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STRONG CONVERGENCE AND STABILITY OF THE SEMI-TAMED AND TAMED EULER SCHEMES FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS UNDER NON-GLOBAL LIPSCHITZ CONDITION

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Abstract. We consider the explicit numerical approximations of stochastic differential equations (SDEs) driven by Brownian process and Poisson jump. It is well known that under non-global Lipschitz condition, Euler Explicit method fails to converge strongly to the exact solution of such SDEs without jumps, while implicit Euler method converges but requires much computational efforts. We investigate the strong convergence, the linear and nonlinear exponential stabilities of tamed Euler and semi-tamed methods for stochastic differential equation driven by Brownian process and Poisson jumps, both in compensated and non compensated forms. We prove that under non-global Lipschitz condition and superlinearly growing drift term, these schemes converge strongly with the standard one-half order. Numerical simulations to substain the theoretical results are provided.

Key words. Stochastic differential equation, strong convergence, linear stability, exponential stability, jump processes, one-sided Lipschitz.

1. Introduction

In this work, we consider jump-diffusion Itô's stochastic differential equations (SDEs) of the form in the interval $[0, T]$

$$dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0. \tag{1}$$

Here $W(t)$ is a m-dimensional Brownian motion, $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $d \in \mathbb{N}$ satisfies the one-sided Lipschitz condition and the polynomial growth condition, the functions $g : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$ and $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfy the globally Lipschitz, and $N(t)$ is a one dimensional Poisson process with parameter λ. Extension to vector-valued jumps with independent entries is straightforward. The one-sided Lipschitz function f can be decomposed as $f = u + v$, where the function $u : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is the global Lipschitz continuous part and $v : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is the non-global Lipschitz continuous part, see e.g. [25]. Using this decomposition, we can rewrite the jump-diffusion SDEs (1) in the following equivalent form

$$X(t) = \left(u(X(t^-) + v(X(t^-))\right) dt + g(X(t^-))dW(t) + h(X(t^-))dN(t). \tag{2}$$

This decomposition will be used only for semi-tamed schemes. Equations of type (1) arise in a range of scientific, engineering and financial applications [3, 1, 14]. Most of such equations do not have explicit solutions and therefore one requires numerical schemes for their approximations. Their numerical analysis has been studied in [6, 24, 5, 19] with implicit and explicit schemes where strong and weak convergence have been investigated. The implementation of implicit schemes requires significantly more computational effort than the explicit Euler-type approximations as Newton method is usually required to solve nonlinear systems at each time iteration in implicit schemes. The standard explicit method for approximating SDEs of type (1) is the Euler-Maruyama method [19]. Recently it has been proved

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(see [13, 11]) that the Euler-Maruyama method often fails to converge strongly to the exact solution of nonlinear SDEs of the form (1) without jump term when at least one of the functions f and g grows superlinearly. To overcome this drawback of the Euler-Maruyama method, numerical approximation, with computational cost close to that of the Euler-Maruyama method and which converges strongly even in the case the function f is superlinearly growing was first introduced in [12] and strong convergence was investigated. Further investigations have been performed in the litterature (see for example [21, 9, 25] and references therein), where in [21] the time step Δt in [12] is replaced by its power $\Delta t^{\alpha},\ \alpha \in (0, 1/2]$ in the denominator of the taming drift term. Recently the work in [21] has been extended for SDEs driven by compensated Levy noise in [2, 15]. The condition $\alpha \in (0, 1/2]$ is key in the convergence proofs in [2, 21, 15], so the proofs cannot be extended for $\alpha \in [1/2, 1]$. Strong and weak convergences are not the only features of numerical techniques. Stability is also a good feature as the information about time step size for which does a particular numerical method replicate the stability properties of the exact solution is valuable. The linear stability is an extension of the deterministic A-stability while exponential stability can guarantee that errors introduced in one time step will decay exponentially in future time steps, exponential stability also implies asymptotic stability [8]. By the Chebyshev inequality and the Borel–Cantelli lemma, it is well known that exponential mean-square stability implies almost sure stability [8]. The stability of classical implicit and explicit methods for (1) are well understood [6, 8, 24]. Although the strong convergence of tamed schemes with and without jump have been studied, a rigorous stability properties have not yet been investigated to the best of our knowledge.

The aim of this paper is to study the strong convergence of tamed schemes driven by Brownian process and Poisson jump for $\alpha \in [1/2, 1]$, and to provide a rigorous study of the linear and exponential stabilities of semi-tamed and tamed schemes for $\alpha \in [0, 1]$. Following closely the breakthrough idea in [12], we provide the strong convergence of the tamed schemes and the corresponding semi tamed schemes both in compensated and non compensated forms for $\alpha \in [1/2, 1]$. The extensions are not straightforward as several technical lemmas are needed. Numerical experiments show that the semi-tamed works better than the tamed and compensated tamed schemes. Numerical results also show that the tamed and the compensated tamed Euler scheme have good stability behavior when α approaches 1. Therefore, our tamed schemes with $\alpha \in [1/2, 1]$ have better stability property than the tamed schemes presented in [2] for $\alpha \in (0, 1/2]$.

The paper is organized as follows. Section 2 presents the classical result of existence and uniqueness of the solution X of (1). The compensated and non compensated tamed schemes and semi-tamed scheme are presented in Section 3 along with their strong convergences. The linear stability of the schemes is provided in Section 4 while the nonlinear exponential stability is provided in Section 5. We end in Section 6 by providing some numerical simulations.

2. Notations, assumptions and well posedness

Throughout this work, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. For all $x, y \in \mathbb{R}^d$, we denote by $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_dy_d$, $\|x\| = \langle x, x\rangle^{1/2}$, $\|A\| = \sup\limits_{x\in\mathbb{R}^d, \|x\|\leq 1} \|Ax\|$ for all $A \in \mathbb{R}^{m\times d}$. $a\vee b$ represents $\max\{a, b\}$. We use also the following convention : $\sum_{i=u}^{n} = 0$ for $u > n$.

We first ensure that SDEs (1) is well-posed. The following assumption is needed.

Assumption 2.1. *We assume that:*
$(A.1)$ *For all $p > 0$, there exists $M_p > 0$ such that $\mathbb{E}\|X_0\|^p \leq M_p$, and $f, g, h \in C^1(\mathbb{R}^d)$.*
$(A.2)$ *The functions g, h and u satisfy the following global Lipschitz condition*

$$\|g(x) - g(y)\| \vee \|h(x) - h(y)\| \vee \|u(x) - u(y)\| \leq C\|x - y\| \quad \forall \quad x, y \in \mathbb{R}^d.$$

$(A.3)$ *The function f satisfies the following one-sided Lipschitz condition*

$$\langle x - y, f(x) - f(y)\rangle \leq C\|x - y\|^2 \quad \forall \quad x, y \in \mathbb{R}^d.$$

$(A.4)$ *The function f satisfies the following superlinear growth condition*

$$\|f(x) - f(y)\| \leq C(1 + \|x\|^c + \|y\|^c)\|x - y\| \quad \forall \quad x, y \in \mathbb{R}^d,$$

where C and c are positive constants.

Remark 2.1. *Note that from Assumption 2.1, u satisfies the global Lipchitz condition, and f satisfies the one-sided Lipschitz condition and the superlinear growth condition, which implies that the function v satisfies the one-sided Lipschitz condition $(A.3)$ and the superlinear growth condition $(A.4)$ in Assumption 2.1.*

Theorem 2.1. *Under the conditions $(A.1)$, $(A.2)$ and $(A.3)$ of Assumption 2.1, the SDE* (1) *has a unique solution with all bounded moments.*

Proof. See [4] for the existence and the uniqueness and [6, Lemma 1] for the boundedness of the moments of the solution. □

3. Numerical Schemes and main results

We consider the SDEs (1) in the current non compensated form. Applying the tamed Euler scheme (as in [12]) in the drift term of (1) yields the following schemes that we will call non compensated tamed scheme (NCTS)

$$X_{n+1}^M = X_n^M + \frac{\Delta t f(X_n^M)}{1 + \Delta t^\alpha \|f(X_n^N)\|} + g(X_n^M)\Delta W_n^M + h(X_n^M)\Delta N_n^M, \tag{3}$$

where $\Delta t = T/M$ is the time step-size, $M \in \mathbb{N}$ is the number of time subdivisions, $\alpha \in [1/2, 1]$, $\Delta W_n^M = W(t_{n+1}) - W(t_n)$ and $\Delta N_n^M = N(t_{n+1}) - N(t_n)$. Applying the semi-tamed Euler scheme (as in [25]) in the non globally Lipschitz part v of the drift term of (2) yields the following scheme that we will call semi-tamed scheme (STS)

$$Z_{n+1}^M = Z_n^M + u(Z_n^M)\Delta t + \frac{\Delta t v(Z_n^M)}{1 + \Delta t^\alpha \|v(Z_n^M)\|} + g(Z_n^M)\Delta W_n + h(Z_n^M)\Delta N_n^M. \tag{4}$$

Recall that the compensated poisson process $\overline{N}(t) := N(t) - \lambda t$ is a martingale and satisfies the the following properties

$$\mathbb{E}\left(\overline{N}(t+s) - \overline{N}(t)\right) = 0 \qquad \mathbb{E}|\overline{N}(t+s) - \overline{N}(t)|^2 = \lambda s, \qquad s, t \geqslant 0. \tag{5}$$

We can easily check that the quadratic variation of $\overline{N}(t)$ is $[\overline{N}, \overline{N}]_t = N(t)$.

We can therefore rewrite the jump-diffusion SDEs (1) in the following equivalent form

$$dX(t) = f_\lambda(X(t^-)dt + g(X(t^-)dW(t) + h(X(t^-)d\overline{N}(t), \tag{6}$$

where $f_\lambda(x) = f(x) + \lambda h(x)$. Note that as f, the function f_λ satisfies the one-sided Lipschitz condition $(A.3)$ and the superlinear growth $(A.4)$. Applying the tamed

Euler scheme in the drift term of (6) as in [12] yields the following updated scheme for jump SDEs (1) that we will call compensated tamed scheme (CTS)

$$Y_{n+1}^M = Y_n^M + \frac{\Delta t f_\lambda(Y_n^M)}{1+\Delta t^\alpha \|f_\lambda(Y_n^M)\|} + g(Y_n^M)\Delta W_n^M + h(Y_n^M)\Delta\overline{N}_n^M, \tag{7}$$

where $\Delta\overline{N}_n^M = \overline{N}(t_{n+1}) - \overline{N}(t_n)$.

Note that if the equivalent model (2) is putting in the compensated form, and the semi-tamed method is applied on the non globally Lipschitz part v of the drift term f, we will obtain the same scheme as in (4).

We define the continuous time interpolations of the discrete numerical approximations of (3), (4) and (7) respectively by

$$\begin{aligned} \overline{X}_t^M &= X_n^M + \frac{(t-n\Delta t)f(X_n^M)}{1+\Delta t^\alpha\|f(X_n^M)\|} + g(X_n^M)(W_t - W_{n\Delta t}) \\ &\quad + h(X_n^M)(N_t - N_{n\Delta t}), \end{aligned} \tag{8}$$

$$\begin{aligned} \overline{Z}_t^M &= Z_n^M + (t-n\Delta t)\left(u(Z_n^M) + \frac{v(Z_n^M)}{1+\Delta t^\alpha\|v(Z_n^M)\|}\right) \\ &\quad + g(Z_n^M)(W_t - W_{n\Delta t}) + h(Z_n^M)(N_t - N_{n\Delta t}), \end{aligned} \tag{9}$$

and

$$\begin{aligned} \overline{Y}_t^M &= Y_n^M + \frac{(t-n\Delta t)f_\lambda(Y_n^M)}{1+\Delta t^\alpha\|f_\lambda(Y_n^M)\|} + g(Y_n^M)(W_t - W_{n\Delta t}) \\ &\quad + h(Y_n^M)(\overline{N}_t - \overline{N}_{n\Delta t}), \end{aligned} \tag{10}$$

for all $t \in [n\Delta t, (n+1)\Delta t)$, $\quad n \in \{0, \cdots, M-1\}$.

