Communications
in
Mathematical
Research
**34**(1)(2018), 77–88

# Unique Common Fixed Points for Two Weakly $C^*$-contractive Mappings on Partially Ordered 2-metric Spaces

Piao Yong-jie
(*Department of Mathematics, College of Science, Yanbian University, Yanji, Jilin, 133002*)

Communicated by Ji You-qing

**Abstract:** In this paper, we give existence theorems of common fixed points for two mappings with a weakly $C^*$-contractive condition on partially ordered 2-metric spaces and give a sufficient condition under which there exists a unique common fixed point.

**Key words:** 2-metric space, weak $C^*$-contraction, common fixed point
**2010 MR subject classification:** 47H05, 47H10, 54E40
**Document code:** A
**Article ID:** 1674-5647(2018)01-0077-12
**DOI:** 10.13447/j.1674-5647.2018.01.08

## 1 Introduction and Preliminaries

Gähler[1]–[3] introduced the definition of 2-metric spaces and discussed the existence problems of fixed points. From then on, many authors discussed and obtained the existence problems of coincidence points and (common) fixed points with a variety of different forms. Especially, there have appeared a lot of useful results in recent years, see the references [4]–[16] and the related papers. All these results generalize and improve the corresponding fixed point theorem in metric spaces.

**Definition 1.1** [1]–[3] *A 2-metric space $(X, d)$ consists of a nonempty set $X$ and a function $d\colon X \times X \times X \to [0, +\infty)$ such that*

(i) *for distant elements $x, y \in X$, there exists a $u \in X$ such that $d(x, y, u) \neq 0$;*

(ii) *$d(x, y, z) = 0$ if and only if at least two elements in $\{x, y, z\}$ are equal;*

(iii) *$d(x, y, z) = d(u, v, w)$, where $\{u, v, w\}$ is any permutation of $\{x, y, z\}$;*

(iv) *$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.*

**Received date:** Dec. 15, 2016.
**Foundation item:** The NSF (11361064) of China.
**E-mail address:** sxpyj@ybu.edu.cn (Piao Y J).

**Definition 1.2** [1]–[3] *A sequence* $\{x_n\}_{n\in\mathbf{N}_+}$ *in* 2*-metric space* $(X,\, d)$ *is said to be a Cauchy sequence if for each* $\varepsilon > 0$ *there exists a positive integer* $N \in \mathbf{N}_+$ *such that* $d(x_n,\, x_m,\, a) < \varepsilon$ *for all* $a \in X$ *and* $n, m > N$*. A sequence* $\{x_n\}_{n\in\mathbf{N}_+}$ *is said to be convergent to* $x \in X$ *if for each* $a \in X$, $\lim\limits_{n\to+\infty} d(x_n,\, x,\, a) = 0$*. And we write that* $x_n \to x$ *and call* $x$ *the limit of* $\{x_n\}_{n\in\mathbf{N}_+}$*. A* 2*-metric space* $(X,\, d)$ *is said to be complete if every Cauchy sequence in* $X$ *is convergent.*

Choudhury[17] introduced the next definition in a real metric space:

**Definition 1.3** [17] *Let* $(X,\, d)$ *be a metric space and* $T\colon X \to X$ *be a map.* $T$ *is said to be weak* $C$*-contraction if there exists a continuous function* $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ *with* $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ *such that*

$$d(Tx,\, Ty) \le \frac{1}{2}[d(x,\, Ty) + d(y,\, Tx)] - \varphi(d(x,\, Ty),\; d(y,\, Tx)), \qquad x, y \in X.$$

Choudhury[17] also proved that any map satisfying the weak $C$-contraction has a unique fixed point on a complete metric space (see [17], Theorem 2.1). Later, the above result was extended to the case in a complete ordered metric spaces (see [18], Theorems 2.1, 2.3 and 3.1).

In 2013, Definition 1.3 was extended to the case in a 2-metric space by Dung and Hang[10] as follows:

**Definition 1.4** [10] *Let* $(X,\, \preceq,\, d)$ *be a ordered* 2*-metric space,* $T\colon X \to X$ *a map.* $T$ *is said to be weak* $C$*-contraction if there exists a continuous function* $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ *with* $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ *such that for any* $x, y, a \in X$ *with* $x \preceq y$ *or* $y \preceq x$,

$$d(Tx,\, Ty,\, a) \le \frac{1}{2}[d(x,\, Ty,\, a) + d(y,\, Tx,\, a)] - \varphi(d(x,\, Ty,\, a),\; d(y,\, Tx,\, a)).$$

Dung and Hang[10] proved that any weakly $C$-contractive map has fixed points on complete ordered 2-metric spaces (see [10], Theorems 2.3, 2.4 and 2.5). The results generalized and improved the corresponding conclusions in [17]–[18].

