and pyroelectric effects, it is important to qualify the effect of heat dissipation on the propagation of wave at low and high frequencies.

The thermo- piezoelectric theory was first proposed by Mindlin [1], later he derived the governing equations of a thermo-piezoelectric plate [2]. The physical laws for the thermo-piezoelectric materials have been discussed by [3, 4]. Chandrasekhariah [5, 6] presented the generalized theory of thermo-piezoelectricity by taking into account the finite speed of propagation of thermal disturbance. Yang and Batra [7] studied the effect of heat conduction on shift in the frequencies of a freely vibrating linear piezoelectric body with the help of perturbation methods. Sharma and Pal [8] discussed the propagation of Lamb waves in a transversely isotropic piezothermoelastic plate. Sharma et al. [9] investigated the free vibration analysis of a homogeneous, transversely isotropic, piezothermoelastic cylindrical panel based on three dimensional piezoelectric thermoelasticity. Sharma and Walia [10] presented the propagation of straight and circular crested waves in generalized piezo thermoelastic materials.

Tang and Xu [11] derived the general dynamic equations, which include mechanical, thermal and electric effects, based on the anisotropic composite laminated plate theory. They also obtained analytical dynamical solutions for the case of general force acting on a simply supported piezo thermoelastic laminated plate and harmonic responses to temperature variation and electric field have been examined as a special case. Tauchert [13] applied thermo-piezoelectricity theory to composite plate. The generalized theory of thermoelasticity was developed by Lord and Shulman [13] involving one relaxation time for isotropic homogeneous media, which is called the first generalization to the coupled theory of elasticity. These equations determine the finite speeds of propagation of heat and displacement distributions, the corresponding equations for an isotropic case were obtained by Dhaliwal and Sherief [14]. The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity, with two relaxation times or the theory of temperature–dependent thermoelasticity. A generalization of this inequality was proposed by Green and Laws [15]. Green and Lindsay [16] obtained an explicit version of the constitutive equations. These equations were also obtained independently by Suhubi [17]. This theory contains two constants that act as relaxation times and modify not only the heat equations, but also all the equations of the coupled theory. The Classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Suhubi [18] studied the longitudinal wave propagation in a generalized thermoplastic infinite cylinder and obtained the dispersion relation for a constant surface temperature of the cylinder.

Sinha et al. [19] have studied the axisymmetric wave propagation in circular cylindrical shell immersed in a fluid, in two parts. In Part I, the theoretical analysis of the propagation modes is discussed and in Part II, the axisymmetric modes excluding tensional modes are obtained theoretically and experimentally and are compared. Berliner and Solecki [20] have studied the wave propagation in a fluid loaded transversely isotropic cylinder. In that paper, Part I consists of the analytical formulation of the frequency equation of the coupled system consisting of the cylinder with inner and outer fluid and Part II gives the numerical results. Guo and Sun [21] discussed the propagation of Bleustein–Gulyaev wave in 6mm piezoelectric materials loaded wwith viscous liquid using the theory of continuum mechanics. Qian et al. [22] analyzed the propagation of Bleustein-Gulyaev waves in 6mm piezoelectric materials loaded with a viscous liquid layer of finite thickness.

Venkatesan and Ponnusamy [23, 24] studied the wave propagation in solid and generalized solid cylinder of arbitrary cross-sections immersed in fluid using the Fourier expansion collocation method. Dayal [25] investigated the free vibrations of a fluid loaded transversely isotropic rod based on uncoupling the radial and axial wave equations by introducing scalar and vector potentials. Nagy [26] studied the propagation of longitudinal guided waves in fluid-loaded transversely isotropic rod based on the superposition of partial waves. Guided waves in a transversely isotropic cylinder immersed in a fluid was analyzed by Ahmad [27]. Ponnusamy [28] and later with Rajagopal [29] have studied, the wave propagation in a generalized thermo elastic solid cylinder of arbitrary cross-section and in a homogeneous transversely isotropic thermo elastic solid cylinder of arbitrary cross-sections respectively using the Fourier expansion collocation method. Ponnusamy [30, 31] studied respectively the wave propagation in thermoelastic plate of arbitrary and polygonal cross-sections. Dispersion analysis of generalized magneto-thermoelastic waves in a transversely isotropic cylindrical panel is analyzed by Ponnusamy and Selvamani [32]. Ponnusamy [33] studied the wave propagation in a piezoelectric solid bar of circular cross-section immersed in fluid.

In this paper, wave propagation in an infinite, homogeneous, transversely isotropic thermo-piezoelectric solid bar of circular cross-sections immersed in fluid is studied using the three-dimensional theory of elasticity. The frequency equations are obtained for longitudinal and flexural modes of vibration and are studied based on Lord-Shulman (LS), Green-Lindsay (GL) and Classical theory (CT) theories of thermo elasticity. The frequency equations of the coupled system consisting of cylinder and fluid is developed under the assumption of perfect-slip boundary conditions at the fluid-solid interfaces, and they are obtained for longitudinal and flexural modes of vibration and are studied numerically for PZT-4 material bar immersed in fluid. The computed non-dimensional frequencies are compared with Lord-Shulman (LS), Green-Lindsay (GL) and Classical theory (CT) theories of thermo elasticity for longitudinal and flexural modes of vibrations.

2 Formulation of the problem

We consider homogeneous transversely isotropic, thermally and electrically conducting piezoelectric circular bar of infinite length with uniform temperature T_0 in the undistributed state initially. The complete equations governing the behavior of piezoelectric cylinder have been considered from Paul [34] and the heat conduction equations taken from Sharma [9]. In cylindrical coordinates (r,θ,z), the equations of motion in the absence of body force are

$$\frac{\partial}{\partial r}\sigma_{rr}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta}+\frac{\partial}{\partial z}\sigma_{rz}+\frac{(\sigma_{rr}-\sigma_{\theta\theta})}{r}=\rho\frac{\partial^2u_r}{\partial t^2}, \tag{2.1a}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta}+\frac{\partial}{\partial z}\sigma_{\theta z}+\frac{2\sigma_{r\theta}}{r}=\rho\frac{\partial^2u_\theta}{\partial t^2}, \tag{2.1b}$$

$$\frac{\partial}{\partial r}\sigma_{rz}+\frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta z}+\frac{\partial}{\partial z}\sigma_{zz}+\frac{\sigma_{rz}}{r}=\rho\frac{\partial^2u_z}{\partial t^2}. \tag{2.1c}$$

The heat conduction equation is

$$\begin{aligned}&K_1\Big(\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2T}{\partial\theta^2}\Big)+K_3\frac{\partial^2T}{\partial z^2}-\rho C_v\Big(\frac{\partial T}{\partial t}+t_0\frac{\partial^2T}{\partial t^2}\Big)\\ =&T_0\Big(\frac{\partial}{\partial t}+\delta_{1k}t_0\frac{\partial^2}{\partial t^2}\Big)\big[\beta_1(S_{rr}+S_{\theta\theta})+\beta_3S_{zz}-p_3E_z\big].\end{aligned} \tag{2.2}$$

The electric displacements D_r, D_θ and D_z satisfy the Gaussian equation

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r)+\frac{1}{r}\frac{\partial D_\theta}{\partial\theta}+\frac{\partial D_z}{\partial r}=0. \tag{2.3}$$

