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$2N + 1$-soliton Solutions of Boussinesq-Burgers Equation

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Abstract: $2N + 1$-soliton solutions of Boussinesq-Burgers equation are obtained by using the Hirota bilinear derivative method and the perturbation technique. Further, we give the graphs of corresponding three- and five-soliton solutions.

Key words: Boussinesq-Burgers equation, Hirota bilinear derivative method, $2N + 1$-soliton solution

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1 Introduction

The research of soliton equations is one of the most important subjects in the field of nonlinear science. However, due to their high nonlinearity, it is very difficult to solve them. Up to now, several systematic methods has been developed to obtain explicit solutions of soliton equations, for instance, the inverse scattering transformation (see [1]–[2]), the Hirota bilinear derivative transformation (see [3]–[4]), the dressing method (see [5]), the Bäcklund and the Darboux transformation (see [6]–[8]), the algebra-geometric method (see [9]–[10]), the nonlinearization approach of eigenvalue problems or Lax pairs (see [11]–[12]), etc. Among the various methods, the Hirota bilinear derivative transformation is a powerful tool to generate exact solutions of the nonlinear evolution equations. And this method has the advantage of being applicable directly upon the nonlinear evolution equations. The key of the method is to transform the equation under consideration into the bilinear derivative

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equation through the dependent variable transformation. Then N-soliton solutions with exponential function form can be obtained with the help of the perturbation expansion and truncation technique.

As is well known, classical Boussinesq-Burgers (CBB) spectral problem is

$$\boldsymbol{\phi}_x = \begin{pmatrix} \lambda - u & \beta u_x + v \\ -1 & -\lambda + u \end{pmatrix} \boldsymbol{\phi}, \qquad \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1.1}$$

where λ is a spectral parameter, β is a constant, u and v are two potentials. The spectral problem was initially introduced by Date[13]. Geng and Wu[14] constructed the finite-band solutions of evolution equations associated with the spectral problem (1.1), and the CBB equation is given as follows:

$$u_t = \frac{1}{2}(\beta - 1)u_{xx} + 2uu_x + \frac{1}{2}v_x, \tag{1.2a}$$

$$v_t = \beta\Big(1 - \frac{1}{2}\beta\Big)u_{xxx} + \frac{1}{2}(1-\beta)v_{xx} + 2(uv)_x. \tag{1.2b}$$

Xu[15] discussed its Darboux transformation and give some explicit solutions. For the special case of the spectral problem (1.1) with $\beta = 0$, this has been studied by Li[16].

In the present paper, we would consider the special case of (1.2) ($\beta = 1$), which is Boussinesq-Burgers equation as follows.

$$u_t = 2uu_x + \frac{1}{2}v_x, \tag{1.3a}$$

$$v_t = \frac{1}{2}u_{xxx} + 2(uv)_x. \tag{1.3b}$$

Based on Hirota bilinear derivative transformation, we obtain three-soliton solutions, five-soliton solutions and $2N+1$-soliton solutions of (1.3). To our knowledge, these solutions are new ones.

2 Bilinear Equation and $2N+1$-soliton Solutions

In this section, firstly, we introduce the logarithm transformations of dependent variables u and v as

$$u = \frac{1}{2}\Big(\ln\frac{g}{f}\Big)_x, \qquad v = \frac{1}{2}(\ln fg)_{xx}. \tag{2.1}$$

Substituting transformations (2.1) into (1.3) and integrating it once with respect to x, (1.1) can be transformed into the following bilinear equations:

$$(2D_t - D_x^2)g \cdot f = 0, \tag{2.2a}$$

$$(2D_tD_x - D_x^3)g \cdot f = 0, \tag{2.2b}$$

where D is the well-known Hirota's bilinear operator that defined as

$$D_x^m D_t^n f(x,t)g(x,t) = (\partial_x - \partial_x')^m(\partial_t - \partial_t')^n f(x,t)g(x',t')|_{x'=x,\,t'=t}. \tag{2.3}$$

In what follows, we would construct multi-soliton solutions of (1.3) based on the perturbation method. We expand f, g as the power series in a small parameter ε that are different from the usual.