The main result of this section is given in the following theorem.

Theorem 3.1. *[Main result]*
Let X_t be the exact solution of (1) and χ_t^M the discrete continuous form of the numerical approximations given by (8),(9) and (10) ($\chi_t^M = \overline{X}_t^M$ for scheme NCTS, $\chi_t^M = \overline{Z}_t^M$ for scheme STS and $\chi_t^M = \overline{Y}_t^M$ for scheme CTS). Under Assumption 2.1, for all $p \in [1, +\infty)$ there exists a constant $C_p > 0$ independent of Δt such that

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t - \chi_t^M\|^p\right]\right)^{1/p} \leq C_p \Delta t^{1/2}, \qquad \Delta t = T/M. \tag{11}$$

3.1. Proof of Theorem 3.1 for $\chi_t^M = \overline{Y}_t^M$. Before giving the proof of Theorem 3.1, some preparatory results are needed. Here we consider the compensated tamed scheme (CTS) given by (7).

3.1.1. Preparatory results. Throughout this work the following notations will be used with slight modification in the next section

$$\alpha_k^M := \mathbb{1}_{\{\|Y_k^M\|\geq 1\}}\left\langle \frac{Y_k^M}{\|Y_k^M\|}, \frac{g(Y_k^M)}{\|Y_k^M\|}\Delta W_k^M\right\rangle, \quad k = 0, \cdots, M$$

$$\beta_k^M := \mathbb{1}_{\{\|Y_k^M\|\geq 1\}}\left\langle \frac{Y_k^M}{\|Y_k^M\|}, \frac{h(Y_k^M)}{\|Y_k^M\|}\Delta\overline{N}_k^M\right\rangle, \quad k = 0, \cdots, M$$

$$\beta := \left(1 + K + 2C + KTC + TC + T\|f_\lambda(0)\| + \|g(0)\| + \|h(0)\|\right)^4,$$

$$D_n^M := (\beta + \|X_0\|)\exp\left(\frac{3\beta}{2} + \sup_{m\in\{0,\cdots,n\}}\sum_{k=m}^{n-1}\left[\frac{3\beta}{2}\|\Delta W_k^M\|^2 + \frac{3\beta}{2}|\Delta\overline{N}_k^M| + \alpha_k^M + \beta_k^M\right]\right),$$

$$\Omega_n^M := \left\{\omega\in\Omega : \sup_{k\in\{0,1,\cdots,n-1\}} D_k^M(\omega) \leq M^{1/2c}, \sup_{k\in\{0,1,\cdots,n-1\}}\left(\|\Delta W_k^M(\omega)\| \vee |\Delta\overline{N}_k^M(\omega)|\right) \leq 1\right\}.$$

The aims of this section is to update all lemmas used in [12] and provide new lemmas for Poisson jump.

Lemma 3.1. *For all $a, b \geq 0$, the following inequality holds*

$$1 + a + b^2 \leq e^{a+\sqrt{2}b}.$$

Proof. See [22, 17]. □

Lemma 3.2. *The following inequality holds for all $M \in \mathbb{N}$ and all $n \in \{0, 1, \cdots, M\}$*

$$\mathbb{1}_{\Omega_n^M}\|Y_n^M\| \leq D_n^M, \tag{12}$$

where D_n^M and Ω_n^M are given in (12).

Proof. The proof follows the same lines as [12, Lemma 3.1]. Details can be found in [22, 17]. □

The proofs of the following can be found in [12, 17].

Lemma 3.3. *The following inequality holds*

$$\sup_{M\in\mathbb{N}, M\geq 4\beta pT} \mathbb{E}\left[\exp\left(\beta p \sum_{k=0}^{M-1}\|\Delta W_k^M\|^2\right)\right] < \infty, \qquad p\in[1,\infty).$$

Lemma 3.4. *Let $\alpha_n^M : \Omega \longrightarrow \mathbb{R}$, $M \in \mathbb{N}$, $n \in \{0, 1, \cdots, M\}$ be the process defined in* (12)*. The following inequality holds*

$$\sup_{z\in\{-1,1\}}\sup_{M\in\mathbb{N}}\left\|\sup_{n\in\{0,1,\cdots,M\}}\exp\left(z\sum_{k=0}^{n-1}\alpha_k^M\right)\right\|_{L^p(\Omega,\mathbb{R})} < \infty, \quad p\in[2,+\infty).$$

Lemma 3.5. *Let $c \in \mathbb{R}$, the following equality holds*

$$\mathbb{E}[\exp(c\Delta\overline{N}_n^M)] = \exp\left[\frac{(e^c + c - 1)\lambda T}{M}\right], \quad M\in\mathbb{N}, \quad n\in\{0,\cdots,M\}.$$

Proof. The proof follows from the moment generating function of a Poisson process. See [22, 17] for details. □

Lemma 3.6. *The following inequality holds*

$$\mathbb{E}\left[\exp\left(pz\mathbb{1}_{\{\|x\|\geq 1\}}\left\langle\frac{x}{\|x\|}, \frac{h(x)}{\|x\|}\Delta\overline{N}_n^M\right\rangle\right)\right] \leq \exp\left[\frac{\lambda\left(e^{p(C+\|h(0)\|)} + p(C+\|h(0)\|\right)}{M}\right],$$

for all $M \in \mathbb{N}$, $z \in \{-1, 1\}$, all $p \in [1, +\infty)$ and all $n \in \{0, \cdots, M\}$.

Proof. The proof follows the same lines as that of [12, (49)] by using Lemma 3.5. See [22, 17] for details. □

Lemma 3.7. *Let $\beta_n^M : \Omega \longrightarrow \mathbb{R}$ be the process defined as in* (12) *for all $M \in \mathbb{N}$ and all $n \in \{0, \cdots, M\}$. The following inequality holds*

$$\sup_{z\in\{-1,1\}}\sup_{M\in\mathbb{N}}\left\|\sup_{n\in\{0,\cdots,M\}}\exp\left(z\sum_{k=0}^{n-1}\beta_k^M\right)\right\|_{L^p(\Omega,\mathbb{R})} < +\infty, \qquad p\in(1,\infty).$$

Proof. The proof follows the same lines as that of [12, Lemma 3.4] by using Lemma 3.6. Details can be found in [22, 17]. □

Lemma 3.8. *The following inequality holds*

$$\sup_{M\in\mathbb{N}} \mathbb{E}\left[\exp\left(p\beta\sum_{k=0}^{M-1}|\Delta\overline{N}_k^M|\right)\right] < +\infty, \quad p\in[1,+\infty).$$

Proof. Using the independence and the stationarity of $\Delta\overline{N}_k^M$, along with Lemma 3.5, it follows that

$$\begin{aligned}\sup_{M\in\mathbb{N}} \mathbb{E}\left[\exp\left(p\beta\sum_{k=0}^{M-1}|\Delta\overline{N}_k^M|\right)\right] &= \sup_{M\in\mathbb{N}}\left(\prod_{k=0}^{M-1}\mathbb{E}[\exp(p\beta|\Delta\overline{N}_k^M|)]\right)\\ &= \sup_{M\in\mathbb{N}}\left[\left(\mathbb{E}[\exp(p\beta|\Delta\overline{N}_k^M|)]\right)^M\right]\\ &= \sup_{M\in\mathbb{N}}\left[\left(\exp\left[\frac{(e^{p\beta}+p\beta-1)\lambda T}{M}\right]\right)^M\right]\\ &= \exp[\lambda T\left(e^{p\beta}+p\beta-1\right)] < +\infty.\end{aligned}$$

□

Inspired by [12, Lemma 3.5], we have the following estimation.

Lemma 3.9. ***[Uniformly bounded moments of the process*** D_n^M***]***
Let $D_n^M : \Omega \longrightarrow [0,\infty)$, $M\in\mathbb{N}$, $n\in\{0,1,\cdots,M\}$ *be the process defined in* (12), *then we have*

$$\sup_{M\in\mathbb{N}, M\geq 8\lambda pT}\left\|\sup_{n\in\{0,1,\cdots,M\}} D_n^M\right\|_{L^p(\Omega,\mathbb{R})} < \infty, \quad p\in[1,\infty).$$

Proof. The proof follows the same lines as that of [12, Lemma 3.5] by using additionally Lemma 3.7. See [22, 17] for details. □

The following lemma is an extension of [12, Lemma 3.6]. Here, we include the jump part.

Lemma 3.10. *Let* $\Omega_M^M\in\mathcal{F}$, $M\in\mathbb{N}$ *be the process defined in* (12). *The following holds*

$$\sup_{M\in\mathbb{N}}\left(M^p\mathbb{P}[(\Omega_M^M)^c]\right) < +\infty, \quad p\in[1,\infty).$$

Proof. The proof follows the same lines as [12, Lemma 3.6]. Details can be found in [22, 17]. □

The following lemma can be found in [20, Theorem 48 pp 193] or in [14, Theorem 1.1, pp 1].

Lemma 3.11. *[Burkholder-Davis-Gundy inequality (BDG)]*

Let M *be a martingale with càdlàg paths and let* $p\geq 1$ *be fixed. Let* $M_t^* = \sup_{s\leq t}\|M_s\|$. *Then there exist constants* c_p *and* C_p *such that*

$$c_p\left[\mathbb{E}\left([M,M]_t\right)^{p/2}\right]^{1/p} \leq [\mathbb{E}(M_t^*)^p]^{1/p} \leq C_p\left[\mathbb{E}\left([M,M]_t\right)^{p/2}\right]^{1/p},$$

for all $0\leq t\leq\infty$, *where* $[M,M]_t$ *stand for the quadratic variation of the process* M.

The proof of the following lemma can be found in [12, Lemma 3.7] or [17].

Lemma 3.12. *Let $k \in \mathbb{N}$ and let $Z : [0,T]\times\Omega \longrightarrow \mathbb{R}^{k\times m}$ be a predictable stochastic process satisfying $\mathbb{P}\left[\int_0^T \|Z_s\|^2 ds < +\infty\right] = 1$. Then we have the following inequality*

$$\left\|\sup_{s\in[0,t]}\left\|\int_0^s Z_u dW_u\right\|\right\|_{L^p(\Omega,\mathbb{R})} \leq C_p\left(\int_0^t \sum_{i=1}^m \|Z_s\vec{e}_i\|^2_{L^p(\Omega,\mathbb{R}^k)} ds\right)^{1/2}$$

for all $t\in[0,T]$ and all $p\in[1,\infty)$, where $(\vec{e}_1,\cdots,\vec{e}_m)$ is the canonical basis of $\mathbb{R}^m$.

The following lemma can be found in [12, Lemma 3.8, pp 16] or [17].