The elastic, the piezo electric, thermal conduction and dielectric matrices of the 6mm crystal class, the piezo thermo elastic relations are

$$\sigma_{rr}=c_{11}S_{rr}+c_{12}S_{\theta\theta}+c_{13}S_{zz}-\beta_1(T+t_1\delta_{2k}\dot{T})+e_{31}E_z, \tag{2.4a}$$
$$\sigma_{\theta\theta}=c_{12}S_{rr}+c_{11}S_{\theta\theta}+c_{13}S_{zz}-\beta_1(T+t_1\delta_{2k}\dot{T})+e_{31}E_z, \tag{2.4b}$$
$$\sigma_{zz}=c_{13}S_{rr}+c_{13}S_{\theta\theta}+c_{33}S_{zz}-\beta_3(T+t_1\delta_{2k}\dot{T})+e_{33}E_z, \tag{2.4c}$$

where

$$\sigma_{r\theta}=2c_{66}S_{r\theta}, \qquad \sigma_{\theta z}=2c_{44}S_{\theta z}-e_{15}E_{\theta}, \qquad \sigma_{rz}=2c_{44}S_{rz}-e_{15}E_r, \tag{2.5a}$$
$$D_r=2e_{15}S_{rz}+\varepsilon_{11}E_r,\ D_{\theta}=2e_{15}S_{\theta z}+\varepsilon_{11}E_{\theta},\ D_z=e_{31}(S_{rr}+S_{\theta\theta})+e_{33}S_{zz}+\varepsilon_{33}E_z+p_3T, \tag{2.5b}$$

where σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, $\sigma_{r\theta}$, $\sigma_{\theta z}$, for σ_{rz} are the stress components, S_{rr}, $S_{\theta\theta}$, S_{zz}, $S_{r\theta}$, $S_{\theta z}$, S_{rz} are the strain components, c_{11}, c_{12}, c_{13}, c_{33}, c_{44} and $c_{66}=(c_{11}-c_{12})/2$ are the five elastic constants, e_{31}, e_{15}, e_{33} are the piezoelectric constants, ε_{11}, ε_{33} are the dielectric constants, T is the temperature change about the equilibrium temperature T_0, C_v is the specific heat capacity, β_1, β_3 are the thermal expansion coefficients, K_1, K_3 are the thermal conductivity, t_0, t_1 are the two thermal relaxation times, ρ is the mass density. The comma notation is used for spatial derivatives, the superposed dot represents time differentiation and δ_{ij} is the Kronecker delta. In addition, $K=1$ for Lord-Shulman (LS) theory and $K=2$ for Green-Linsay (GL) theory. The thermal relation times t_0 and t_1 satisfy the inequalities $t_0\geq t_1\geq 0$ for GL theory only and we assume that $\rho>0$, $T_0>0$ and $C_v>0$.

The strain S_{ij} are related to the displacements are given by

$$S_{rr}=\frac{\partial u_r}{\partial r}, \qquad S_{\theta\theta}=\Big(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\Big), \qquad S_{zz}=\frac{\partial u_z}{\partial z}, \tag{2.6a}$$
$$S_{\theta z}=\frac{1}{2}\Big(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\Big), \quad S_{r\theta}=\frac{1}{2}\Big(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\Big), \quad S_{rz}=\frac{1}{2}\Big(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\Big), \tag{2.6b}$$
$$E_r=-\frac{\partial E}{\partial r}, \qquad E_\theta=-\frac{1}{r}\frac{\partial E}{\partial \theta}, \qquad E_z=-\frac{\partial E}{\partial z}. \tag{2.6c}$$

3 Lord-Shulman (LS) Theory

The Lord-Shulman theory of heat conduction equation and normal stresses for a three dimensional piezo thermoelasticity is obtained by substituting $K=1$ in the Eqs. (2.2) and (2.4), we get

$$\begin{aligned}&K_1\Big(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\Big)+K_3\frac{\partial^2 T}{\partial z^2}-\rho C_v\Big(\frac{\partial T}{\partial t}+t_0\frac{\partial^2 T}{\partial t^2}\Big)\\ =&T_0\Big(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\Big)\big[\beta_1(S_{rr}+S_{\theta\theta})+\beta_3S_{zz}-p_3E_z\big],\end{aligned} \tag{3.1}$$

and

$$\sigma_{rr}=c_{11}S_{rr}+c_{12}S_{\theta\theta}+c_{13}S_{zz}-\beta_1 T+e_{31}E_z, \tag{3.2}$$
$$\sigma_{\theta\theta}=c_{12}S_{rr}+c_{11}S_{\theta\theta}+c_{13}S_{zz}-\beta_1 T+e_{31}E_z, \tag{3.3}$$
$$\sigma_{zz}=c_{13}S_{rr}+c_{13}S_{\theta\theta}+c_{33}S_{zz}-\beta_3 T+e_{33}E_z. \tag{3.4}$$

Substituting Eqs. (2.5a), (2.6) and (3.2) in Eqs. (2.1) and (3.1), the displacement equations of motions are obtained as

$$\begin{aligned}
&c_{11}\Big(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\Big)+\frac{(c_{12}+c_{66})}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}+(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial r\partial z}-\frac{(c_{11}+c_{66})}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{c_{66}}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\\
&\quad+c_{44}\frac{\partial^2 u_r}{\partial z^2}-\beta_1\frac{\partial T}{\partial r}+(e_{15}+e_{31})E_{,rz}=\rho\frac{\partial^2 u_r}{\partial t^2}, && (3.5a)\\
&c_{66}\Big(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\Big)+\frac{c_{11}}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+c_{44}\frac{\partial^2 u_\theta}{\partial z^2}+\frac{(c_{66}+c_{12})}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{(c_{66}+c_{11})}{r^2}\frac{\partial u_r}{\partial\theta}\\
&\quad+\frac{(c_{44}+c_{13})}{r}\frac{\partial^2 u_z}{\partial\theta\partial z}-\frac{\beta_1}{r}\frac{\partial T}{\partial\theta}+\frac{(e_{31}+e_{15})}{r}E_{,\theta z}=\rho\frac{\partial^2 u_\theta}{\partial t^2}, && (3.5b)\\
&c_{44}\Big(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\Big)+c_{33}\frac{\partial^2 u_z}{\partial z^2}+\frac{(c_{13}+c_{44})}{r}\Big(\frac{\partial u_r}{\partial z}+\frac{\partial^2 u_\theta}{\partial\theta\partial z}\Big)+(c_{13}+c_{44})\frac{\partial^2 u_r}{\partial r\partial z}\\
&\quad+e_{33}E_{,zz}+e_{15}\Big(\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E}{\partial\theta^2}\Big)-\beta_3 T=\rho\frac{\partial^2 u_z}{\partial t^2}, && (3.5c)\\
&K_1\nabla^2 T+K_3\frac{\partial^2 T}{\partial z^2}=T_0\Big(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\Big)\Big[\beta_1\Big(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\Big)+\beta_3\frac{\partial u_z}{\partial z}-p_3\frac{\partial E}{\partial z}\Big]\\
&\quad+\rho C_v\Big(\frac{\partial T}{\partial t}+t_0\frac{\partial^2 T}{\partial t^2}\Big), && (3.5d)\\
&e_{15}\Big(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\Big)+(e_{15}+e_{31})\Big(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\Big)\\
&\quad-\varepsilon_{11}\Big(\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E}{\partial\theta^2}\Big)+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\varepsilon_{33}\frac{\partial^2 E}{\partial z^2}+p_3\frac{\partial T}{\partial z}=0. && (3.5e)
\end{aligned}$$

4 Solution of the field equation

To obtain the propagation of harmonic waves in piezo thermoelastic circular bar, we assume the solutions of the displacement components to be expressed in terms of derivatives of potentials are taken from Paul [34]. Thus, we seek the solution of the Eq. (3.5a) in the form of Paul [34] are

$$u_r(r,\theta,z,t)=\Big(\phi_{,r}+\frac{1}{r}\psi_{,\theta}\Big)e^{i(kz+\omega t)},\qquad u_\theta(r,\theta,z,t)=\Big(\frac{1}{r}\phi_{,\theta}-\psi_{,r}\Big)e^{i(kz+\omega t)}, \tag{4.1a}$$
$$u_z(r,\theta,z,t)=\Big(\frac{i}{a}\Big)We^{i(kz+\omega t)},\qquad E(r,\theta,z,t)=\Big(\frac{ic_{44}}{ae_{33}}\Big)Ee^{i(kz+\omega t)}, \tag{4.1b}$$

$$T(r,\theta,z,t)=\frac{c_{44}}{\beta_3 a^2}Te^{i(kz+\omega t)}, \qquad E_r(r,\theta,z,t)=-E_{,r}e^{i(kz+\omega t)}, \tag{4.1c}$$

$$E_\theta(r,\theta,z,t)=-\frac{1}{r}E_{,\theta}e^{i(kz+\omega t)}, \qquad E_z(r,\theta,z,t)=E_{,z}e^{i(kz+\omega t)}, \tag{4.1d}$$

where $i=\sqrt{-1}$, k is the wave number, ω is the angular frequency, $\phi(r,\theta)$, $\psi(r,\theta)$, $E(r,\theta)$ and $T(r,\theta)$ are the displacement potentials and a is the geometrical parameter of the bar.