$$f = 1 + f^{(1)}\varepsilon + f^{(2)}\varepsilon^2 + \cdots + f^{(j)}\varepsilon^j + \cdots, \tag{2.4a}$$

$$g = 1 + g^{(1)}\varepsilon + g^{(2)}\varepsilon^2 + \cdots + g^{(j)}\varepsilon^j + \cdots \tag{2.4b}$$

Substituting (2.4) into (2.2) and collecting terms of each order of ε yields the following recursion relations

$$-2f_t^{(1)} - f_{xx}^{(1)} + 2g_t^{(1)} - g_{xx}^{(1)} = 0, \tag{2.5a}$$

$$-2f_t^{(2)} - f_{xx}^{(2)} + 2g_t^{(2)} - g_{xx}^{(2)} = (D_x^2 - 2D_t)g^{(1)} \cdot f^{(1)}, \tag{2.5b}$$

$$-2f_t^{(3)} - f_{xx}^{(3)} + 2g_t^{(3)} - g_{xx}^{(3)} = (D_x^2 - 2D_t)(g^{(1)} \cdot f^{(2)} + g^{(2)} \cdot f^{(1)}), \tag{2.5c}$$

$$-2f_t^{(4)} - f_{xx}^{(4)} + 2g_t^{(4)} - g_{xx}^{(4)} = (D_x^2 - 2D_t)(g^{(1)} \cdot f^{(3)} + g^{(2)} \cdot f^{(2)} + g^{(3)} \cdot f^{(1)}), \tag{2.5d}$$

$$\cdots\cdots$$

$$2f_{tx}^{(1)} + f_{xxx}^{(1)} + 2g_{tx}^{(1)} - g_{xxx}^{(1)} = 0, \tag{2.6a}$$

$$2f_{tx}^{(2)} + f_{xxx}^{(2)} + 2g_{tx}^{(2)} - g_{xxx}^{(2)} = (D_x^3 - 2D_tD_x)g^{(1)} \cdot f^{(1)}, \tag{2.6b}$$

$$2f_{tx}^{(3)} + f_{xxx}^{(3)} + 2g_{tx}^{(3)} - g_{xxx}^{(3)} = (D_x^3 - 2D_tD_x)(g^{(1)} \cdot f^{(2)} + g^{(2)} \cdot f^{(1)}), \tag{2.6c}$$

$$2f_{tx}^{(4)} + f_{xxx}^{(4)} + 2g_{tx}^{(4)} - g_{xxx}^{(4)} = (D_x^3 - 2D_tD_x)(g^{(1)} \cdot f^{(3)} + g^{(2)} \cdot f^{(2)} + g^{(3)} \cdot f^{(1)}), \tag{2.6d}$$

$$\cdots\cdots$$

(1) Three-soliton solutions. Integrating (2.6a) once with respect to x and selecting the constant of integration to be zero, we have

$$2f_t^{(1)} + f_{xx}^{(1)} + 2g_t^{(1)} - g_{xx}^{(1)} = 0, \tag{2.7}$$

(2.5a) plus (2.7), we get

$$2g_t^{(1)} - g_{xx}^{(1)} = 0. \tag{2.8}$$

Then we can select a special solution

$$g^{(1)} = \mathrm{e}^{\xi_1}, \quad \xi_1 = k_1x + \omega_1t + \xi_1^{(0)}, \quad 2\omega_1 - k_1^2 = 0. \tag{2.9}$$

Similarly, (2.7) minus (2.5a), we arrive at

$$f^{(1)} = \mathrm{e}^{\eta_1}, \quad \eta_1 = l_1x + \sigma_1t + \eta_1^{(0)}, \quad 2\sigma_1 + l_1^2 = 0. \tag{2.10}$$

Substituting $g^{(1)} = \mathrm{e}^{\xi_1}$, $f^{(1)} = \mathrm{e}^{\eta_1}$ into (2.5b) and (2.6b), after a direct calculation, we have

$$g^{(2)} = \frac{k_1^2}{(k_1 + l_1)^2}\mathrm{e}^{\xi_1+\eta_1}, \qquad f^{(2)} = \frac{l_1^2}{(k_1 + l_1)^2}\mathrm{e}^{\xi_1+\eta_1}. \tag{2.11}$$