Lemma 3.13. *Let $k \in \mathbb{N}$ and let $Z_l^M : \Omega \longrightarrow \mathbb{R}^{k\times m}$, $l \in \{0,1,\cdots,M-1\}$, $M\in\mathbb{N}$ be a familly of mappings such that Z_l^M is $\mathcal{F}_{lT/M}/\mathcal{B}(\mathbb{R}^{k\times m})$-measurable for all $l\in\{0,1,\cdots,M-1\}$ and $M\in\mathbb{N}$. Then the following inequality holds:*

$$\left\|\sup_{j\in\{0,1,\cdots,n\}}\left\|\sum_{l=0}^{j-1} Z_l^M \Delta W_l^M\right\|\right\|_{L^p(\Omega,\mathbb{R})} \leq C_p\left(\sum_{l=0}^{n-1}\sum_{i=1}^m ||Z_l^M.\vec{e}_i||^2_{L^p(\Omega,\mathbb{R}^k)}\frac{T}{M}\right)^{1/2}, \quad p\geq 1.$$

Lemma 3.14. *Let $k \in \mathbb{N}$ and $Z : [0,T]\times\Omega \longrightarrow \mathbb{R}^k$ be a predictable stochastic process satisfying $\mathbb{P}\left[\int_0^T \|Z_s\|^2 ds < +\infty\right] = 1$. Then the following inequality holds:*

$$\left\|\sup_{s\in[0,T]}\left\|\int_0^s Z_u d\overline{N}_u\right\|\right\|_{L^p(\Omega,\mathbb{R})} \leq C_p\left(\int_0^T \|Z_s\|^2_{L^p(\Omega,\mathbb{R}^k)} ds\right)^{1/2}, \quad t\in[0,T], \quad p\in[1,+\infty).$$

Proof. Since $\overline{N}$ is a martingale with càdlàg paths satisfying $d[\overline{N},\overline{N}]_s = dN_s$, it follows from the property of the quadratic variation (see [14, (8.21), pp 219]) that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t Z_s d\overline{N}_s, \int_0^t Z_s d\overline{N}_s\right]\\
(13)\qquad = \ &\mathbb{E}\left[\int_0^t \|Z_s\|^2 dN_s\right] = \mathbb{E}\left[\int_0^t \|Z_s\|^2 d\overline{N}_s\right] + \lambda\mathbb{E}\left[\int_0^t \|Z_s\|^2 ds\right].
\end{aligned}$$

The first term of (13) vanishes as the compensated Poisson process is a martingale. Therefore, we have

$$\mathbb{E}\left[\int_0^t Z_s d\overline{N}_s, \int_0^t Z_s d\overline{N}_s\right] = \lambda\mathbb{E}\left[\int_0^t \|Z_s\|^2 ds\right]. \tag{14}$$

The proof follows from BDG inequality and Minkowski's inequality. In fact Applying Lemma 3.11 with $M_t = \sup\limits_{0\leq t\leq T}\int_0^t Z_s d\overline{N}_s$ and using (14) leads to

$$\left[\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\int_0^t Z_u d\overline{N}_u\right\|^p\right]\right]^{1/p} \leq C_p\left[\mathbb{E}\left(\int_0^T \|Z_s\|^2 ds\right)^{p/2}\right]^{1/p}, \tag{15}$$

where C_p is a positive constant depending on p and λ.

Using the definition of $\|X\|_{L^p(\Omega,\mathbb{R}^d)}$ for any random variable X, it follows from (15) that

$$\left\|\sup_{s\in[0,T]}\left\|\int_0^s Z_u d\overline{N}_u\right\|\right\|_{L^p(\Omega,\mathbb{R})} \leq C_p\left\|\int_0^T \|Z_s\|^2 ds\right\|^{1/2}_{L^{p/2}(\Omega,\mathbb{R})}. \tag{16}$$

Using Minkowski's inequality in its integral form yields

$$\begin{aligned}\left\|\sup_{s\in[0,T]}\left\|\int_0^s Z_u d\overline{N}_u\right\|\right\|_{L^p(\Omega,\mathbb{R})} &\leq C_p\left(\int_0^T \left\|\|Z_s\|^2\right\|_{L^{p/2}(\Omega,\mathbb{R})} ds\right)^{1/2}\\ &= C_p\left(\int_0^T \|Z_s\|^2_{L^p(\Omega,\mathbb{R}^k)} ds\right)^{1/2}.\end{aligned}$$

□

Lemma 3.15. *Let $k\in\mathbb{N}$, $M\in\mathbb{N}$ and $Z_l^M:\Omega\longrightarrow\mathbb{R}^k, l\in\{0,1,\cdots,M-1\}$ be a family of mappings such that Z_l^M is $\mathcal{F}_{lT/M}/\mathcal{B}(\mathbb{R}^k)$-measurable for all $l\in\{0,1,\cdots,M-1\}$, then $\forall\, n\in\{0,1\cdots,M\}$ the following inequality holds*

$$\left\|\sup_{j\in\{0,1,\cdots,n\}}\left\|\sum_{l=0}^{j-1} Z_l^M\Delta\overline{N}_l^M\right\|\right\|_{L^p(\Omega,\mathbb{R})}\leq C_p\left(\sum_{j=0}^{n-1}||Z_j^M||^2_{L^p(\Omega,\mathbb{R}^k)}\frac{T}{M}\right)^{1/2},\quad p\geq 1.$$

Proof. Let us define $\overline{Z}^M:[0,T]\times\Omega\longrightarrow\mathbb{R}^k$ such that $\overline{Z}_s^M := Z_l^M$ for all $s\in\left[\frac{lT}{M},\frac{(l+1)T}{M}\right)$, $l\in\{0,1,\cdots,M-1\}$.

Using the definition of stochastic integral and Lemma 3.14, it follows that

$$\begin{aligned}\left\|\sup_{j\in\{0,1,\cdots,n\}}\left\|\sum_{l=0}^{j-1} Z_l^M\Delta\overline{N}_l^M\right\|\right\|_{L^p(\Omega,\mathbb{R})} &= \left\|\sup_{j\in\{0,1,\cdots,n\}}\left\|\int_0^{jT/M}\overline{Z}_u^M d\overline{N}_u^M\right\|\right\|_{L^p(\Omega,\mathbb{R})}\\ &\leq \left\|\sup_{s\in[0,nT/M]}\left\|\int_0^s \overline{Z}_u^M d\overline{N}_u\right\|\right\|_{L^p(\Omega,\mathbb{R}^k)}\\ &\leq C_p\left(\int_0^{nT/M}\|\overline{Z}_u^M\|^2_{L^p(\Omega,\mathbb{R}^k)}ds\right)^{1/2}\\ &= C_p\left(\sum_{j=0}^{n-1}||Z_j^M||^2_{L^p(\Omega,\mathbb{R}^k)}\frac{T}{M}\right)^{1/2}.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.16. *Let $Y_n^M:\Omega\longrightarrow\mathbb{R}^d$ be defined by (7) for $n\in\{0,\cdots,M\}$ and all $M\in\mathbb{N}$. The following inequality holds*

$$\sup_{M\in\mathbb{N}}\sup_{n\in\{0,\cdots,M\}}\mathbb{E}\left[\|Y_n^M\|^p\right]<+\infty,\quad p\in[1,\infty).$$

Proof. The proof follows the same lines as [12, Lemma 3.10]. See [22, 17] for details. □

Lemma 3.17. *Let Y_n^M be defined by (7) for all $M\in\mathbb{N}$ and all $n\in\{0,1,\cdots,M\}$, then we have*

$$\sup_{M\in\mathbb{N}}\sup_{n\in\{0,1,\cdots,M\}}\left(\mathbb{E}\left[\|f_\lambda(Y_n^M)\|^p\right]\vee\mathbb{E}\left[\left\|g(Y_n^M)\right\|^p\right]\vee\mathbb{E}\left[\left\|h(Y_n^M)\right\|^p\right]\right)<+\infty,\quad p\in[1,\infty).$$

Proof. The proof follows the same lines as that of [12, Lemma 3.10]. See [22, 17] for details. □

In the sequel, for all $s\in[0,T]$ we denote by $\lfloor s\rfloor$ the greatest grid point less than s.

Lemma 3.18. *Let $\overline{Y}_t^M$ be the time continuous approximation given by* (10)*, there exists a constant C_p such that the following inequalities hold*

$$\sup_{t\in[0,T]} \left\|\overline{Y}_t^M - \overline{Y}_{\lfloor t\rfloor}^M\right\|_{L^p(\Omega,\mathbb{R}^d)} \leq C_p \Delta t^{1/2}, \tag{17}$$

$$\sup_{M\in\mathbb{N}} \sup_{t\in[0,T]} \left\|\overline{Y}_t^M\right\|_{L^p(\Omega,\mathbb{R}^d)} < \infty, \tag{18}$$

$$\sup_{t\in[0,T]} \left\|f_\lambda(\overline{Y}_t^M) - f_\lambda(\overline{Y}_{\lfloor t\rfloor}^M)\right\|_{L^p(\Omega,\mathbb{R}^d)} \leq C_p \Delta t^{1/2}. \tag{19}$$

for all $p \in [1,\infty)$.

Proof. The proof follows the same lines as that of [12, (67), (68), (70)]. Details can be found in [22, 17]. □

Now we are ready to give the proof of Theorem 3.1.

3.1.2. Main part of the proof of Theorem 3.1 for $\chi_t^M = \overline{Y}_t^M$. Recall that for $s \in [0,T]$, $\lfloor s\rfloor$ denote the greatest grid point less than s. The time continuous solution (10) can be written in its integral form as bellow

$$\begin{aligned}\overline{Y}_s^M &= X_0 + \int_0^s \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}du + \int_0^s g(\overline{Y}_{\lfloor u\rfloor}^M)dW_u \\ &\quad + \int_0^s h(\overline{Y}_{\lfloor u\rfloor}^M)d\overline{N}_u,\end{aligned} \tag{20}$$

for all $s \in [0,T]$ almost surely and all $M \in \mathbb{N}$.

Let us estimate first the quantity $\|X_s - \overline{Y}_s^M\|^2$, where X_s is the exact solution of (1).

$$\begin{aligned}X_s - \overline{Y}_s &= \int_0^s \left(f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right)du \\ &\quad + \int_0^s \left(g(X_u) - g(\overline{Y}_{\lfloor u\rfloor}^M)\right)dW_u + \int_0^s \left(h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right)d\overline{N}_u.\end{aligned}$$

Using the relation $d\overline{N}_u = dN_u - \lambda du$, it follows that

$$\begin{aligned}X_s - \overline{Y}_s &= \int_0^s \left[\left(f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right) - \lambda\left(h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right)\right]du \\ &\quad + \int_0^s \left(g(X_u) - g(\overline{Y}_{\lfloor u\rfloor}^M)\right)dW_u + \int_0^s \left(h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right)dN_u.\end{aligned}$$

The function $k : \mathbb{R}^d \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|^2$ is twice differentiable. Applying Itô's formula for jumps process ([18, pp. 6-9]) to the process $X_s - \overline{Y}_s^M$ leads to

$$\begin{aligned}&\left\|X_s - \overline{Y}_s^M\right\|^2 \\ &= 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du \\ &\quad -2\lambda\int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle du + \sum_{i=1}^m \int_0^s \|g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\quad +2\sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle dW_u^i \\ &\quad + \int_0^s \left[\|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2\right]dN_u.\end{aligned}$$

Using again the relation $dN_u = d\overline{N}_u + \lambda du$ leads to

$$
\begin{aligned}
& \left\|X_s - \overline{Y}_s^M\right\|^2 \\
= \ & 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du \\
& -2\lambda\int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle du + \sum_{i=1}^m \int_0^s \|g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\
& +2\sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle dW_u^i \\
& +\int_0^s \left[\|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2\right] d\overline{N}_u \\
& +\lambda\int_0^s \left[\|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2\right] du \\
= \ & A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \qquad (21)
\end{aligned}
$$

In the next step, we give some useful estimations of A_1, A_2, A_3 and A_6.