Introducing the dimensionless quantities such as

$$\begin{aligned}
&x=r/a, && \zeta=ka, && \Omega^2=\rho\omega^2a^2/c_{44},\\
&\bar{c}_{11}=c_{11}/c_{44}, && \bar{c}_{13}=c_{13}/c_{44}, && \bar{c}_{33}=c_{33}/c_{44},\\
&\bar{c}_{66}=c_{66}/c_{44}, && \bar{\beta}=\beta_1/\beta_3, && \bar{K}_i=K_ic_{44}/\omega\beta_3^2T_0a^2,\\
&\bar{d}=\rho C_v c_{44}/\beta_3^2T_0, && \bar{p}_3=p_3c_{44}/\beta_3e_{33}, && \bar{\varepsilon}_{11}=\varepsilon_{11}c_{44}/e_{33}^2,\\
&\bar{e}_{31}=e_{31}/e_{33}, && \bar{e}_{15}=e_{15}/e_{33},
\end{aligned}$$

and substituting Eq. (4.1) in Eq. (3.5a), we obtain

$$[\bar{c}_{11}\nabla^2+(\Omega^2-\zeta^2)]\phi-\zeta(1+\bar{c}_{13})W-\zeta(\bar{e}_{15}+\bar{e}_{31})E-\bar{\beta}T=0, \tag{4.2a}$$

$$\zeta(1+\bar{c}_{13})\nabla^2\phi+[\nabla^2+(\Omega^2-\zeta^2\bar{c}_{33})]W+(\bar{e}_{15}\nabla^2-\zeta^2)E-\zeta T=0, \tag{4.2b}$$

$$\tau_0\bar{\beta}\nabla^2\phi-\tau_0\zeta W+\tau_0\zeta\bar{p}_3E+[\tau_0\bar{d}+i\bar{K}_1\nabla^2-i\bar{K}_3\zeta^2]T=0, \tag{4.2c}$$

$$\zeta(\bar{e}_{15}+\bar{e}_{31})\nabla^2\phi+(\bar{e}_{15}\nabla^2-\zeta^2)W+(\zeta^2\bar{\varepsilon}_{33}-\bar{\varepsilon}_{11}\nabla^2)E+\bar{p}_3\zeta T=0, \tag{4.2d}$$

and

$$(\bar{c}_{66}\nabla^2+(\Omega^2-\zeta^2))\psi=0, \tag{4.3}$$

Eq. (4.2) can be written as

$$\begin{vmatrix}
\bar{c}_{11}\nabla^2+g_3 & -\zeta g_6 & -\zeta g_5 & -\bar{\beta}\\
\zeta g_6\nabla^2 & \nabla^2+g_1 & (\bar{e}_{15}\nabla^2-\zeta^2) & -\zeta\\
\zeta g_5\nabla^2 & (\bar{e}_{15}\nabla^2-\zeta^2) & (\zeta^2\bar{\varepsilon}_{33}-\bar{\varepsilon}_{11}\nabla^2) & \bar{p}_3\zeta\\
\tau_0\bar{\beta}\nabla^2 & -\tau_0\zeta & \tau_0\zeta\bar{p}_3 & (i\bar{K}_1\nabla^2+g_2)
\end{vmatrix}(\phi,W,E,T)=0, \tag{4.4}$$

where $g_1=\Omega^2-\zeta^2\bar{c}_{33}$, $g_2=\tau_0\bar{d}-i\bar{K}_3\zeta^2$, $g_3=\Omega^2-\zeta^2$, $g_4=\bar{\varepsilon}_{11}^2+\bar{e}_{15}^2$, $g_5=\bar{e}_{31}+\bar{e}_{15}$ and $g_6=1+\bar{c}_{13}$.

Evaluating the determinant given in Eq. (4.4), we obtain a partial differential equation of the form

$$(A\nabla^8+B\nabla^6+C\nabla^4+D\nabla^2+E)(\phi,W,E,T)=0. \tag{4.5}$$

Where

$$A=-i\overline{K}_1\overline{c}_{11}g_4, \tag{4.6a}$$

$$\begin{aligned}B=&\,i\overline{K}_1\overline{c}_{11}(\zeta^2(\overline{\varepsilon}_{33}+\overline{e}_{15})-\overline{\varepsilon}_{11}g_1)-\overline{c}_{11}\overline{\varepsilon}_{11}g_2-i\overline{K}_1g_3g_4\\&-\zeta^2g_5g_6i\overline{K}_1\overline{e}_{15}-\zeta^2g_5i\overline{K}_1(\overline{e}_{15}g_6-g_5)-\overline{e}_{15}^2\tau_0\overline{\beta}^2,\end{aligned} \tag{4.6b}$$

$$\begin{aligned}C=&\,\overline{c}_{11}g_1(i\overline{K}_1\zeta^2\overline{\varepsilon}_{33}-\overline{\varepsilon}_{11}g_2)-\overline{c}_{11}\overline{e}_{15}[\overline{e}_{15}g_2+\zeta^2(-i\overline{K}_1+i\overline{K}_3\zeta^2+\tau_0(\overline{p}_3\overline{e}_{15}-\overline{d}-\overline{\beta}_3\overline{p}_3\zeta^2))]\\&+\overline{c}_{11}[g_2(\zeta^2\overline{\varepsilon}_{33}+\overline{e}_{15})+\zeta^2(\tau_0\overline{p}_3\overline{e}_{15}-i\overline{K}_1)+\tau_0(-\overline{p}_3^2-\overline{p}_3\overline{e}_{15}\\&+\overline{\varepsilon}_{11})]+i\overline{K}_3g_3[\zeta^2(\overline{\varepsilon}_{33}+\overline{e}_{15})-\overline{\varepsilon}_{11}g_1]-\overline{\varepsilon}_{11}g_2g_3-\zeta^2g_6\overline{e}_{15}(g_2g_5-\tau_0\overline{\beta}\overline{p_3})\\&-\zeta^2g_5g_6\overline{e}_{15}(g_2-i\overline{K}_1\zeta^2)+\zeta^2(i\overline{K}_1(\zeta^2g_5g_6+g_1g_5^2)-g_2g_5^2)\\&+\tau_0\overline{\beta}\zeta^2(-\overline{\varepsilon}_{11}g_6+g_5(-\overline{p}_3+g_1\overline{e}_{15}))+\zeta^2g_6\tau_0\overline{\beta}(\overline{e}_{15}\overline{p}_3-\overline{\varepsilon}_{11})\\&+\tau_0\overline{\beta}(-g_1\overline{\varepsilon}_{11}\overline{\beta}+\overline{e}_{15}\zeta^2(\overline{\beta}-g_5))+\tau_0\overline{\beta}^2(\zeta^2\overline{e}_{15}-\overline{\varepsilon}_{11}),\end{aligned} \tag{4.6c}$$

$$\begin{aligned}D=&\,\overline{c}_{11}\zeta^2[g_1(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2\tau_0)+\zeta^2(\tau_0(\overline{d}+\overline{p}_3-\overline{\varepsilon}_{33})+\zeta^2(\tau_0\overline{p}_3-i\overline{K}_3))]+g_1g_3(i\overline{K}_1\zeta^2\overline{\varepsilon}_{33}\\&-\overline{\varepsilon}_{11}g_2)-\overline{e}_{15}g_3[\overline{e}_{15}g_2+\zeta^2(-i\overline{K}_1-g_2+\tau_0\overline{p}_3(\overline{e}_{15}-\zeta^2))+\zeta^2g_3[g_2(\overline{\varepsilon}_{33}+\overline{e}_{15})-i\overline{K}_1\zeta^2\\&+\tau_0(\overline{p}_3(\zeta^2\overline{e}_{15}-\overline{p}_3-\overline{e}_{15})+\overline{\varepsilon}_{11})]+\zeta^4g_6^2(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2g_5)+\zeta^2g_1g_5(g_2g_5-\tau_0\overline{p}_3\overline{\beta})\\&-\tau_0\zeta^4(g_5(\overline{\beta}-g_5)-g_6\overline{\beta}\overline{\varepsilon}_{33}\overline{p}_3)-\zeta^2\tau_0\overline{\beta}(g_5\overline{p}_3-\overline{\varepsilon}_{33}\overline{\beta})-\zeta^4\tau_0\overline{\beta}(\overline{\beta}-g_5),\end{aligned} \tag{4.6d}$$

$$\begin{aligned}E=&\,g_3\zeta^2[g_1(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2\tau_0)+\zeta^2(\tau_0(\overline{d}+\overline{p}_3-\overline{\varepsilon}_{33})+\zeta^2(\tau_0\overline{p}_3-i\overline{K}_3))]\\&+\zeta^4g_6(g_5(g_2-\tau_0\overline{p}_3)-\tau_0\overline{p}_3\overline{\beta})-g_1\overline{\beta}\zeta^2\tau_0(\overline{p}_3g_5-\overline{\varepsilon}_{33}\overline{\beta}).\end{aligned} \tag{4.6e}$$