Putting (2.9), (2.10), (2.11) into (2.5c) and (2.6c), they return to (2.5a) and (2.6a) by using superscript (1.3) instead of (1.1), then we can choose

$$g^{(3)} = f^{(3)} = 0.$$

Take

$$g^{(4)} = f^{(4)} = g^{(5)} = f^{(5)} = \cdots = 0.$$

Select $\varepsilon = 1$. Thus, f and g are truncated as

$$g_1 = 1 + \mathrm{e}^{\xi_1} + k_1^2\mathrm{e}^{\xi_1+\eta_1+\theta_{13}}, \tag{2.12a}$$

$$f_1 = 1 + \mathrm{e}^{\eta_1} + l_1^2\mathrm{e}^{\xi_1+\eta_1+\theta_{13}}, \tag{2.12b}$$

$\mathrm{e}^{\theta_{13}} = (k_1 + l_1)^{-2}$. With the help of (2.1), we get a three-soliton solutions of (1.3)

$$u = \frac{1}{2}\left(\ln \frac{1 + \mathrm{e}^{\xi_1} + k_1^2\mathrm{e}^{\xi_1+\eta_1+\theta_{13}}}{1 + \mathrm{e}^{\eta_1} + l_1^2\mathrm{e}^{\xi_1+\eta_1+\theta_{13}}}\right)_x, \tag{2.13a}$$

$$v = \frac{1}{2}[\ln(1 + e^{\xi_1} + k_1^2 e^{\xi_1+\eta_1+\theta_{13}})(1 + e^{\eta_1} + l_1^2 e^{\xi_1+\eta_1+\theta_{13}})]_{xx}. \tag{2.13b}$$

The corresponding graphs are shown in Fig. 2.1.

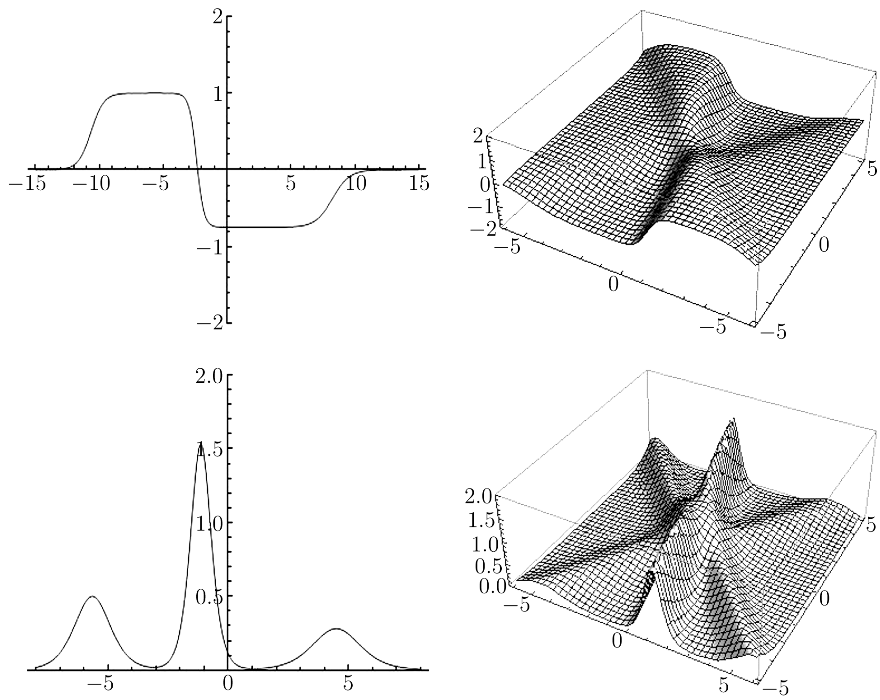

Fig. 2.1 The shape of the three-soliton solutions of (2.13) with $k_1 = 2$, $l_1 = 1.5$, $\xi_1^{(0)} = 2$, $\eta_1^{(0)} = 4$

(2) Five-soliton solutions. Obviously,

$$g^{(1)} = e^{\xi_1} + e^{\xi_2}, \qquad f^{(1)} = e^{\eta_1} + e^{\eta_2}, \tag{2.14}$$

with

$$\xi_i = k_i x + \omega_i t + \xi_i^{(0)}, \quad 2\omega_i - k_i^2 = 0, \qquad i = 1, 2, \tag{2.15}$$

$$\eta_i = l_i x + \sigma_i t + \eta_i^{(0)}, \quad 2\sigma_i + l_i^2 = 0, \qquad i = 1, 2 \tag{2.16}$$

are still solutions of (2.5a) and (2.6a).