$$
\begin{aligned}
A_1 & := 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor})}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du \\
& = 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - f_\lambda(\overline{Y}_u^M)\right\rangle du \\
& \quad +2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du, \\
& = A_{11} + A_{12}.
\end{aligned}
$$

Using the one-sided Lipschitz condition satisfied by f_λ leads to

$$
A_{11} := 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(X_u) - f_\lambda(\overline{Y}_u^M)\right\rangle du \leq 2C\int_0^s \|X_u - \overline{Y}_u^M\|^2 du. \qquad (22)
$$

Moreover, using the inequality $2\langle a, b\rangle \leq 2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$ for all $a, b \in \mathbb{R}^d$ leads to

$$
\begin{aligned}
A_{12} & = 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du \\
& = 2\int_0^s \left\langle X_u - \overline{Y}_u^M, f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle ds \\
& \quad +2\Delta t^\alpha\int_0^s \left\langle X_u - \overline{Y}_u^M, \frac{f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}{1+\Delta t^\alpha\|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|}\right\rangle du \\
& \leq \int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\
& \quad +\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \frac{T^{2\alpha}}{M^{2\alpha}}\int_0^s \|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^4 du \\
& \leq 2\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor})\|^2 du \\
& \quad +\frac{T^{2\alpha}}{M^{2\alpha}}\int_0^s \|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^4 du. \qquad (23)
\end{aligned}
$$

Combining (22) and (23) give the following estimation of A_1

$$\begin{aligned} A_1 \quad &\leq \quad (2C+2)\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor})\|^2 du \\ &\quad + \frac{T^{2\alpha}}{M^{2\alpha}}\int_0^s \|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^4 du. \end{aligned} \tag{24}$$

Using again the inequality $2\langle a,b\rangle \leq 2\|a\|\|b\| \leq \|a\|^2 + \|b\|^2$ for all $a, b \in \mathbb{R}^d$ and the global Lipschitz condition satisfied by h leads to

$$\begin{aligned} A_2 \quad &:= \quad -2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle du \\ &= \quad -2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(X_u) - h(\overline{Y}_u^M)\right\rangle du \\ &\quad\quad -2\lambda \int_0^s \left\langle X_u - \overline{Y}_u^M, h(\overline{Y}_u^M) - h(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle du \\ &\leq \quad (2\lambda + \lambda C^2)\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + \lambda C^2 \int_0^s \|\overline{Y}_u^M - \overline{Y}_{\lfloor u\rfloor}^M\|^2 du. \end{aligned} \tag{25}$$

Using the inequalities $\|g_i(x) - g_i(y)\| \leq \|g(x) - g(y)\|$ and $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$ for all $a, b \in \mathbb{R}^d$ and the global Lipschitz condition we have

$$\begin{aligned} A_3 \quad &:= \quad \sum_{i=1}^m \int_0^s \|g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad m\int_0^s \|g(X_u) - g(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad m\int_0^s \|g(X_u) - g(\overline{Y}_u^M) + g(\overline{Y}_u^M) - g(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad 2m\int_0^s \|g(X_u) - g(\overline{Y}_u^M)\|^2 du + 2m\int_0^s \|g(\overline{Y}_u^M) - g(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad 2mC^2\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + 2mC^2 \int_0^s \|\overline{Y}_u^M - \overline{Y}_{\lfloor u\rfloor}^M\|^2 du. \end{aligned} \tag{26}$$

Using once again inequality $\|a+b\|^2 \leq 2\|a\|^2 + \|b\|^2$ for all $a, b \in \mathbb{R}^d$ we obtain the following estimation of A_6

$$\begin{aligned} A_6 \quad &:= \quad \lambda\int_0^s \left[\|X_u - \overline{Y}_u^M + h(\overline{Y}_u^M) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2\right] du \\ &\leq \quad \lambda\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + 2\lambda\int_0^s \|h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad \lambda\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + 4\lambda\int_0^s \|h(X_u) - h(\overline{Y}_u^M)\|^2 du \\ &\quad\quad +4\lambda\int_0^s \|h(\overline{Y}_u^M) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\ &\leq \quad (\lambda + 4\lambda C^2)\int_0^s \|X_u - \overline{Y}_u^M\|^2 du + 4\lambda C^2 \int_0^s \|\overline{Y}_u^M - \overline{Y}_{\lfloor u\rfloor}^M\|^2 du. \end{aligned} \tag{27}$$

Inserting (24), (25), (26) and (27) in (21) we obtain

$$
\begin{aligned}
&\left\|X_s - \overline{Y}_s^M\right\|^2 \\
\leq\ & (2C + 2 + 2mC^2 + 3\lambda + 5\lambda C^2)\int_0^s \|X_u - \overline{Y}_u^M\|^2 du \\
&+(2mC^2 + 5\lambda C^2)\int_0^s \|\overline{Y}_u^M - \overline{Y}_{\lfloor u\rfloor}^M\|^2 du + \int_0^s \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du \\
&+\frac{T^{2\alpha}}{M^{2\alpha}}\int_0^s \|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^4 du + 2\sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle dW_u^i \\
&+\int_0^s \left[\|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 - \|X_u - \overline{Y}_u^M\|^2\right] d\overline{N}_u.
\end{aligned}
$$

Taking the supremum in both sides of the previous inequality leads to

$$
\begin{aligned}
&\sup_{s\in[0,t]}\left\|X_s - \overline{Y}_s^M\right\|^2 \\
\leq\ & (2C + 2 + 2mC^2 + 3\lambda + 5\lambda C^2)\int_0^t \|X_u - \overline{Y}_u^M\|^2 du \\
&+(2mC^2 + 5\lambda C^2)\int_0^t \|\overline{Y}_u^M - \overline{Y}_{\lfloor u\rfloor}^M\|^2 du \\
&+\int_0^t \|f_\lambda(\overline{Y}_u^M) - f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^2 du + \frac{T^{2\alpha}}{M^{2\alpha}}\int_0^t \|f_\lambda(\overline{Y}_{\lfloor u\rfloor}^M)\|^4 du \\
&+2\sup_{s\in[0,t]}\left|\sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle dW_u^i\right| \\
&+\sup_{s\in[0,t]}\left|\int_0^s \left[\|X_u - \overline{Y}_u^M + h(X_u) - h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2\right] d\overline{N}_u\right| \\
&+\sup_{s\in[0,t]}\left|\int_0^s \|X_u - \overline{Y}_u^M\|^2 d\overline{N}_u\right|.
\end{aligned}
\tag{28}
$$

Using Lemma 3.12 we have the following estimation for all $p \geq 2$

$$
\begin{aligned}
B_1 &:= \left\|2\sup_{s\in[0,t]}\left|\sum_{i=1}^m \int_0^s \left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle dW_u^i\right|\right\|_{L^{p/2}(\Omega,\mathbb{R})} \\
&\leq C_p\left(\int_0^t \sum_{i=1}^m \left\|\left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle\right\|^2_{L^{p/2}(\Omega,\mathbb{R})} ds\right)^{1/2}.
\end{aligned}
$$

Moreover, using the inequalities $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$ and $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, we have the following estimations for all $p \geq 2$

$$
\begin{aligned}
B_1 &\leq C_p\left(\int_0^t \sum_{i=1}^m \left\|\left\langle X_u - \overline{Y}_u^M, g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\right\rangle\right\|^2_{L^{p/2}(\Omega,\mathbb{R})} du\right)^{1/2} \\
&\leq C_p\left(\int_0^t \sum_{i=1}^m \|X_u - \overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R})}\|g_i(X_u) - g_i(\overline{Y}_{\lfloor u\rfloor}^M)\|^2_{L^p(\Omega,\mathbb{R}^d)} du\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C_p}{\sqrt{2}}\left(\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|_{L^p(\Omega,\mathbb{R}^d)}\right)\left(2C^2m\int_0^t\|X_s-\overline{Y}_{\lfloor s\rfloor}^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\right)^{1/2}\\
&\leq \frac{1}{4}\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|^2_{L^p(\Omega,\mathbb{R}^d)}+C_p^2m\int_0^t\|X_s-\overline{Y}_{\lfloor s\rfloor}^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\\
&\leq \frac{1}{4}\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|^2_{L^p(\Omega,\mathbb{R}^d)}+2C_p^2m\int_0^t\|X_s-\overline{Y}_s^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\\
&\quad +2C_p^2m\int_0^t\|\overline{Y}_s^M-\overline{Y}_{\lfloor s\rfloor}^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds. \qquad (29)
\end{aligned}
$$

Using Lemma 3.14 and the inequality $(a+b)^4\leq 16a^4+16b^4$, it follows that

$$
\begin{aligned}
B_2 &:= \left\|\sup_{s\in[0,t]}\left|\int_0^s\|X_u-\overline{Y}_u^M+h(X_u)-h(\overline{Y}_{\lfloor u\rfloor}^M)\|^2d\overline{N}_u\right|\right\|_{L^{p/2}(\Omega,\mathbb{R}^d)}\\
&\leq C_p\left(\int_0^t\|X_u-\overline{Y}_u^M+h(X_u)-h(\overline{Y}_{\lfloor u\rfloor}^M)\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}\\
&\leq C_p\left(\int_0^t 16\|X_u-\overline{Y}_u^M\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}+16\|h(X_u)-h(\overline{Y}_{\lfloor u\rfloor}^M)\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2},
\end{aligned}
$$

for all $p\geq 2$.

Using the inequality $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for $a,b\in\mathbb{R}^+$, it follows that

$$
\begin{aligned}
B_2 &\leq 2C_p\left(\int_0^t\|X_u-\overline{Y}_u^M\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}\\
&\quad +2C_p\left(\int_0^t\|h(X_u)-h(\overline{Y}_{\lfloor u\rfloor}^M)\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}\\
&= B_{21}+B_{22}. \qquad (30)
\end{aligned}
$$

Using Holder's inequality, it follows that

$$
\begin{aligned}
B_{21} &:= 2C_p\left(\int_0^t\|X_u-\overline{Y}_u^M\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}\\
&\leq 2C_p\left(\int_0^t\|X_u-\overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R}^d)}\|X_u-\overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R}^d)}du\right)^{1/2}\\
&\leq \frac{1}{4}\sup_{u\in[0,t]}\|X_u-\overline{Y}_u^M\|_{L^p(\Omega,\mathbb{R}^d)}\ 8C_p\left(\int_0^t\|X_u-\overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R}^d)}du\right)^{1/2}.
\end{aligned}
$$

Using the inequality $2ab\leq a^2+b^2$ for $a,b\in\mathbb{R}$ leads to

$$
B_{21} \leq \frac{1}{16}\sup_{u\in[0,t]}\|X_u-\overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R}^d)}+16C_p^2\int_0^t\|X_u-\overline{Y}_u^M\|^2_{L^p(\Omega,\mathbb{R}^d)}du. \qquad (31)
$$

Using the inequalities $(a+b)^4 \leq 4a^4+4b^4$ and $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for $a, b \in \mathbb{R}^+$, we obtain

$$\begin{aligned}
& B_{22} := 2C_p\left(\int_0^t \|h(X_u)-h(\overline{Y}^M_{\lfloor u\rfloor})\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2} \\
\leq\ & 2C_p\left(\int_0^t \left[4\|h(X_u)-h(\overline{Y}^M_u)\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}+4\|h(\overline{Y}^M_u)-h(\overline{Y}^M_{\lfloor u\rfloor})\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}\right]du\right)^{1/2} \\
\leq\ & 4C_p\left(\int_0^t \|h(X_u)-h(\overline{Y}^M_u)\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2} \\
& +4C_p\left(\int_0^t \|h(\overline{Y}^M_u)-h(\overline{Y}^M_{\lfloor u\rfloor})\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}.
\end{aligned}$$