Solving the Eq. (4.5), we get the solution for a circular cylinder as

$$\phi=\sum_{i=1}^4 A_iJ_n(\alpha_i ax)\cos n\theta, \qquad W=\sum_{i=1}^4 a_iA_iJ_n(\alpha_i ax)\cos n\theta, \tag{4.7a}$$

$$E=\sum_{i=1}^4 b_iA_iJ_n(\alpha_i ax)\cos n\theta, \qquad T=\sum_{i=1}^4 c_iA_iJ_n(\alpha_i ax)\cos n\theta, \tag{4.7b}$$

where $(\alpha_i a)^2>0$, $(i=1,2,3,4)$, are the roots of the algebraic equation

$$A(\alpha a)^8-B(\alpha a)^6+C(\alpha a)^4-D(\alpha a)^2+E=0. \tag{4.8}$$

The solutions corresponding to the root $(\alpha_i a)^2=0$ is not considered here, since $J_n(0)$ is zero, except for $n=0$. The Bessel function J_n is used when the roots $(\alpha_i a)^2$, $(i=1,2,3,4)$ are real or complex and the modified Bessel function I_n is used when the roots $(\alpha_i a)^2$, $(i=1,2,3,4)$ are imaginary.

The constants a_i, b_i and c_i defined in Eq. (4.7) can be calculated from the following equations

$$\begin{aligned}a_i&=\zeta g_6(\overline{e}_{15}\nabla^2-\zeta^2)-\zeta g_5(\nabla^2+g_1)/H,\\b_i&=-\zeta^2g_5-\overline{\beta}(\overline{e}_{15}\nabla^2-\zeta^2)/H,\\c_i&=-\overline{\beta}\zeta g_6+\zeta(\overline{c}_{11}\nabla^2+g_3)/H,\end{aligned}$$

where

$$H=\bar{c}_{11}\nabla^2+g_3(\nabla^2+g_1)+\zeta^2 g_6^2\nabla^2. \tag{4.9}$$

Solving the Eq. (4.3), we obtain

$$\psi=A_5 J_n(\alpha_5 ax)\sin n\theta, \tag{4.10}$$

where $(\alpha_5 a)^2=(\Omega^2-\zeta^2)/\bar{c}_{66}$. If $(\alpha_5 a)^2<0$, the Bessel function J_n is replaced by the modified Bessel function I_n.

5 Equations of motion of the fluid

In cylindrical polar coordinates r, θ and z the acoustic pressure and radial displacement equation of motion for an invicid fluid are of the form Achenbach [36] as

$$p^f=-B^f(u^f_{r,r}+r^{-1}(u^f_r+u^f_{\theta,\theta})+u^f_{z,z}), \tag{5.1}$$

and

$$\bar{c}_f^2 u^f_{r,tt}=\Delta_{,r}, \tag{5.2}$$

respectively, where B^f is the adiabatic bulk modulus, ρ^f is the density, $c_f=\sqrt{B^f/\rho^f}$ is the acoustic phase velocity in the fluid, and

$$\Delta=(u^f_{r,r}+r^{-1}(u^f_r+u^f_{\theta,\theta})+u^f_{z,z}). \tag{5.3}$$

Substituting

$$u^f_r=\phi^f_{,r},\quad u^f_\theta=r^{-1}\phi^f_{,\theta}\quad\text{and}\quad u^f_z=\phi^f_{,z}, \tag{5.4}$$

and seeking the solution of Eq. (4.7) in the form

$$\phi^f(r,\theta,z,t)=\sum_{n=0}^{\infty}\phi^f(r)\cos n\theta e^{i(kz+\omega t)}. \tag{5.5}$$

The fluid that represents the oscillary wave propagating away is given as

$$\phi^f=A_6 H_n^{(1)}(\alpha_6 ax), \tag{5.6}$$

where $(\alpha_6 a)^2=\Omega^2/\bar{\rho}\overline{B}^f-\zeta^2$, in which $\bar{\rho}=\rho/\rho^f$, $\overline{B}^f=B^f/c_{44}$, $H_n^{(1)}$ is the Hankel function of first kind. If $(\alpha_6 a)^2<0$, then the Hankel function of first kind is to be replaced by K_n, where K_n is the modified Bessel function of the second kind. By substituting Eq. (5.5) in Eq. (5.1) along with Eq. (5.6), the acoustic pressure for the fluid can be expressed as

$$p^f=\sum_{n=0}^{\infty}A_6\Omega^2\bar{\rho}H_n^{(1)}(\alpha_6 ax)\cos n\theta e^{i(\zeta z+\Omega T_a)}. \tag{5.7}$$

6 Boundary conditions and frequency equations

The free vibration of transversely isotropic piezo thermoelastic solid bar of circular cross-section is considered in this problem.

i. Mechanical boundary condition. The mechanical boundary condition for an infinite cylindrical circular bar is

$$\sigma_{rr}=-p^f=0, \quad \sigma_{r\theta}=\sigma_{rz}=0 \quad \text{at } r=a. \tag{6.1}$$

ii. Thermal boundary condition

$$T=0. \tag{6.2}$$

iii. Electrical boundary condition

$$E=0. \tag{6.3}$$

Substituting the solution given in the Eqs. (4.7) and (4.10) along with Eq. (5.7) in the boundary condition in the Eq. (5.3), we obtain a system of five linear algebraic equation as follows:

$$[A]\{X\}=\{0\}, \tag{6.4}$$

where $[A]$ is a 5×5 matrix of unknown wave amplitudes and $\{X\}$ is an 5×1 column vector of the unknown amplitude coefficients A_1, A_2, A_3, A_4, A_5. The components of $[A]$ are defined in the Appendix. The solution of Eq. (5.7) is nontrivial when the determinant of the coefficient of the wave amplitudes $\{X\}$ vanishes, that is

$$|A|=0. \tag{6.5}$$

Eqs. (6.1)-(6.3) is the frequency equation of the coupled system consisting of transversely isotropic piezo thermoelastic solid circular bar.

7 Green-Linsay (GL) theory

The basic governing equations of motion and heat conduction in the absence of body force and heat source for a piezo thermoelastic cylinder of circular bar, in the context of GL theory is obtained by substituting $K=2$ in the heat conduction equation Eq. (2.2) and in the stress-strain relation Eq. (2.4), thus we obtain

$$\begin{aligned}&K_1\Big(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\Big)+K_3\frac{\partial^2 T}{\partial z^2}-\rho C_v\Big(\frac{\partial T}{\partial t}+t_0\frac{\partial^2 T}{\partial t^2}\Big)\\ =&T_0\frac{\partial}{\partial t}[\beta_1(S_{rr}+S_{\theta\theta})+\beta_3 S_{zz}-p_3E_z],\end{aligned} \tag{7.1}$$

and

$$\sigma_{rr}=c_{11}S_{rr}+c_{12}S_{\theta\theta}+c_{13}S_{zz}-\beta_1(T+t_1\dot{T})-e_{31}E_z, \tag{7.2a}$$
$$\sigma_{\theta\theta}=c_{12}S_{rr}+c_{11}S_{\theta\theta}+c_{13}S_{zz}-\beta_1(T+t_1\dot{T})-e_{31}E_z, \tag{7.2b}$$
$$\sigma_{zz}=c_{13}S_{rr}+c_{13}S_{\theta\theta}+c_{33}S_{zz}-\beta_3(T+t_1\dot{T})-e_{33}E_z. \tag{7.2c}$$