Substituting (2.14) into (2.5b) and (2.6b), and integrating (2.6b) once with respect to x and selecting the constant of integration to be zero, we can get

$$2f_t^{(2)} + f_{xx}^{(2)} - 2g_t^{(2)} + g_{xx}^{(2)} = 2\sum_{i,j=1,2} k_i l_j e^{\xi_i+\eta_j}, \tag{2.17a}$$

$$2f_t^{(2)} + f_{xx}^{(2)} + 2g_t^{(2)} - g_{xx}^{(2)} = -2\sum_{i,j=1,2} \frac{k_i l_j (k_i - l_j)}{k_i + l_j} e^{\xi_i+\eta_j}. \tag{2.17b}$$

Letting (2.17b) minus or plus (2.17a), we can obtain

$$2g_t^{(2)} - g_{xx}^{(2)} = -2\sum_{i,j=1,2} \frac{k_i^2 l_j}{k_i + l_j} e^{\xi_i+\eta_j}, \tag{2.18a}$$

$$2f_t^{(2)} + f_{xx}^{(2)} = 2\sum_{i,j=1,2} \frac{k_i l_j^2}{k_i + l_j} \mathrm{e}^{\xi_i+\eta_j}. \tag{2.18b}$$

Then the linear equations (2.18) have a special solution

$$\begin{aligned} g^{(2)} &= \sum_{i,j=1,2} \frac{k_i^2}{(k_i+l_j)^2} \mathrm{e}^{\xi_i+\eta_j} \\ &= k_1^2 \mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + k_1^2 \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + k_2^2 \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + k_2^2 \mathrm{e}^{\xi_2+\eta_2+\theta_{24}}. \end{aligned} \tag{2.19a}$$

$$\begin{aligned} f^{(2)} &= \sum_{i,j=1,2} \frac{l_j^2}{(k_i+l_j)^2} \mathrm{e}^{\xi_i+\eta_j} \\ &= l_1^2 \mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + l_1^2 \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + l_2^2 \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + l_2^2 \mathrm{e}^{\xi_2+\eta_2+\theta_{24}}, \end{aligned} \tag{2.19b}$$

where $\mathrm{e}^{\theta_{j(s+2)}} = (k_j + l_s)^{-2}$ $(j, s = 1, 2)$.

Similarly, utilizing (2.14) and (2.19), from (2.5c) and (2.6c) through complicated calculation, we get

$$\begin{aligned} g^{(3)} &= \frac{l_1^2 (k_1-k_2)^2}{(k_1+l_1)^2 (k_2+l_1)^2} \mathrm{e}^{\xi_1+\xi_2+\eta_1} + \frac{l_2^2 (k_1-k_2)^2}{(k_1+l_2)^2 (k_2+l_2)^2} \mathrm{e}^{\xi_1+\xi_2+\eta_2} \\ &= l_1^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{13}+\theta_{23}} + l_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{14}+\theta_{24}}, \end{aligned} \tag{2.20a}$$

$$\begin{aligned} f^{(3)} &= \frac{k_1^2 (l_1-l_2)^2}{(k_1+l_1)^2 (k_1+l_2)^2} \mathrm{e}^{\xi_1+\eta_1+\eta_2} + \frac{k_2^2 (l_1-l_2)^2}{(k_2+l_1)^2 (k_2+l_2)^2} \mathrm{e}^{\xi_2+\eta_1+\eta_2} \\ &= k_1^2 \mathrm{e}^{\xi_1+\eta_1+\eta_2+\theta_{13}+\theta_{14}+\theta_{34}} + k_2^2 \mathrm{e}^{\xi_2+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{34}}, \end{aligned} \tag{2.20b}$$

where $\mathrm{e}^{\theta_{12}} = (k_1 - k_2)^2$, $\mathrm{e}^{\theta_{34}} = (l_1 - l_2)^2$.