Using the global Lipschitz condition, leads to

$$\begin{aligned}
B_{22} \leq\ & 4C_p\left(\int_0^t C\|X_u-\overline{Y}^M_u\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2} \\
& +4C_p\left(\int_0^t C\|\overline{Y}^M_u-\overline{Y}^M_{\lfloor u\rfloor}\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}.
\end{aligned}$$

Using the same estimations as for B_{21}, it follows that :

$$\begin{aligned}
B_{22} \leq\ & \frac{1}{16}\sup_{s\in[0,t]}\|X_s-\overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)}+64C_p\int_0^t\|X_u-\overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)}du \\
& +\frac{1}{4}\sup_{s\in[0,t]}\|\overline{Y}^M_s-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}+64C_p\int_0^t\|\overline{Y}^M_u-\overline{Y}^M_{\lfloor u\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}du.
\end{aligned}$$

Taking the supremum under the integrand in the last term of the above inequality and using the fact that C_p is an arbitrary constant leads to

$$\begin{aligned}
B_{22} \leq\ & \frac{1}{16}\sup_{s\in[0,t]}\|X_s-\overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)}+64C_p\int_0^t\|X_u-\overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)}du \\
& +C_p\sup_{s\in[0,t]}\|\overline{Y}^M_s-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}. \qquad (32)
\end{aligned}$$

Inserting (31) and (32) into (30) gives

$$\begin{aligned}
B_2 \leq\ & \frac{1}{8}\sup_{s\in[0,t]}\|X_s-\overline{Y}^M_s\|^2_{L^p(\Omega,\mathbb{R}^d)}+C_p\int_0^t\|X_u-\overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)}du \\
& +C_p\sup_{s\in[0,t]}\|\overline{Y}^M_s-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}. \qquad (33)
\end{aligned}$$

Using again Lemma 3.14 leads to

$$\begin{aligned}
B_3 :=\ & \left\|\sup_{u\in[0,t]}\left(\int_0^s\|X_u-\overline{Y}^M_u\|^2 d\overline{N}_u\right)^{1/2}\right\|_{L^{p/2}(\Omega,\mathbb{R}^d)} \\
\leq\ & C_p\left(\int_0^t\|X_u-\overline{Y}^M_u\|^4_{L^{p/2}(\Omega,\mathbb{R}^d)}du\right)^{1/2}.
\end{aligned}$$

Using the same argument as for B_{21}, we obtain

$$B_3 \leq \frac{1}{8}\sup_{u\in[0,t]}\|X_u-\overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)}+C_p\int_0^t\|X_u-\overline{Y}^M_u\|^2_{L^p(\Omega,\mathbb{R}^d)}du. \qquad (34)$$

Taking the L^p norm in both side of (28), inserting inequalities (29), (33), (34) and using Minkowski's inequality in its integral form leads to

$$\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|^2_{L^p(\Omega,\mathbb{R})} = \left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|^2\right\|_{L^{p/2}(\Omega,\mathbb{R})},$$

so

$$\begin{aligned}
&\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|^2_{L^p(\Omega,\mathbb{R})}\\
\leq\;& C_p\int_0^t\|X_s-\overline{Y}_s^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds+C_p\int_0^t\|\overline{Y}_s^M-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\\
&+\int_0^t\|f_\lambda(X_s)-f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^2_{L^p(\Omega,\mathbb{R}^d)}ds+C_p\sup_{u\in[0,t]}\|\overline{Y}_u^M-\overline{Y}^M_{\lfloor u\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}\\
&+\frac{T^{2\alpha}}{M^{2\alpha}}\int_0^t\|f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^4_{L^{2p}(\Omega,\mathbb{R}^d)}ds+2C_p\int_0^t\|\overline{Y}_s^M-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\\
&+\frac{1}{2}\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|^2_{L^p(\Omega,\mathbb{R})},
\end{aligned}\tag{35}$$

for all $t\in[0,T]$ and all $p\in[2,+\infty)$.

Inequality (35) can be rewritten in the following appropriate form

$$\begin{aligned}
&\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|^2_{L^p(\Omega,\mathbb{R})}\\
\leq\;& C_p\int_0^t\|X_s-\overline{Y}_s^M\|^2_{L^p(\Omega,\mathbb{R}^d)}ds+C_p\int_0^t\|\overline{Y}_s^M-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\\
&+\int_0^t\|f_\lambda(X_s)-f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^2_{L^p(\Omega,\mathbb{R}^d)}ds+C_p\sup_{u\in[0,t]}\|\overline{Y}_u^M-\overline{Y}^M_{\lfloor u\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}\\
&+\frac{T^{2\alpha}}{M^{2\alpha}}\int_0^t\|f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^4_{L^{2p}(\Omega,\mathbb{R}^d)}ds+2C^2m\int_0^t\|\overline{Y}_s^M-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}ds.
\end{aligned}\tag{36}$$

Applying Gronwall's lemma to (36) leads to

$$\begin{aligned}
&\frac{1}{2}\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|^2_{L^p(\Omega,\mathbb{R})}\\
\leq\;& C_pe^{C_p}\left(\int_0^T\|f_\lambda(\overline{Y}_s^M)-f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^2_{L^p(\Omega,\mathbb{R}^d)}ds+C_p\sup_{u\in[0,t]}\|\overline{Y}_u^M-\overline{Y}^M_{\lfloor u\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}\right.\\
&\left.+\frac{T^{2\alpha}}{M^{2\alpha}}\int_0^T\|f_\lambda(\overline{Y}^M_{\lfloor s\rfloor})\|^4_{L^{2p}(\Omega,\mathbb{R}^d)}ds+C_p\int_0^T\|\overline{Y}_s^M-\overline{Y}^M_{\lfloor s\rfloor}\|^2_{L^p(\Omega,\mathbb{R}^d)}ds\right).
\end{aligned}\tag{37}$$

From (37) and the inequality $\sqrt{a+b+c} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$ for all $a,b,c \in \mathbb{R}^+$, it follows that

$$\begin{aligned}
&\frac{1}{2}\left\|\sup_{s\in[0,t]}\|X_s-\overline{Y}_s^M\|\right\|_{L^p(\Omega,\mathbb{R})}\\
\leq\quad & C_pe^{C_p}\left(\sup_{t\in[0,T]}\|f_\lambda(\overline{Y}_t^M)-f_\lambda(\overline{Y}_{\lfloor t\rfloor}^M)\|_{L^p(\Omega,\mathbb{R}^d)}+C_p\sup_{t\in[0,T]}\|\overline{Y}_t^M-\overline{Y}_{\lfloor t\rfloor}^M\|_{L^p(\Omega,\mathbb{R}^d)}\right.\\
&\left.+\frac{T^\alpha}{M^\alpha}\left[\sup_{n\in\{0,\cdots,M\}}\|f_\lambda(Y_n^M)\|^2_{L^p(\Omega,\mathbb{R}^d)}\right]+C_p\sup_{t\in[0,T]}\|\overline{Y}_t^M-\overline{Y}_{\lfloor t\rfloor}^M\|_{L^p(\Omega,\mathbb{R}^d)}\right), \qquad (38)
\end{aligned}$$

for all $p\in[2,\infty)$.

Using Lemma 3.17, Lemma 3.18 and the inequality $\dfrac{T^\alpha}{M^\alpha}\leq C_\alpha\,\Delta t^{1/2}$, it follows from (38) that

$$\left\|\sup_{t\in[0,T]}\|X_t-\overline{Y}_t^M\|\right\|_{L^p(\Omega,\mathbb{R})}=\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|X_t-\overline{Y}_t^M\right\|^p\right]\right)^{1/p}\leq C_p\Delta t^{1/2}, \qquad (39)$$

for all $p\in[2,\infty)$ and all $M\in\mathbb{N}$. Using Holder's inequality, one can prove that (39) holds for $p\in[1,2]$. The proof of the theorem is complete.

3.2. Proof of Theorem 3.1 for STS scheme ($\chi_t^M=\overline{Z}_t^M$). After replacing the increment of the poisson process ΔN_n^M by its compensated form $\Delta\overline{N}_n^M$ in STS (4), we obtain an equivalent scheme similar to the compensated tamed scheme (CTS). Therefore, the proof of the strong convergence of the STS follows exactly the one of compensated tamed scheme (CTS) (7) in Section 3.1. Here we should make the following changes for our semi-tamed scheme

$$\alpha_k^M := \mathbb{1}_{\{\|Z_k^M\|\geq 1\}}\left\langle\frac{Z_k^M+u_\lambda(Z_n^M)\Delta t}{\|Z_k^M\|},\frac{g(Z_k^M)}{\|Z_k^M\|}\Delta W_k^M\right\rangle,$$

$$\beta_k^M := \mathbb{1}_{\{\|Z_k^M\|\geq 1\}}\left\langle\frac{Z_k^M+u_\lambda(Z_n^M)\Delta t}{\|Z_k^M\|},\frac{h(Z_k^M)}{\|Z_k^M\|}\Delta\overline{N}_k^M\right\rangle,$$

where $u_\lambda=u+\lambda h$. The function v which is one-side Lipschitz (see Remark 2.1) should replace the function f_λ in the proof of the compensated tamed scheme (CTS). It follows from the proof in Section 3.1 that there exists a constant $C_p>0$ such that

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|X_t-\overline{Z}_t^M\right\|^p\right]\right)^{1/p}\leq C_p\Delta t^{1/2}, \qquad (40)$$

for all $p\in[1,\infty)$. Details can also be found in [17].

3.3. Proof of Theorem 3.1 for NCTS scheme ($\chi_t^M=\overline{X}_t^M$). Using the relation $\Delta\overline{N}_n^M=\Delta N_n^M-\lambda\Delta t$, the continuous interpolation of (8) can be expressed in the following form

$$\begin{aligned}
\overline{X}_t^M \quad = \quad & X_n^M+\lambda(t-n\Delta t)h(X_n^M)+\frac{(t-n\Delta t)f(X_n^M)}{1+\Delta t^\alpha\|f(X_n^M)\|}+g(X_n^M)(W_t-W_{n\Delta t})\\
&+h(X_n^M)(\overline{N}_t-\overline{N}_{n\Delta t}),
\end{aligned}$$

for all $t\in[n\Delta t,(n+1)\Delta t)$.

The numerical solution of non compensated tamed scheme (NCTS) (3) is also equivalent to

$$
\begin{aligned}
X_{n+1}^M &= X_n^M + \frac{\Delta t f(X_n^M)}{1+\Delta t^{\alpha}\|f(X_n^M)\|} + g(X_n^M)\Delta W_n^M + h(X_n^M)\Delta N_n^M \\
&= X_n^M + \lambda h(X_n^M)\Delta t + \frac{\Delta t f(X_n^M)}{1+\Delta t^{\alpha}\|f(X_n^M)\|} + g(X_n^M)\Delta W_n^M \\
&\quad + h(X_n^M)\Delta \overline{N}_n^M. \qquad (41)
\end{aligned}
$$

The functions λh and f in the numerical solution of the scheme NCTS given by (3) (or (41)) satisfy respectively the same conditions as the u_λ and v in the numerical solution of the STS given by (4). Hence, it follows from the proof in Section 3.2 that there exists a constant $C_p > 0$ such that

$$
\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|X_t - \overline{X}_t^M\right\|^p\right]\right)^{1/p} \leq C_p \Delta t^{1/2}, \qquad (42)
$$

for all $p \in [1, \infty)$.