Substituting Eqs. (2.5a), (2.5b), (6.5) in Eqs. (2.1) along with Eq. (6.4), we get the displacement equations of motion as follows:

$$\begin{aligned}
&c_{11}\Big(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\Big)+\frac{(c_{12}+c_{66})}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}+(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial r\partial z}-\frac{(c_{11}+c_{66})}{r^2}\frac{\partial u_\theta}{\partial\theta}\\
&\quad+\frac{c_{66}}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+c_{44}\frac{\partial^2 u_r}{\partial z^2}-\beta_1\Big(\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r\partial t}\Big)+(e_{15}+e_{31})E_{,rz}=\rho\frac{\partial^2 u_r}{\partial t^2}, &(7.3a)\\
&c_{66}\Big(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\Big)+\frac{c_{11}}{r^2}\frac{\partial^2 u_\theta}{\partial r^2}+c_{44}\frac{\partial^2 u_\theta}{\partial z^2}+\frac{(c_{66}+c_{12})}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{(c_{66}+c_{11})}{r}\frac{\partial u_r}{\partial\theta}\\
&\quad+\frac{(c_{44}+c_{13})}{r}\frac{\partial^2 u_z}{\partial\theta\partial z}-\frac{\beta_1}{r}(T+t_1\dot{T})+\frac{(e_{31}+e_{15})}{r}E_{,\theta z}=\rho\frac{\partial^2 u_\theta}{\partial t^2}, &(7.3b)\\
&c_{44}\Big(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\Big)+c_{33}\frac{\partial^2 u_z}{\partial z^2}+\frac{(c_{13}+c_{44})}{r}\Big(\frac{\partial u_r}{\partial z}+\frac{\partial^2 u_\theta}{\partial\theta\partial z}\Big)\\
&\quad+(c_{13}+c_{44})\frac{\partial^2 u_r}{\partial r\partial z}+e_{33}E_{,zz}+e_{15}\Big(E_{,rr}+\frac{1}{r}E_{,r}+\frac{1}{r^2}E_{,\theta\theta}\Big)-\beta_3(T+t_1\dot{T})_{,z}=\rho\frac{\partial^2 u_z}{\partial t^2}, &(7.3c)\\
&e_{15}\Big(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\Big)+(e_{15}+e_{31})\Big(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\Big)\\
&\quad-\varepsilon_{11}\Big(E_{,rr}+\frac{1}{r}E_{,r}+\frac{1}{r^2}E_{,\theta\theta}\Big)+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\varepsilon_{33}E_{,zz}+p_3T_{,z}=0, &(7.3d)\\
&K_1\nabla^2 T+K_3T_{,zz}=T_0\frac{\partial}{\partial t}\Big[\beta_1\Big(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\Big)+\beta_3\frac{\partial u_z}{\partial z}-p_3\frac{\partial E}{\partial z}\Big]\\
&\quad+\rho C_v\Big(\frac{\partial T}{\partial t}+t_0\frac{\partial^2 T}{\partial t^2}\Big). &(7.3e)
\end{aligned}$$

Substituting the solution given in Eq. (4.1) in Eq. (7.1) along with the dimensionless quantities, and applying the same procedure as discussed in the previous section, we get

$$\begin{vmatrix}
(\bar{c}_{11}\nabla^2+g_3) & -\zeta g_6 & -\zeta g_5 & -\bar{\beta}\tau_1\\
\zeta g_6\nabla^2 & \nabla^2+g_1 & (\bar{e}_{15}\nabla^2-\zeta^2) & -\zeta\\
\zeta g_5 & \bar{e}_{15}\nabla^2-\zeta & \zeta^2\bar{\varepsilon}_{33}-\bar{\varepsilon}_{11}\nabla^2 & \bar{p}_3\zeta\\
\tau_0{}'\bar{\beta}\nabla^2 & -\tau_0{}'\zeta & \tau_0{}'\zeta\bar{p}_3 & i\bar{K}_1\nabla^2+g_2
\end{vmatrix}(\phi',W',E',T')=0. \tag{7.4}$$

Simplifying the determinant given in Eq. (7.2), we obtain

$$\big(P\nabla^8+Q\nabla^6+R\nabla^4+S\nabla^2+U\big)(\phi',W',E',T')=0. \tag{7.5}$$

Where

$$
\begin{aligned}
P=&-i\overline{K}_1\overline{c}_{11}g_4,\\
Q=&i\overline{K}_1\overline{c}_{11}(\zeta^2(\overline{\varepsilon}_{33}+\overline{e}_{15})-\overline{\varepsilon}_{11}g_1)-\overline{c}_{11}\overline{\varepsilon}_{11}g_2-i\overline{K}_1g_3g_4-\zeta^2g_5g_6i\overline{K}_1\overline{e}_{15}\\
&-\zeta^2g_5i\overline{K}_1(\overline{e}_{15}g_6-g_5)-\overline{e}_{15}^2\tau_0{}'\overline{\beta}^2,\\
R=&\overline{c}_{11}g_1(i\overline{K}_1\zeta^2\overline{\varepsilon}_{33}-\overline{\varepsilon}_{11}g_2)-\overline{c}_{11}\overline{e}_{15}[\overline{e}_{15}g_2+\zeta^2(-i\overline{K}_1+i\overline{K}_3\zeta^2-\tau_0\overline{d}+\tau_0{}'(\overline{p}_3\overline{e}_{15}-\overline{\beta}_3\overline{p}_3\zeta^2))]\\
&+\overline{c}_{11}[g_2(\zeta^2\overline{\varepsilon}_{33}+\overline{e}_{15})+\zeta^2(\tau_0{}'\overline{p}_3\overline{e}_{15}-i\overline{K}_1)+\tau_0{}'(-\overline{p}_3^2-\overline{p}_3\overline{e}_{15}+\overline{\varepsilon}_{11})]\\
&+i\overline{K}_3g_3[\zeta^2(\overline{\varepsilon}_{33}+\overline{e}_{15})-\overline{\varepsilon}_{11}g_1]-\overline{\varepsilon}_{11}g_2g_3-\zeta^2g_6\overline{e}_{15}(g_2g_5-\tau_0{}'\overline{\beta}\overline{p_3})\\
&-\zeta^2g_5g_6\overline{e}_{15}(g_2-i\overline{K}_1\zeta^2)+\zeta^2(i\overline{K}_1(\zeta^2g_5g_6+g_1g_5^2)-g_2g_5^2)\\
&+\tau_0{}'\overline{\beta}\zeta^2(-\overline{\varepsilon}_{11}g_6+g_5(-\overline{p}_3+g_1\overline{e}_{15}))+\zeta^2g_6\tau_0{}'\tau_1\overline{\beta}(\overline{e}_{15}\overline{p}_3-\overline{\varepsilon}_{11})\\
&+\tau_0{}'\tau_1\overline{\beta}(-g_1\overline{\varepsilon}_{11}\overline{\beta}+\overline{e}_{15}\zeta^2(\overline{\beta}-g_5))+\tau_0{}'\tau_1\overline{\beta}^2(\zeta^2\overline{e}_{15}-\overline{\varepsilon}_{11}),\\
S=&\overline{c}_{11}\zeta^2[g_1(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2\tau_0)+\zeta^2(\tau_0\overline{d}+\tau_0{}'(\overline{p}_3-\overline{\varepsilon}_{33})+\zeta^2(\tau_0{}'\overline{p}_3-i\overline{K}_3))]+g_1g_3(i\overline{K}_1\zeta^2\overline{\varepsilon}_{33}-\overline{\varepsilon}_{11}g_2)\\
&-\overline{e}_{15}g_3[\overline{e}_{15}g_2+\zeta^2(-i\overline{K}_1-g_2+\tau_0{}'\overline{p}_3(\overline{e}_{15}-\zeta^2))+\zeta^2g_3[g_2(\overline{\varepsilon}_{33}+\overline{e}_{15})-i\overline{K}_1\zeta^2\\
&+\tau_0{}'(\overline{p}_3(\zeta^2\overline{e}_{15}-\overline{p}_3-\overline{e}_{15})+\overline{\varepsilon}_{11})]+\zeta^4g_6^2(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2g_5)+\zeta^2g_1g_5(g_2g_5-\tau_0{}'\tau_1\overline{p}_3\overline{\beta})\\
&-\tau_0{}'\zeta^4(g_5(\overline{\beta}-g_5)-g_6\overline{\beta}(\overline{\varepsilon}_{33}-\overline{p}_3))-\zeta^2\tau_0{}'\tau_1\overline{\beta}(g_5\overline{p}_3-\overline{\varepsilon}_{33}\overline{\beta})-\zeta^4\tau_0{}'\tau_1\overline{\beta}(\overline{\beta}-g_5),\\
U=&g_3\zeta^2[g_1(\overline{\varepsilon}_{33}g_2-\overline{p}_3^2\tau_0{}')+\zeta^2(\tau_0\overline{d}+\tau_0{}'(\overline{p}_3-\overline{\varepsilon}_{33})+\zeta^2(\tau_0{}'\overline{p}_3-i\overline{K}_3))]\\
&+\zeta^4g_6(g_5(g_2-\tau_0{}'\overline{p}_3)-\tau_0{}'\overline{p}_3\overline{\beta})-g_1\overline{\beta}\zeta^2\tau_0{}'\tau_1(\overline{p}_3g_5-\overline{\varepsilon}_{33}\overline{\beta}).
\end{aligned}
$$