Substituting (2.14), (2.19) and (2.20) into (2.5c) and (2.6c), after tedious calculation with the help of the maple, we arrive at

$$\begin{aligned} g^{(4)} &= \frac{k_1^2 k_2^2 (k_1-k_2)^2 (l_1-l_2)^2}{(k_1+l_1)^2 (k_1+l_2)^2 (k_2+l_1)^2 (k_2+l_2)^2} \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2} \\ &= k_1^2 k_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}, \end{aligned} \tag{2.21a}$$

$$\begin{aligned} f^{(4)} &= \frac{l_1^2 l_2^2 (l_1-l_2)^2 (k_1-k_2)^2}{(k_1+l_1)^2 (k_2+l_1)^2 (k_1+l_2)^2 (k_2+l_2)^2} \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2} \\ &= l_1^2 l_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}. \end{aligned} \tag{2.21b}$$

Putting (2.14), (2.19)–(2.21) into (2.5d) and (2.6d), we can obtain a special solution

$$g^{(5)} = f^{(5)} = 0.$$

Take $g^{(6)} = f^{(6)} = g^{(7)} = f^{(7)} = \cdots = 0$ and $\varepsilon = 1$. Then the functions f, g are reduced to

$$\begin{aligned} g_2 = 1 &+ \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + k_1^2 \mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + k_1^2 \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + k_2^2 \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + k_2^2 \mathrm{e}^{\xi_2+\eta_2+\theta_{24}} \\ &+ l_1^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{13}+\theta_{23}} + l_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{14}+\theta_{24}} \\ &+ k_1^2 k_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}, \end{aligned} \tag{2.22a}$$

$$\begin{aligned} f_2 = 1 &+ \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + l_1^2 \mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + l_1^2 \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + l_2^2 \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + l_2^2 \mathrm{e}^{\xi_2+\eta_2+\theta_{24}} \\ &+ k_1^2 \mathrm{e}^{\xi_1+\eta_1+\eta_2+\theta_{13}+\theta_{14}+\theta_{34}} + k_2^2 \mathrm{e}^{\xi_2+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{34}} \end{aligned}$$

$$+ l_1^2 l_2^2 \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}. \tag{2.22b}$$

Substituting (2.14), (2.19)–(2.21) into (2.1), we work out a five-soliton solutions of the Boussinesq-Burgers equation (1.3). The corresponding graphs are shown in Fig. 2.2.