4. Linear mean-square stability

The goal of this section is to find the time step-size limit for which the tamed Euler scheme and the semi-tamed Euler scheme are stable in the linear mean-square sense. For the scalar linear test problem, the concept of A-stability of a numerical method may be interpreted as "problem stable $\Rightarrow$ method stable for all Δt". We consider the following linear test equation with real and scalar coefficients

$$
dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t), \qquad X(0) = X_0, \qquad (43)
$$

where X_0 satisfied $\mathbb{E}\|X_0\|^2 < \infty$. In the sequel of this paper we take $\alpha \in [0,1]$. It has been proved in [6] that the exact solution of (43) is mean-square stable if and only if

$$
\lim_{t\to\infty} \mathbb{E}(X(t)^2) = 0 \Leftrightarrow l := 2a + b^2 + \lambda c(2+c) < 0. \qquad (44)
$$

Using the discrete form of (43), the numerical schemes (4) and (3) will be mean-square stable if $l < 0$ and

$$
\lim_{n\to\infty} \mathbb{E}(Y_n^2) = \lim_{n\to\infty} \mathbb{E}(X_n^2) = 0. \qquad (45)
$$

The following result provides the time step-size limit for which the semi-tamed scheme (STS) (4) is mean-square stable.

Theorem 4.1. *Assume that $l < 0$, then the semi-tamed scheme (4) is mean-square stable if and only if*

$$
\Delta t < \frac{-l}{(a+\lambda c)^2}.
$$

Proof. The proof is straightforward. Details can be found in [22, 17]. □

The following result provides the time step-size limit for which the non compensated tamed scheme (NCTS) (3) is stable.

Theorem 4.2. *Assume that $l < 0$, then the tamed Euler scheme (3) is mean-square stable if one of the following conditions is satisfied*

(i) $a(1+\lambda c\Delta t) \leq 0$, $2a - l > 0$ *and* $\Delta t < \dfrac{2a-l}{a^2+\lambda^2 c^2}$.

(ii) $a(1+\lambda c\Delta t) > 0$ *and* $\Delta t < \dfrac{-l}{(a+\lambda c)^2}$.

Proof. The proof is straightforward. See [22, 17] for details. $\square$

Remark 4.1. *In Theorem 4.2, we can easily check that if $l < 0$, we have*

$$\left\{\begin{array}{l} a(1+\lambda c\Delta t) \leq 0, \\ 2a - l > 0 \\ \Delta t < \dfrac{2a-l}{a^2+\lambda^2c^2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a \in (l/2, 0], c \geq 0, \\ \Delta t < \dfrac{2a-l}{a^2+\lambda^2c^2} \end{array}\right. \bigcup \left\{\begin{array}{l} a \in (l/2, 0), c < 0, \\ \Delta t < \dfrac{2a-l}{a^2+\lambda^2c^2} \\ \Delta t \leq \dfrac{-1}{\lambda c} \end{array}\right.$$
$$\bigcup \left\{\begin{array}{l} a > 0, c < 0 \\ \Delta t < \dfrac{2a-l}{a^2+\lambda^2c^2} \\ \Delta t \geq \dfrac{-1}{\lambda c} \end{array}\right.$$

$$\left\{\begin{array}{l} a(1+\lambda c\Delta t) > 0, \\ \Delta t < \dfrac{-l}{(a+\lambda c)^2} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a > 0, c > 0, \\ \Delta t < \dfrac{-l}{(a+\lambda c)^2} \end{array}\right. \bigcup \left\{\begin{array}{l} a > 0, c < 0, \\ \Delta t < \dfrac{-l}{(a+\lambda c)^2} \wedge \dfrac{-1}{\lambda c} \end{array}\right.$$
$$\bigcup \left\{\begin{array}{l} a < 0, c < 0 \\ \Delta t < \dfrac{-l}{(a+\lambda c)^2} \\ \Delta t > \dfrac{-1}{\lambda c}. \end{array}\right.$$

Remark 4.2. *Note that from the above studies, we can deduce the linear stabilities of schemes* (3) *and* (4) *for SDEs without jump by just take $c = 0$ in* (43)*, Theorem 4.1 and Theorem 4.2.*

5. Nonlinear mean-square stability

In this section, we focus on the exponential mean-square stability of the approximation (4). We follow closely [25, 6] and assume that $f(0) = u(0) = v(0) = g(0) = h(0) = 0$ and $\mathbb{E}\|X_0\|^2 < \infty$. It has been proved in [6] that under the following conditions

$$\langle x - y, f(x) - f(y)\rangle \leq \mu\|x-y\|^2, \tag{46}$$
$$\|g(x) - g(y)\|^2 \leq \sigma\|x-y\|^2, \tag{47}$$
$$\|h(x) - h(y)\|^2 \leq \gamma\|x-y\|^2, \tag{48}$$

for all $x, y \in \mathbb{R}^d$, where μ, σ and γ are constants, the exact solution of SDE (1) is nonlinear mean-square stable if $\alpha := 2\mu + \sigma + \lambda\sqrt{\gamma}(\sqrt{\gamma}+2) < 0$. Indeed under the above assumptions, we have [6, Theorem 4]

$$\mathbb{E}\|X(t)\|^2 \leq \mathbb{E}\|X_0\|^2 e^{\alpha t}.$$

So, if $\alpha < 0$ we have $\lim\limits_{t\to\infty} \mathbb{E}\|X(t)\|^2 = 0$ and the exact solution X is exponentially mean-square stable.

In the sequel of this section, we will use some weaker assumptions, which of course imply that the conditions (46)-(48) hold. More precisely, for nonlinear stability of the semi-tamed scheme (STS), we also make the following assumptions.

Assumption 5.1. *There exist some positive constants ρ, β,$\overline{\beta}$, K, C, θ and $a > 1$ such that*

$$\begin{aligned} \langle x-y, u(x)-u(y)\rangle &\leq -\rho\|x-y\|^2, & \|u(x)-u(y)\| &\leq K\|x-y\|, \\ \langle x-y, v(x)-v(y)\rangle &\leq -\beta\|x-y\|^{a+1}, & \|v(x)\| &\leq \overline{\beta}\|x\|^a, \\ \|g(x)-g(y)\| &\leq \theta\|x-y\|, & \|h(x)-h(y)\| &\leq C\|x-y\|. \end{aligned}$$

We denote by $\alpha_1 := -2\rho+\theta^2+\lambda C(C+2)$ and we will always assume that $\alpha_1 < 0$ to ensure the stability of the exact solution. The nonlinear stability result for the scheme (STS) is given in the following theorem.

Theorem 5.1. *Under Assumptions 5.1 and the further hypothesis $2\beta - \overline{\beta} > 0$, for any stepsize Δt such that $\Delta t < \dfrac{-\alpha_1}{(K+\lambda C)^2} \wedge \dfrac{2\beta}{[2(K+\lambda C)+\overline{\beta}]\overline{\beta}} \wedge \dfrac{2\beta-\overline{\beta}}{2(K+\lambda c)\overline{\beta}}$, there exists a constant $\gamma = \gamma(\Delta t) > 0$ such that*

$$\mathbb{E}\|Y_n\|^2 \leq \mathbb{E}\|X_0\|^2 e^{-\gamma\, t_n},\ t_n = n\,\Delta t, \qquad \lim_{\Delta t\to 0}\gamma(\Delta t) = -\alpha_1,$$

and the numerical solution (4) *is exponentiallly mean-square stable.*

Proof. The proof follows the same lines as [25, Theorem 4.2] by using additional techniques related to the compensated Poisson process. Details can be found in [22, 17]. □

To analyse the nonlinear mean-square stability of the tamed Euler scheme (NCTS), we make the following assumption.

Assumption 5.2. *There positive constants β, $\overline{\beta}$, θ, μ, K, ρ, C and $a > 1$ such that :*

$$\begin{aligned} &\langle x-y, f(x)-f(y)\rangle \leq -\rho\|x-y\|^2 - \beta\|x-y\|^{a+1}, \\ &\|f(x)\| \leq \overline{\beta}\|x\|^a + K\|x\|, \\ &\|g(x)-g(y)\| \leq \theta\|x-y\|, \qquad \|h(x)-h(y)\| \leq C\|x-y\|, \\ &\langle x-y, h(x)-h(y)\rangle \leq -\mu\|x-y\|^2. \end{aligned} \tag{49}$$

Remark 5.1. *Assumption 5.2 is a consequence of Assumption 5.1, except* (49)*.*

Using Assumption 5.2, we can easily check that the exact solution of (1) is exponentiallly mean-square stable if $\alpha_2 := -2\rho + \theta^2 + \lambda C(2 + C) < 0$.

Theorem 5.2. *Under Assumption 5.2, if $\alpha_3 := K + \theta^2 + \lambda C^2 - 2\lambda\mu C < 0$ and $\overline{\beta}(1 + 2C) - 2\beta < 0$ for any stepsize*

$$\Delta t < \frac{-\alpha_3}{2K^2 + \lambda^2 C^2 + 2\lambda CK} \wedge \frac{\beta - C\overline{\beta}}{\overline{\beta}^2},$$

there exists a constant $\gamma = \gamma(\Delta t) > 0$ such that

$$\mathbb{E}\|X_n\|^2 \leq \mathbb{E}\|X_0\|^2 e^{-\gamma t_n},\ t_n = n\,\Delta t, \lim_{\Delta t\to 0}\gamma(\Delta t) = -\alpha_3.$$

and the numerical solution (3) *is exponentiallly mean-square stable.*

Proof. From equation (3), we have

$$
\begin{aligned}
& \|X_{n+1}\|^2 \\
= \; & \|X_n\|^2 + \frac{\Delta t^2 \|f(X_n)\|^2}{(1+\Delta t^{\alpha}\|f(X_n)\|)^2} + \|g(X_n)\Delta W_n\|^2 + \|h(X_n)\Delta N_n\|^2 \\
& +2\left\langle X_n, \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}\right\rangle + 2\left\langle X_n + \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle \\
& +2\left\langle X_n + \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, h(X_n)\Delta N_n\right\rangle + 2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle. \qquad (50)
\end{aligned}
$$

Using Assumption 5.2, it follows that

$$
\begin{aligned}
2\left\langle X_n, \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}\right\rangle &\leq \frac{-2\Delta t\rho\|X_n\|^2}{1+\Delta t^{\alpha}\|f(X_n)\|} - \frac{2\beta\Delta t\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|} \\
&\leq -\frac{2\beta\Delta t\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}. \\
\|g(X_n)\Delta W_n\|^2 &\leq \theta^2\|X_n\|^2\|\Delta W_n\|^2 \\
\|h(X_n)\Delta N_n\|^2 &\leq C^2\|X_n\|^2|\Delta N_n|^2. \\
2\langle X_n, h(X_n)\Delta N_n\rangle &= 2\langle X_n, h(X_n)\rangle\Delta N_n \leq -2\mu\|X_n\|^2|\Delta N_n|, \\
2\left\langle \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, h(X_n)\Delta N_n\right\rangle &\leq \frac{2\Delta t\|f(X_n)\|\|h(X_n)\||\Delta N_n|}{1+\Delta t^{\alpha}\|f(X_n)\|} \\
&\leq \frac{2\Delta t C\overline{\beta}\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}|\Delta N_n| + 2CK\Delta t\|X_n\|^2|\Delta N_n|
\end{aligned}
$$