Solving the Eq. (7.5), we get the solution for a circular cylinder as

$$\phi'=\sum_{i=1}^{4}A_iJ_n(\beta_iax)\cos n\theta,\qquad W'=\sum_{i=1}^{4}d_iA_iJ_n(\beta_iax)\cos n\theta, \tag{7.6a}$$

$$E'=\sum_{i=1}^{4}e_iA_iJ_n(\beta_iax)\cos n\theta,\qquad T'=\sum_{i=1}^{4}f_iA_iJ_n(\beta_iax)\cos n\theta, \tag{7.6b}$$

where $(\beta_ia)^2>0$, $(i=1,2,3,4)$, are the roots of the algebraic equation

$$P(\beta a)^8-Q(\beta a)^6+R(\beta a)^4-S(\beta a)^2+T=0. \tag{7.7}$$

The solutions corresponding to the root $(\beta_ia)^2=0$ is not considered here, since $J_n(0)$ is zero, except for $n=0$. The Bessel function J_n is used when the roots $(\beta_ia)^2$, $(i=1,2,3,4)$, are real or complex and the modified Bessel function I_n is used when the roots $(\beta_ia)^2$, $(i=1,2,3,4)$, are imaginary.

The constants d_i, e_i and f_i defined in Eq. (4.7) can be calculated from the following equations

$$d_i=\zeta g_6(\overline{e}_{15}\nabla^2-\zeta^2)-\zeta g_5(\nabla^2+g_1)/H', \tag{7.8a}$$

$$e_i = -\zeta^2 g_5 - \overline{\beta}\tau_1(\bar{e}_{15}\nabla^2 - \zeta^2)/H', \tag{7.8b}$$
$$f_i = -\overline{\beta}\tau_1\zeta g_6 + \zeta(\bar{c}_{11}\nabla^2 + g_3)/H', \tag{7.8c}$$

where

$$H' = \bar{c}_{11}\nabla^2 + g_3(\nabla^2 + g_1) + \zeta^2 g_6^2 \nabla^2. \tag{7.9}$$

Solving the Eq. (4.3), we obtain

$$\psi = A_5 J_n(\beta_5 ax)\sin n\theta, \tag{7.10}$$

where $(\alpha_5 a)^2 = (\Omega^2 - \zeta^2)/\bar{c}_{66}$. If $(\beta_5 a)^2 < 0$, the Bessel function J_n is replaced by the modified Bessel function I_n.

Substituting the solution given in the Eqs. (7.1), (7.5) along with the Eq. (5.7) in the boundary condition in the Eqs. (6.1)-(6.3), we obtain a system of five linear algebraic equation as follows:

$$[B]\{X\} = \{0\}, \tag{7.11}$$

where $[B]$ is a 5×5 matrix of unknown wave amplitudes and $\{X\}$ is an 5×1 column vector of the unknown amplitude coefficients B_1, B_2, B_3, B_4, B_5. The solution of Eq. (5.7) is nontrivial when the determinant of the coefficient of the wave amplitudes $\{X\}$ vanishes, that is

$$|B| = 0. \tag{7.12}$$

Eq. (7.7) is the frequency equation of the coupled system consisting of transversely isotropic thermo-piezoelectric solid circular bar.

$$\begin{aligned}
b_{1i} &= 2\bar{c}_{66}\{n(n-1)J_n(\beta_i ax) + (\beta_i ax)J_{n+1}(\beta_i ax)\} - x^2[(\beta_i a)^2\bar{c}_{11} + \zeta\bar{c}_{13}d_i \\
&\quad + \zeta e_i + \overline{\beta}\tau_1 f_i]J_n(\beta_i ax), \quad i = 1,2,3,4, \\
b_{15} &= 2\bar{c}_{66}\{n(n-1)J_n(\beta_5 ax) - (\beta_5 ax)J_{n+1}(\beta_5 ax)\}, \\
b_{2i} &= 2n\{(\beta_i ax)J_{n+1}(\beta_i ax) - (n-1)J_n(\beta_i ax)\}, \quad i = 1,2,3,4, \\
b_{25} &= \{[(\beta_5 ax)^2 - 2n(n-1)]J_n(\beta_5 ax) - 2(\beta_5 ax)J_{n+1}(\beta_5 ax)\}, \quad i = 1,2,3,4, \\
a_{3i} &= (\zeta + d_i + \bar{e}_{15}e_i)[nJ_n(\beta_i ax) - (\beta_i ax)J_{n+1}(\beta_i ax)], \quad i = 1,2,3,4, \\
b_{35} &= n\zeta J_n(\beta_5 ax), \\
b_{4i} &= (\bar{e}_{15}\zeta d_i - \bar{\varepsilon}_{11}e_i)\{nJ_n(\beta_i ax) - (\beta_i ax)J_{n+1}(\beta_i ax)\}, \quad i = 1,2,3,4, \\
b_{45} &= \bar{e}_{15}\zeta n J_n(\beta_5 ax), \\
b_{5i} &= f_i\{nJ_n(\beta_i ax) - (\beta_i ax)J_{n+1}(\beta_i ax)\}, \quad i = 1,2,3,4, \\
b_{55} &= 0.
\end{aligned}$$

8 Particular case

The frequency equation for a piezoelectric circular bar of infinite length is obtained by omitting the thermal and fluid medium in the corresponding equations and solutions, the resulting frequency equation is compared with the frequency equations of Paul and Raju [35], which matches well with the frequency equations of the author.

8.1 Specific loss

The interval energy of a material is defined by specific loss. The specific loss is the ratio of the amount of energy (ΔE) dissipated in a specimen through a stress cycle to the elastic energy (E) stored in that specimen at a maximum strain. According to Kolsky [37], specific loss $(\Delta E/E)$ is equal to 4π times the absolute value of the imaginary part of α to the real part of α, therefore we have

$$|\Delta E/E| = 4\pi |Im(\zeta)/Re(\zeta)| = |v_q/\omega|. \tag{8.1}$$

The real and imaginary parts of the wave number is obtained from the relation $\zeta = R + iq$, where $R = \omega/v$ and the wave speed v and the attenuation coefficient (q) are real numbers.

8.2 Thermo-elastic damping factor

The thermo-elastic damping factor (Q^{-1}) is defined with respect to the angular frequency ω as

$$Q^{-1} = 2\Big|\frac{Im(\omega)}{Re(\omega)}\Big|. \tag{8.2}$$

Where $Re(\omega)$ is the real part of ω and $Im(\omega)$ is the imaginary part of ω, respectively.

8.3 Attenuation coefficient

The attenuation coefficient is the quantity that characterizes how easily a material or medium can be penetrated by a beam of light, sound particles or matters. It is defined as

$$1/c = 1/s + iq/\omega, \tag{8.3}$$

where c is the phase velocity, ω is the angular frequency and q is the attenuation coefficient.

9 Numerical results and discussion

The frequency equation given in Eq. (6.4) is transcendental in nature with unknown frequency and wavenumber. The solutions of the frequency equation are obtained numerically by fixing the wave number. The material chosen for the numerical calculation is

PZT-4. The material properties of PZT-4 are taken form Berlincourt et al. [38] are used for the numerical calculation is given below:

$$
\begin{aligned}
&c_{11}=13.9\times 10^{10}Nm^{-2}, && c_{12}=7.78\times 10^{10}Nm^{-2}, && c_{13}=7.43\times 10^{10}Nm^{-2},\\
&c_{33}=11.5\times 10^{10}Nm^{-2}, && c_{44}=2.56\times 10^{10}Nm^{-2}, && c_{66}=3.06\times 10^{10}Nm^{-2},\\
&e_{31}=-5.2Cm^{-2}, && e_{33}=15.1Cm^{-2}, && e_{15}=12.7Cm^{-2},\\
&\varepsilon_{11}=6.46\times 10^{-9}C^2N^{-1}m^{-2}, && \varepsilon_{33}=5.62\times 10^{-9}C^2N^{-1}m^{-2}, && \rho=7500Kgm^{-2},
\end{aligned}
$$

and for fluid the density $\rho^f=1000Kgm^{-3}$, phase velocity $c=1500m\sec^{-1}$ and used for the numerical calculations.