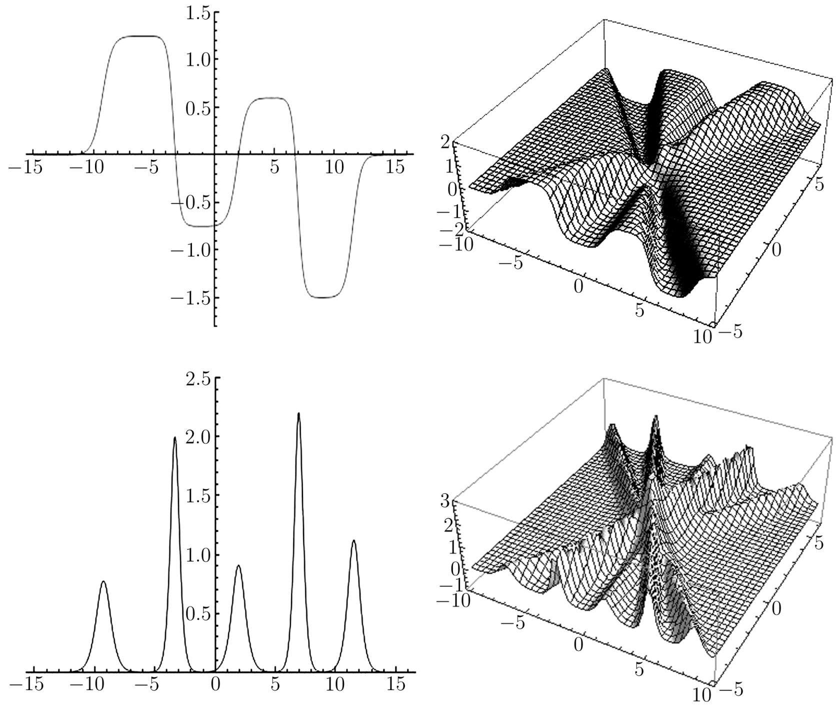

Fig. 2.2 The shape of the five-soliton solutions of (2.13) with $k_1 = 1.5$, $l_1 = 1.2$, $k_2 = 3$, $l_2 = 2.5$, $\xi_1^{(0)} = 2$, $\eta_1^{(0)} = 2$, $\xi_2^{(0)} = 1$, $\eta_2^{(0)} = 4$

(3) $2N+1$-soliton solutions. In general, we have

$$\begin{aligned} g_N(t,x) = & \sum_{\mu=0,1} A_1(\mu)\exp\left\{\sum_{j=1}^{N}\mu_j(\xi_j + 2\ln k_j) + \sum_{j=N+1}^{2N}\mu_j\xi_j + \sum_{1\le j<s}^{2N}\mu_j\mu_s\theta_{js}\right\} \\ & + \sum_{\mu=0,1} A_2(\mu)\exp\left\{\sum_{j=1}^{N}\mu_j\xi_j + \sum_{j=N+1}^{2N}\mu_j(\xi_j + 2\ln l_{j-N}) + \sum_{1\le j<s}^{2N}\mu_j\mu_s\theta_{js}\right\}, \end{aligned} \tag{2.23a}$$

$$\begin{aligned} f_N(t,x) = & \sum_{\mu=0,1} A_1(\mu)\exp\left\{\sum_{j=1}^{N}\mu_j\xi_j + \sum_{j=N+1}^{2N}\mu_j(\xi_j + 2\ln l_{j-N}) + \sum_{1\le j<s}^{2N}\mu_j\mu_s\theta_{js}\right\} \\ & + \sum_{\mu=0,1} A_3(\mu)\exp\left\{\sum_{j=1}^{N}\mu_j(\xi_j + 2\ln k_j) + \sum_{j=N+1}^{2N}\mu_j\xi_j + \sum_{1\le j<s}^{2N}\mu_j\mu_s\theta_{js}\right\}, \end{aligned} \tag{2.23b}$$

where

$$\begin{aligned} &\xi_{N+j} = \eta_j, && j = 1,2,\cdots,N, \\ &\mathrm{e}^{\theta_{j(N+s)}} = (k_j + l_s)^{-2}, && j,s = 1,2,\cdots,N, \\ &\mathrm{e}^{\theta_{js}} = (k_j - k_s)^2, && j < s = 2,\cdots,N, \\ &\mathrm{e}^{\theta_{(N+j)(N+s)}} = (l_j - l_s)^2, && j < s = 2,\cdots,N, \end{aligned}$$

the notations A_1, A_2 and A_3 stand for all possible combinations of $\mu_j = 0, 1$ $(j = 1, 2, \cdots, N)$ and satisfy the conditions

$$\sum_{j=1}^{N}\mu_j = \sum_{j=1}^{N}\mu_{N+j}, \qquad \sum_{j=1}^{N}\mu_j = 1 + \sum_{j=1}^{N}\mu_{N+j}, \qquad 1 + \sum_{j=1}^{N}\mu_j = \sum_{j=1}^{N}\mu_{N+j},$$

respectively. Substituting (2.23) into (2.1), we arrive at $2N + 1$-soliton solutions of the Boussinesq-Burgers equation (1.3). And they are new soliton solutions of (1.3).

3 Conclusions

In the present paper, we obtain $2N + 1$-soliton solutions for Boussinesq-Burgers equation based on Hirota bilinear derivative method and the perturbation technique. However, whether the Boussinesq-Burgers equation have Wronskian determinant form of solution remains an open problem, and we will discuss it elsewhere.

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