So from Assumption 5.2, we have

$$
\begin{aligned}
2\left\langle X_n, \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}\right\rangle &\leq -\frac{2\beta\Delta t\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|} \\
\|g(X_n)\Delta W_n\|^2 &\leq \theta^2\|X_n\|^2\|\Delta W_n\|^2 \\
\|h(X_n)\Delta N_n\|^2 &\leq C^2\|X_n\|^2|\Delta N_n|^2 \qquad (51) \\
2\langle X_n, h(X_n)\Delta N_n\rangle &\leq -2\mu\|X_n\|^2|\Delta N_n| \\
2\left\langle \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, h(X_n)\Delta N_n\right\rangle &\leq \frac{2\Delta t C\overline{\beta}\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}|\Delta N_n| + 2CK\|X_n\|^2|\Delta N_n|
\end{aligned}
$$

Let $\Omega_n := \{w \in \Omega : \|X_n(\omega)\| > 1\}$, on Ω_n, using Assumption 5.2, we have

$$
\begin{aligned}
\frac{\Delta t^2\|f(X_n)\|^2}{(1+\Delta t^{\alpha}\|f(X_n)\|)^2} &\leq \frac{\Delta t\|f(X_n)\|}{1+\Delta t^{\alpha}\|f(X_n)\|} \leq \frac{\Delta t\overline{\beta}\|X_n\|^{a}}{1+\Delta t^{\alpha}\|f(X_n)\|} + K\Delta t\|X_n\| \\
&\leq \frac{\Delta t\overline{\beta}\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|} + K\Delta t\|X_n\|^2. \qquad (52)
\end{aligned}
$$

Therefore substituting (51) and (52) in (50) yields

$$
\begin{aligned}
\|X_{n+1}\|^2 \leq \; & \|X_n\|^2 + K\Delta t\|X_n\|^2 + \theta^2\|X_n\|^2\|\Delta W_n\|^2 + C^2\|X_n\|^2|\Delta N_n|^2 \\
& +2\left\langle X_n + \frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle \\
& -2\mu\|X_n\|^2|\Delta N_n| + 2CK\Delta t|\Delta N_n| \\
& +2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle + \frac{\left[-2\beta\Delta t + \overline{\beta}\Delta t + 2\overline{\beta}C\Delta t\right]\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}. \qquad (53)
\end{aligned}
$$

Since $\overline{\beta}(1+2C)-2\beta<0$, (53) becomes

$$\begin{aligned}\|X_{n+1}\|^2 &\leq \|X_n\|^2+K\Delta t\|X_n\|^2+\theta^2\|X_n\|^2\|\Delta W_n\|^2+C^2\|X_n\|^2|\Delta N_n|^2\\ &+2\left\langle X_n+\frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle-2\mu\|X_n\|^2|\Delta N_n|\\ &+2CK\Delta t|\Delta N_n|+2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle. \qquad (54)\end{aligned}$$

On Ω_n^c, using Assumption 5.2 and the inequality $(a+b)^2\leq 2a^2+2b^2$, we have

$$\begin{aligned}\frac{\Delta t^2\|f(X_n)\|^2}{(1+\Delta t^{\alpha}\|f(X_n)\|)^2} &\leq \frac{\Delta t^2\|f(X_n)\|^2}{1+\Delta t^{\alpha}\|f(X_n)\|}\\ &\leq \frac{2\Delta t^2\overline{\beta}^2\|X_n\|^{2a}}{1+\Delta t^{\alpha}\|f(X_n)\|}+2K^2\Delta t^2\|X_n\|^2\\ &\leq \frac{2\Delta t^2\overline{\beta}^2\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}+2K^2\Delta t^2\|X_n\|^2. \qquad (55)\end{aligned}$$

Therefore, using (51) and (55), (50) becomes

$$\begin{aligned}\|X_{n+1}\|^2 &\leq \|X_n\|^2+2K^2\Delta t^2\|X_n\|^2+\theta^2\|X_n\|^2\|\Delta W_n\|^2+C^2\|X_n\|^2|\Delta N_n|^2\\ &+2\left\langle X_n+\frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle-2\mu\|X_n\|^2|\Delta N_n|\\ &+2CK\Delta t|\Delta N_n|+2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle\\ &+\frac{\left[2C\overline{\beta}\Delta t-2\beta\Delta t+2\overline{\beta}^2\Delta t^2\right]\|X_n\|^{a+1}}{1+\Delta t^{\alpha}\|f(X_n)\|}.\end{aligned}$$

Note that

$$\Delta t<\frac{\beta-C\overline{\beta}}{\overline{\beta}^2}\Leftrightarrow 2C\overline{\beta}\Delta t-2\beta\Delta t+2\overline{\beta}^2\Delta t^2<0.$$

Therefore (56) becomes

$$\begin{aligned}\|X_{n+1}\|^2 &\leq \|X_n\|^2+2K^2\Delta t^2\|X_n\|^2+\theta^2\|X_n\|^2\|\Delta W_n\|^2+C^2\|X_n\|^2|\Delta N_n|^2\\ &+2\left\langle X_n+\frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle\\ &-2\mu\|Y_n\|^2|\Delta N_n|+2CK\Delta t|\Delta N_n|+2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle \qquad (56)\end{aligned}$$

From the above discussion on Ω_n and Ω_n^c, using the fact that $\overline{\beta}(1+2C)-2\beta<0$ and $\Delta t<\dfrac{\beta-C\overline{\beta}}{\overline{\beta}^2}$ that on Ω, we have

$$\begin{aligned}\|X_{n+1}\|^2 &\leq \|X_n\|^2+K\Delta t\|X_n\|^2+2K^2\Delta t^2\|X_n\|^2+\theta^2\|X_n\|^2\|\Delta W_n\|^2\\ &+C^2\|X_n\|^2|\Delta N_n|^2+2\left\langle X_n+\frac{\Delta t f(X_n)}{1+\Delta t^{\alpha}\|f(X_n)\|}, g(X_n)\Delta W_n\right\rangle\\ &-2\mu\|X_n\|^2|\Delta N_n|+2CK\Delta t|\Delta N_n|+2\langle g(X_n)\Delta W_n, h(X_n)\Delta N_n\rangle. \qquad (57)\end{aligned}$$

Taking the expectation in both sides of (57), using independence of ΔW_n and ΔN_n together with the relation $\mathbb{E}\Delta W_n=0$, $\mathbb{E}\|\Delta W_n\|^2=\Delta t$, $\mathbb{E}|\Delta N_n|=\lambda\Delta t$ and $\mathbb{E}|\Delta N_n|^2=\lambda^2\Delta t^2+\lambda\Delta t$ leads to

$$\begin{aligned}\mathbb{E}\|X_{n+1}\|^2 &\leq \mathbb{E}\|X_n\|^2+K\Delta t\mathbb{E}\|X_n\|^2+2K^2\Delta t^2\mathbb{E}\|X_n\|^2+\theta^2\Delta t\mathbb{E}\|X_n\|^2\\ &+\lambda^2C^2\Delta t^2\mathbb{E}\|X_n\|^2+\lambda C^2\Delta t\mathbb{E}\|X_n\|^2\\ &-2\mu\lambda\Delta t\mathbb{E}\|X_n\|^2+2\lambda CK\Delta t^2\mathbb{E}\|X_n\|^2\\ &=\left[1+(2K^2+\lambda^2C^2+2\lambda CK)\Delta t^2+(K+\theta^2+\lambda C^2-2\mu\lambda)\Delta t\right]\mathbb{E}\|X_n\|^2.\end{aligned}$$

Iterating the last inequality leads to

$$\mathbb{E}\|X_n\|^2 \leq \left[1 + (2K^2 + \lambda^2 C^2 + 2\lambda CK)\Delta t^2 + (K + \theta^2 + \lambda C^2 - 2\mu\lambda)\Delta t\right]^n \mathbb{E}\|X_0\|^2.$$

To have the stability of the NCTS scheme, we should also have

$$1 + (2K^2 + \lambda^2 C^2 + 2\lambda CK)\Delta t^2 + (K + \theta^2 + \lambda C^2 - 2\mu\lambda)\Delta t < 1.$$

That is

$$\Delta t < \frac{-[K + \theta^2 + \lambda C^2 - 2\mu\lambda]}{2K^2 + \lambda^2 C^2 + 2\lambda CK},$$

and there exists a constant $\gamma = \gamma(\Delta t) > 0$ such that

$$\mathbb{E}\|X_n\|^2 \leq \mathbb{E}\|X_0\|^2 e^{-\gamma\, t_n}, \; t_n = n\,\Delta t.$$

As in the proof of Theorem 5.1, we obviously have $\lim\limits_{\Delta t \to 0} \gamma(\Delta t) = -(K + \theta^2 + \lambda C^2 - 2\mu\lambda) = -\alpha_3$. □

6. Numerical simulations

6.1. Convergence. In this section, we present some numerical experiments to illustrate our theoretical strong convergence result. We consider the following stochastic differential equations

$$dX(t) = (-4X(t) - X^3(t))dt + X(t)dW(t) + X(t)dN_t, \tag{58}$$

$$\begin{aligned} dX(t) &= (-4X(t) - X^3(t))dt + X(t)dW_1(t) + 2X(t)dW_2(t), \\ &\quad +X(t)dN_1(t) - 2X(t)dN_2(t). \end{aligned} \tag{59}$$

Note that W, W_1 and W_2 are independent Brownian motions and N, N_1 and N_2 are independent Poisson processes. Here $u(x) = -4x$. It is obvious to check that u, v, g and h satisfy Assumption 2.1. Indeed $\langle x - y, f(x) - f(x)\rangle \leq c(x-y)^2$ for all $c \geqslant 0$.

Figure 1 shows the strong errors for equation (58) with different values of α. As you can observe in Figure 1, all schemes have strong convergence order 0.5, which confirm the theoretical result in Theorem 3.1.

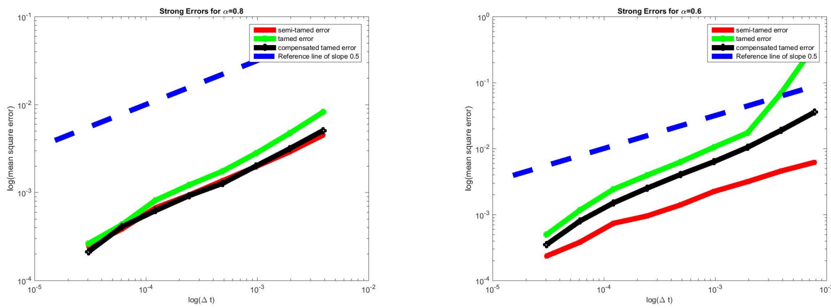
Remember that we have assumed scalar Poisson jump just for simplicity. In Figure 2, the convergence of our schemes with vector-valued jumps and vector-valued Brownian motions is investigated. The errors graph corresponding to the equation (59) are given at Figure 2(a) and Figure 2(b). All schemes have strong convergence order 0.5, which also confirm the theoretical result in Theorem 3.1.

6.2. Linear stability. The goal of this section is to provide some practical examples to sustain our theoretical results in the previous section. We compare the stability behaviors of the tamed Euler and the compensated tamed Euler schemes with the one of semi-tamed Euler scheme. We denote by Y_n all the approximated solutions from those schemes. Here we consider the following linear stochastic differential equation

$$\begin{cases} dX(t) = aX(t)dt + bX(t)dW(t) + cX(t)dN(t), \\ X(0) = 1, \end{cases} \tag{60}$$

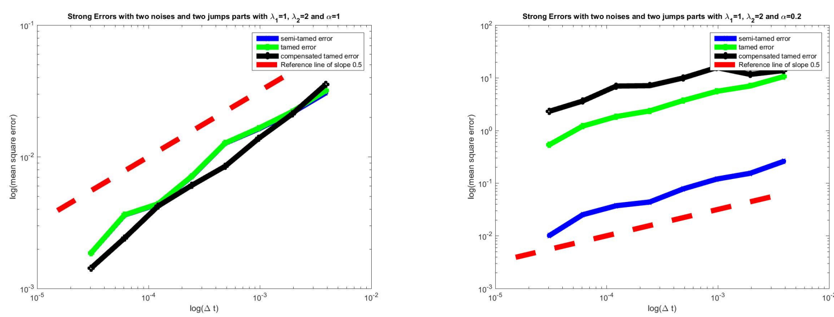
with the following two parameters

- Case I. $a = -1$, $b = 2$, $c = -0.9$ and $\lambda = 9$.
- Case II. $a = 2$, $b = 2$, $c = -0.9$ and $\lambda = 9$.