In this problem, there are two kinds of basic independent modes of wave propagation have been considered, namely, the longitudinal and flexural modes. By choosing respectively $n=0$ and $n=1$, we can obtain the non-dimensional frequencies of longitudinal and flexural modes of vibrations.

9.1 Comparison table

A comparison is made between non-dimensional wave number for thermo-piezoelectric cylindrical immersed in fluid and in space for longitudinal modes of vibration and is shown in Table 1. From Table 1, it is observed that the wave numbers of GL theory is higher than the wave numbers of both LS and CT theory. Also it is observed that the wave numbers of LS theory is higher than the wave numbers of CT theory. In addition, the wave numbers of cylinder immersed in fluid is higher than the wave numbers of cylinder in space for all the three theories. A comparison is made between non-dimensional wave number for thermo piezoelectric cylinder immersed in fluid and in space for flexural modes of vibration and is shown in Table 2. From Table 2, it is observed that the wave numbers of CT theory is higher than the wave numbers of both GL and LS theories for cylinder immersed in fluid and the wave numbers of CL theory is lesser than the wave numbers of both GL and LS theories for cylinder in space. Also it is observed that the wave number of LS theory for the case of immersed in fluid and it is reverse for the case of cylindrical bar in space. Further the wave numbers of cylindrical bar in space is greater than the wave number of cylinder immersed in fluid for all three theories.

Table 1: Comparison between non-dimensional wave numbers for thermo-piezoelectric cylinder immersed in fluid and in space for longitudinal modes of vibration.

Model	Immersed in fluid			In space		
	LS	GL	CT	LS	GL	CT
1	1.4145	1.4156	1.4026	1.2374	1.2358	0.7636
2	2.8288	2.8313	2.8284	2.7261	2.7520	2.8188
3	4.2473	4.2479	4.2468	4.2715	4.9672	4.2268
4	5.6563	5.6625	5.6521	7.7065	3.2459	5.6273
5	7.0786	7.0753	7.0774	4.0662	5.0004	7.0138

Table 2: Comparison between non-dimensional wave numbers for thermo-piezoelectric cylinder immersed in fluid and in space for flexural antisymmetric modes of vibration.

Model	Immersed in fluid			In space		
	LS	GL	CT	LS	GL	CT
1	1.4156	1.2539	1.4000	1.4144	1.4223	1.4230
2	2.0943	2.5807	2.8138	2.8283	2.8266	2.8223
3	3.1903	3.9568	4.2241	4.2483	4.2459	4.2322
4	4.0725	5.3512	5.6327	5.6578	5.6653	5.6400
5	5.0216	6.7540	7.0396	7.0755	7.0781	7.0446

9.2 Dispersion curves

The computed non-dimensional wave numbers are plotted in the form of dispersion curves. The notations used in the figures namely, LM, FM respectively denotes longitudinal and flexural modes of vibrations in all the graphs. The 1 refers the first mode and 2 the second and so on. A comparison graph is drawn between non-dimensional frequencies Ω versus dimensionless wave number $|\varsigma|$ of longitudinal and flexural modes of vibrations for LS and GL theory of thermo-piezoelectric cylindrical bar immersed in fluid are shown respectively in the Figs. 1 and 2. From Figs. 1 and 2, it is observed that the dimensionless wave number increases as non-dimensional frequency increases for both modes of vibration, further it is observed that the wave numbers of LS theory is higher than the wave numbers of GL theory. A Similar comparison graph is drawn between non-dimensional frequencies Ω versus attenuation coefficient of longitudinal modes for LS and GL theory of thermo-piezoelectric cylindrical bar immersed in fluid is shown in Fig. 3. From Fig. 3, it is observed that the attenuation coefficient increases as non-dimensional frequency increases for every modes of vibration, also, the attenuation coefficient of LS theory is higher than the attenuation coefficient of GL theory for any modes of vibration. A graph is drawn between non-dimensional frequencies Ω versus specific heat of longitudinal modes for LS and GL theory of thermo-piezoelectric cylin-

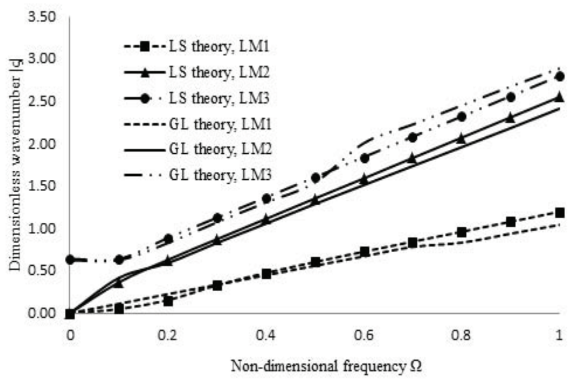

Figure 1: Comparison between non-dimensional frequency Ω versus dimensionless wavenumber $|\varsigma|$ of longitudinal modes of thermo-piezoelectric cylinderical bar immersed in fluid.

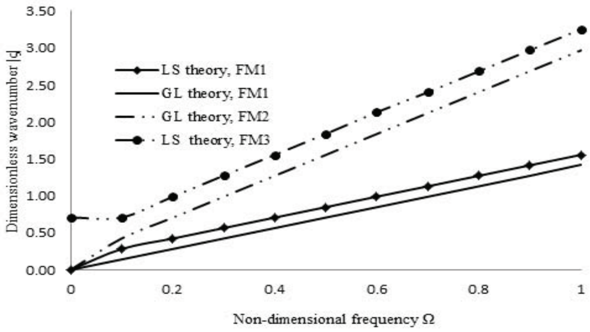

Figure 2: Comparison between non-dimensional frequency Ω versus dimensionless wavenumber $|\varsigma|$ of lflexural modes of thermo-piezoelectric cylinderical bar immersed in fluid.

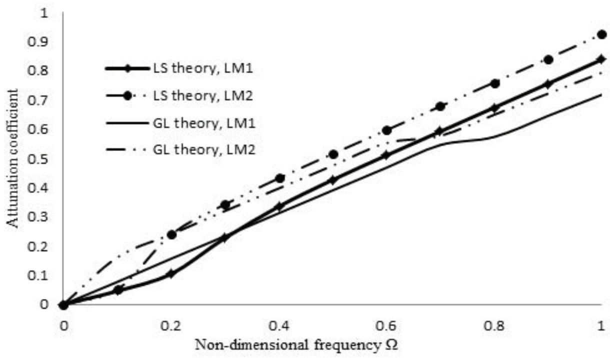

Figure 3: Comparison between non-dimensional frequency Ω versus attunation coefficient of longitudinal modes of thermo-piezoelectric cylinderical bar immersed in fluid.

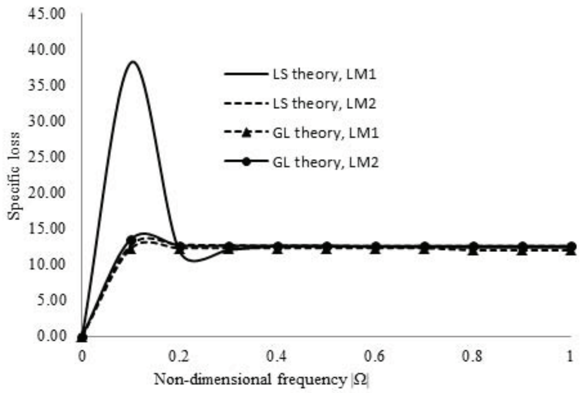

Figure 4: Comparison between non-dimensional frequency $|\Omega|$ versus specific loss of longitudinal modes of thermo-piezoelectric cylindrical bar immersed in fluid.

drical bar immersed in fluid and is shown in Fig. 4. From Fig. 4, it is observed that specific loss increases as non-dimensional frequency increases and finally it is asymptotic to the horizontal axis.