(a) (b)

FIGURE 1. Strong convergence of the compensated tamed scheme (CTS), the non compensated tamed scheme (NCTS) and the semi-tamed scheme (STS) for different values of α for SDEs (58). For each value of α we use 5000 sample paths and the reference solutions are the numerical solutions with step size $\Delta t = 2^{-16}$. The initial solution is $X_0 = 1$ and the parameter of the scalar Poisson $\lambda = 1$ and $T = 1$. Graph (a) corresponds to $\alpha = 0.8$ and graph (b) corresponds to $\alpha = 0.6$.



(a) (b)

FIGURE 2. Strong convergence of the compensated tamed scheme (CTS), the non compensated tamed scheme (NCTS) and the semi-tamed scheme (STS) for multiple noise terms and jumps terms. The initial solution is $X_0 = 1$, $T = 1$. We use 5000 sample paths and the reference solutions are the numerical solutions with step size $\Delta t = 2^{-16}$. Figures 2(a) and 2(b) are for SDEs (59). Graphs (a) corresponds to $\alpha = 1$, $\lambda_1 = 1$ and $\lambda_2 = 2$.

We can easily check that in both cases $l < 0$, which ensure the linear mean-square stable of the exact solution in the two situations. We can also easily check from the theoretical result that the semi-tamed and the tamed Euler scheme reproduce the linear mean-square property of the exact solution in the first case for all $\Delta t < 0.048$

and in the second case for all $\Delta t < 0.0245$. In Figure 3, we illustrate the mean-square stability of the tamed Euler, the compensated tamed Euler and the semi-tamed Euler schemes for different stepsizes. We observe from Figure 3 that the semi-tamed scheme works better than the tamed and compensated tamed schemes.

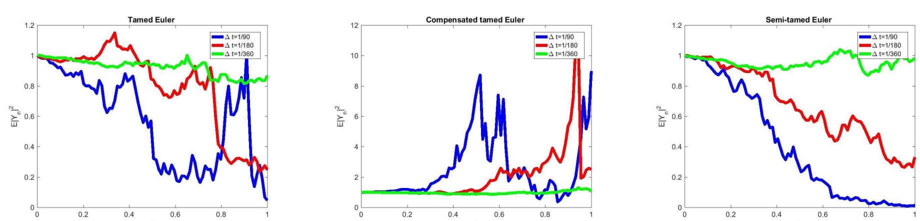
6.3. Nonlinear stability. For nonlinear stability, we consider the following nonlinear stochastic differential equation

$$(61)\quad \begin{cases} dX(t) = \left(-2X(t) - X(t)^3\right) dt + \sqrt{2}X(t)dW(t) - \dfrac{1}{4}X(t)dN(t), \\ X(0) = 1. \end{cases}$$

The Poisson process intensity is $\lambda = 1$, $f(x) = -2x - x^3$, $g(x) = \sqrt{2}x$, $h(x) = -\dfrac{1}{4}x$ and $T = 2$. We take $u(x) = -2x$ and $v(x) = -x^3$. Indeed, we obviously have

$$\langle x - y, f(x) - f(y)\rangle \leq -2(x-y)^2$$
$$|g(x) - g(y)|^2 \leq 2(x-y)^2, \quad |h(x) - h(y)|^2 \leq \frac{1}{16}(x-y)^2.$$

Then $\mu = -2$, $\sigma = 2$, $\gamma = \dfrac{1}{16}$ and $\alpha = 2\mu + \sigma + \lambda\sqrt{\gamma}(\sqrt{\gamma} + 2) = -\dfrac{23}{16} < 0$. It follows that the exact solution is exponentially mean-square stable. One can easily check from theoretical results that for $\Delta t < 0.22$, the semi-tamed Euler schemes reproduces the exponential mean-square stability property of the exact solution. Figure 5 illustrates the stability of the tamed scheme, compensated tamed scheme and the semi-tamed scheme for different step-sizes. We take $\Delta t = 1/6$, $\Delta t = 1/12$ and $\Delta t = 1/24$ and generate 7×10^3 samples for each numerical method. We observe that the semi-tamed scheme works better than the tamed and the compensated tamed Euler schemes. We observe also that when α approaches 1 the tamed and compensated tamed Euler scheme are more stable.



(a) (b) (c)

FIGURE 3. Linear stability of with $\alpha = 1$ with different stepsizes for SDE (61) with Case II (a) Tamed Euler scheme, (b) Compensated tamed Euler scheme (c) Semi-tamed Euler scheme. This reveals that the semi-tamed Euler scheme works better than the tamed Euler and the compensated tamed Euler schemes.

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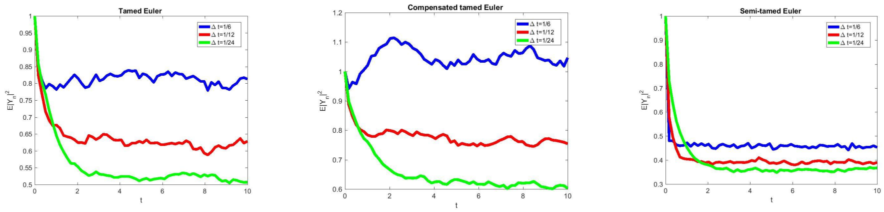
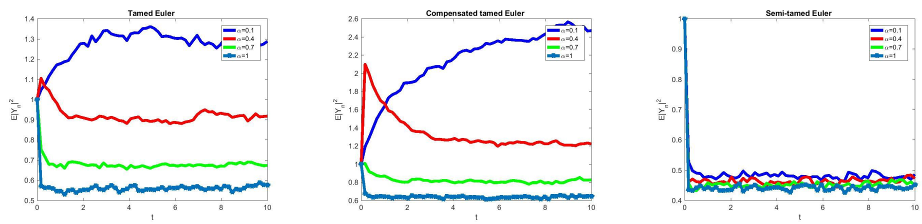


(a) (b) (c)

Figure 4. Nonlinear stability with $\alpha = 0.5$ with different stepsizes, (a) Tamed Euler scheme, (b) Semi-tamed Euler scheme, (c) Compensated tamed Euler scheme (d) for 7×10^3 samples of each numerical scheme. This illustrate that semi-taned Euler scheme works better than the tamed and the compensated tamed Euler schemes.



(a) (b) (c)

Figure 5. Nonlinear stability with different values of α for with $\Delta t = 1/6$ with 7×10^3 samples paths. (a) Tamed Euler scheme, (b) Compensated tamed Euler scheme, (c) Semi-tamed Euler scheme (d). This reveals that when α approaches 1 the tamed Euler and the compensated tamed Euler schemes are more stable and behave like the semi-tamed Euler scheme.

References

[1] R. Cont and P. Tankov, Modelling With Jump Processes, Chapman & Hall/CRC, Florida, 2004.

[2] K. Dareiotis, C. Kumar and S. Sabanis. On Tamed Euler Approximations of SDEs Driven by Lévy Noise with Applications to Delay Equations, SIAM J. Numer. Anal., 54(3) (2016), pp. 1840–1872.

[3] I. I. Gikhman and A. V. Skorokhod, Stochastic Differential Equations, Springer-Verlag, Berlin, 1972. SIAM J. Numer. Anal. (2016).

[4] I. Gyongy and N. V. Krylov On Stochastic Equations with Respect to Semimartingales I, Stochastics, 4:121, (1980).

[5] D. J. Higham and P. E. Kloeden, Convergence and Stability of Implicit Methods for Jump-diffusion Systems, Int. J. Numer. Anal. Model., 3(2006), pp. 125–140.

[6] D. J. Higham, P.E. Kloeden. Numerical methods for nonlinear stochastic differential equations with jumps, Numer. Math., 101 (2005), pp. 101–119.

[7] D. J. Higham, X. Mao and C. Yuan. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Numer. Anal., 45(2), pp. 592–609.

[8] C. Huang. Exponential mean square stability of numerical methods for systems of stochastic differential equations, J. Comput. Appl. Math., 236 (2012), pp. 4016–4026.

[9] M. Hutzenthaler and A. Jentzen, Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients, Mem. Amer. Math. Soc., 236 (2015), no. 1112, 99 pages.

[10] M. Hutzenthaler, A. Jentzen. Convergence of the stochastic Euler scheme for locally Lipschitz coefficients, Found. Comput. Math., 11(2011), pp. 657–706.

[11] M. Hutzenthaler, A. Jentzen and P. E. Kloeden. Divergence of the multilevel Monte Carlo Euler Euler Method for nonlinear Stochastic Differential Equations, Ann. Appl. Probab., 23(5)(2013), pp. 1913–1966.

[12] M. Hutzenthaler, A. Jentzen and P.E. Kloeden, Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients, Ann. Appl. Probab., 22, (2012), pp. 1611–1641.

[13] M. Hutzenthaler, A. Jentzen and P.E. Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467 (2011), pp. 1563–1576.

[14] F. C. Klebaner. Introduction to stochastic calculus with applications. Second edition, Imperial College Press, London WC2H 9HE, 2004.

[15] C. Kumar and S. Sabanis. On Tamed Milstein Schemes of SDEs Driven by Lévy Noise, arXiv: 1407.5347v3, 2015.

[16] Q. Li, and S. Gan. Almost sure exponential stability of numerical solutions for stochastic delay differential equations with jumps, J. Appl. Math. Comput., 37(1)(2011), pp. 541–557.

[17] J. D. Mukam. Stochastic Calculus with Jumps Processes : Theory and Numerical Techniques. Master Thesis, arXiv: 1510.01236v2 [math.NA], AIMS-Sengeal, 2015.

[18] B. Oksendal and A. Sulem, Applied Stochastic Control of Jump Diffusions, Third Edition, Springer, Berlin Heidelberg New York, Hong Kong London , 2009.

[19] E. Platen and N. Bruti-Liberati, Solution of Stochastic Differential Equations with Jumps in Finance, Spinger-Verlag, Berlin, 2010.

[20] P. E. Protter, Stochastic Integration and Differential Equations, Second Edition, ISBN 3-540-00313-4 Springer-Verlag Berlin Heidelberg New York, 2004.

[21] S. Sabanis, A note on tamed Euler approximations, Electron. Commun. Probab., 18 (2013), pp. 1–10.

[22] A. Tambue and J. D. Mukam, Strong convergence of the tamed and the semi-tamed Euler schemes for stochastic differential equations with jumps under non-global Lipschitz condition, arXiv: 1510.04729v2 (2015).

[23] X. Wang and S. Gan. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Diff. Equat. Appl., 19 (3) (2013), pp. 466–490.

[24] X. Wang and S. Gan, Compensated stochastic theta methods for stochastic differential equations with jumps, Appl. Numer. Math., 60(2010), pp. 877–887.

[25] X. Zong, F. Wu, and C. Huang, Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-lipschitz continous coefficients, Appl. Math. Comput., 228 (2014), pp. 240–250.

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