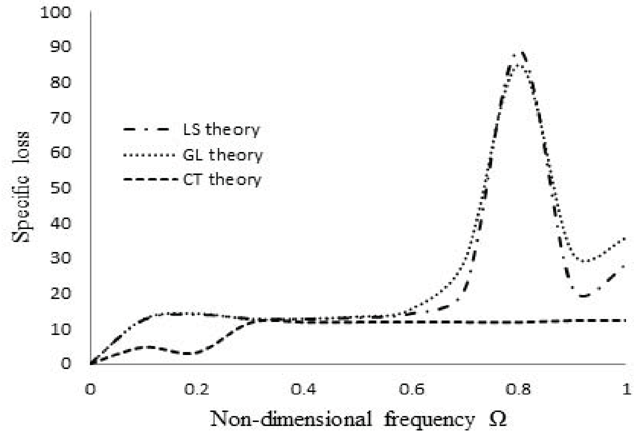

Figure 5: Non-dimensional frequency versus specific loss for longitudinal modes of thermo-piezoelectric cylindrical bar in space.

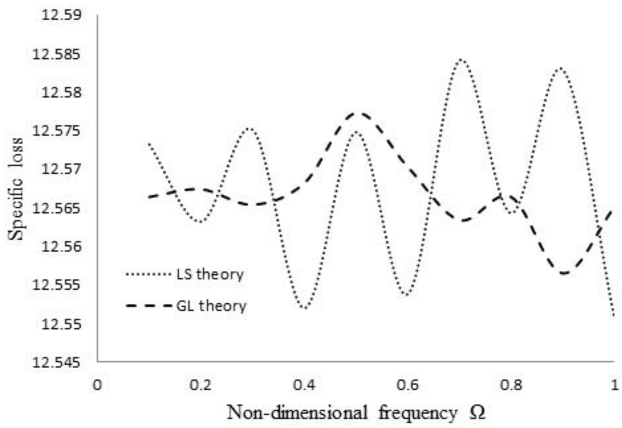

Figure 6: Non-dimensional frequency versus specific loss for flexural modes of thermo-piezoelectric cylindrical bar in space.

A dispersion curve is drawn to compare LS, GL and CT theories of thermo-piezoelectric cylindrical bar in space; here the graph is drawn between non-dimensional frequency Ω versus specific loss. The Figs. 5 and 6 respectively represent the specific loss of the material in longitudinal and flexural modes of vibrations. From the Fig. 6, it is observed that the behavior of material in LS and GL theory will be the same as compared with the CT theory of thermo-piezoelectric cylindrical bar, also, it is observed that, the non-dimensional frequencies increases the specific losses are also increases in both LS and GL theories of thermo elasticity. A similar graph to represents the flexural modes of vibration is shown in the Fig. 6. From the Fig. 6, it is observed that, the vibration and displacements are higher in the LS theory as compared to GL theory thermo elasticity. The cross over points in all the graphs represents the transfer of heat energy between any two modes of vibrations. The dispersion curve shown in the Fig. 7 represents the specific

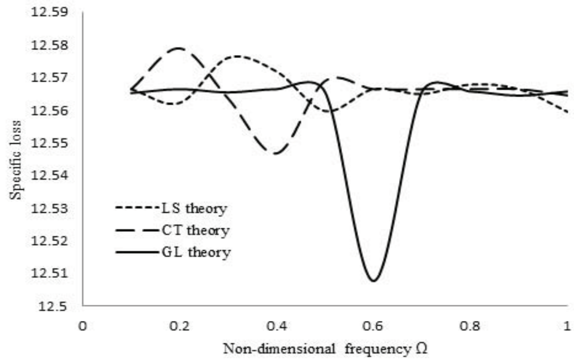

Figure 7: Non-dimensional frequency versus specific loss for longitudinal modes of thermo-piezoelectric cylindrical bar immersed in fluid.

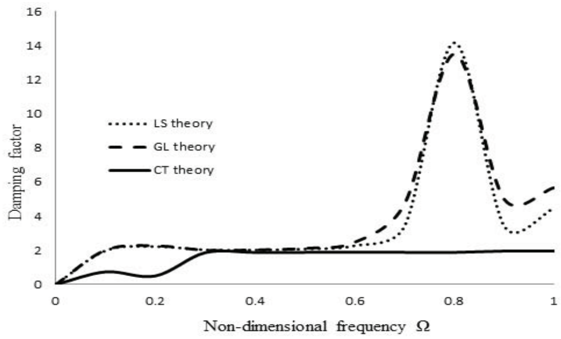

Figure 8: Non-dimensional frequency versus dimensionless daming factor for longitudinal modes of thermo-piezoelectric cylinderical bar in space.

loss occurred when the thermo-piezoelectric cylindrical bar immersed in fluid. From the Fig. 7, it is observed that the specific loss decrease by increasing the wave number, also after $\Omega=0.6$, the specific loss will get same in all the LS, GL and CT theories of thermo elasticity. A graph is drawn between non-dimensional frequency Ω versus damping factor for longitudinal modes thermo-piezoelectric cylindrical bar in space is shown in Fig. 8. From the Fig. 8, it is observed that the damping factor is increases by increasing the non-dimensional wave numbers in both LS and GL theories of thermo elasticity, where as in CT theory, the trend line linearly increases along the horizontal axis.

10 Conclusions

The propagation in a thermo-piezoelectric solid bar of circular cross-section immersed in fluid is discussed using three-dimensional theory of piezoelectricity. Three displacement potential functions are introduced to uncouple the equations of motion, electric conduction. The frequency equation of the coupled system consisting of cylinder and fluid is developed under the assumption of perfect-slip boundary conditions at the fluid-solid interfaces. The frequency equations are obtained for longitudinal and flexural modes of vibration and are studied numerically for PZT-4 material bar immersed in fluid based on Lord-Shulman (LS), Green-Lindsay (GL) and Classical theory (CT) theories of thermo elasticity. The computed non-dimensional frequencies are compared with Lord-Shulman (LS), Green-Lindsay (GL) and Classical theory (CT) theories of thermo elasticity for longitudinal and flexural modes of vibrations and the frequencies are tabulated. The dispersion curves are drawn for longitudinal and flexural modes of vibrations, the dispersion of specific loss and damping factors are also analyzed for longitudinal and flexural modes of vibrations.

Appendix

$$
\begin{aligned}
a_{1i}=&2\bar{c}_{66}\{n(n-1)J_n(\alpha_i ax)+(\alpha_i ax)J_{n+1}(\alpha_i ax)\}-x^2[(\alpha_i a)^2\bar{c}_{11}+\zeta\bar{c}_{13}a_i\\
&+\zeta b_i+\bar{\beta}c_i]J_n(\alpha_i ax), \qquad i=1,2,3,4,\\
a_{15}=&2\bar{c}_{66}\{n(n-1)J_n(\alpha_5 ax)-(\alpha_5 ax)J_{n+1}(\alpha_5 ax)\},\\
a_{2i}=&2n\{(\alpha_i ax)J_{n+1}(\alpha_i ax)-(n-1)J_n(\alpha_i ax)\}, \quad -i=1,2,3,4,\\
a_{25}=&\{[(\alpha_5 ax)^2-2n(n-1)]J_n(\alpha_5 ax)-2(\alpha_5 ax)J_{n+1}(\alpha_5 ax)\},\\
a_{3i}=&(\zeta+a_i+\bar{e}_{15}b_i)[nJ_n(\alpha_i ax)-(\alpha_i ax)J_{n+1}(\alpha_i ax)], \quad i=1,2,3,4,\\
a_{35}=&n\zeta J_n(\alpha_5 ax),\\
a_{4i}=&(\bar{e}_{15}\zeta a_i-\bar{\varepsilon}_{11}b_i)\{nJ_n(\alpha_i ax)-(\alpha_i ax)J_{n+1}(\alpha_i ax)\}, \quad i=1,2,3,4,\\
a_{45}=&\bar{e}_{15}\zeta nJ_n(\alpha_5 ax),\\
a_{5i}=&c_i\{nJ_n(\alpha_i ax)-(\alpha_i ax)J_{n+1}(\alpha_i ax)\}, \quad i=1,2,3,4,\\
a_{55}=&0.
\end{aligned}
$$

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