**Definition 1.5** *Let* $(X,\, \preceq,\, d)$ *be a ordered* 2*-metric space and* $S, T\colon X \to X$ *be two maps.* $S$, $T$ *are said to be weakly* $C^*$*-contractive maps if there exists a continuous function* $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ *with* $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ *such that for any* $x, y, a \in X$ *with* $x \preceq y$ *or* $y \preceq x$,

$$d(Sx,\, Ty,\, a) \le kd(x,\, y,\, a) + l[d(x,\, Ty,\, a) + d(y,\, Sx,\, a)] - \varphi(d(x,\, Ty,\, a),\; d(y,\, Sx,\, a)),$$

*where* $k$ *and* $l$ *are two real numbers satisfying* $l > 0$ *and* $0 < k + l \le 1 - l$*.*

Obviously, if $S = T$ and $k = 0$ and $l = \dfrac{1}{2}$, then Definition 1.5 becomes Definition 1.3.

**Definition 1.6** [10] *Let* $(X,\, d)$ *be a* 2*-metric space and* $a, b \in X$, $r > 0$*. The set*

$$B(a,\, b;\, r) = \{x \in X : d(a,\, b,\, x) < r\}$$

*is said to be a* 2*-ball with centers* $a$ *and* $b$ *and radius* $r$*. Each* 2*-metric* $d$ *on* $X$ *generalizes a topology* $\tau$ *on* $X$ *whose base is the family of* 2*-balls.* $\tau$ *is said to be a* 2*-metric topology.*

**Lemma 1.1** [13]–[14] *If a sequence* $\{x_n\}_{n\in\mathbf{N}_+}$ *in a 2-metric space* $(X, d)$ *is convergent to* $x$, *then*

$$\lim_{n\to+\infty} d(x_n, b, c) = d(x, b, c), \qquad b, c \in X.$$

**Lemma 1.2** [19] *Let* $\{x_n\}_{n\in\mathbf{N}_+}$ *be a sequence in* $(X, d)$ *satisfying* $\lim\limits_{n\to\infty} d(x_n, x_{n+1}, a) = 0$ *for any* $a \in X$. *If* $\{x_n\}$ *is not Cauchy, then there exists an* $a \in X$ *and an* $\epsilon > 0$ *such that for any* $i \in \mathbf{N}_+$, *there exist* $m(i), n(i) \in \mathbf{N}_+$ *with* $m(i) > n(i) > i$ *such that* $d(x_{m(i)}, x_{n(i)}, a) > \epsilon$, *but* $d(x_{m(i)-1}, x_{n(i)}, a) \le \epsilon$.

**Lemma 1.3** [6] $\lim\limits_{n\to\infty} x_n = x$ *in* 2*-metric space* $(X, d)$ *if and only if* $\lim\limits_{n\to\infty} x_n = x$ *in* 2*-metric topology space* $X$.

**Lemma 1.4** [6] *Let* $X$ *and* $Y$ *be two* 2*-metric spaces and* $T\colon X \to Y$ *be a map. If* $T$ *is continuous, then* $\lim\limits_{n\to\infty} x_n = x$ *implies* $\lim\limits_{n\to\infty} Tx_n = Tx$.

**Lemma 1.5** [6] *Each* 2*-metric space is* $T_2$*-space.*

The purpose of this paper is to use the method in [10] to discuss and study the existence problems of common fixed points for two maps satisfying weakly $C^*$-contractive condition on ordered 2-metric spaces and give a sufficient condition under which there exists a unique common fixed point.

## 2 Unique Common Fixed Points

Let $\varphi\colon [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function with $\varphi(x, y) = 0 \Leftrightarrow x = y = 0$. $\varphi(x, y) = \dfrac{x+y}{2}$ and $\varphi(x, y) = \dfrac{\max\{x, y\}}{2}$ for any $x, y \in [0, \infty)$ satisfy the above conditions.

Now, we discuss the existence problems of unique common fixed point for two maps on non-complete 2-metric spaces without ordered relation.

**Theorem 2.1** *Let* $(X, d)$ *be a* 2*-metric space and* $S, T\colon X \to X$ *be two maps. Suppose that*

$$\begin{aligned} d(Sx, Ty, a) \le{}& kd(x, y, a) + l[d(x, Ty, a) + d(y, Sx, a)] \\ & - \varphi(d(x, Ty, a),\ d(y, Sx, a)), \qquad x, y, a \in X, \end{aligned} \tag{2.1}$$

*where* $k$, $l$ *are two real numbers such that* $l > 0$ *and* $0 < k + l \le 1 - l$. *If* $S(X)$ *or* $T(X)$ *is complete, then* $S$ *and* $T$ *have a unique common fixed point.*

*Proof.* Take any element $x_0 \in X$ and construct a sequence $\{x_n\}$ satisfying

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \qquad n = 0, 1, 2, \cdots$$

For any $n = 0, 1, 2, \cdots$ and $a \in X$, by (2.1), we can get

$$d(x_{2n+1}, x_{2n+2}, a)$$

$$
\begin{aligned}
&= d(Sx_{2n},\, Tx_{2n+1},\, a)\\
&\le kd(x_{2n},\, x_{2n+1},\, a) + l[d(x_{2n},\, x_{2n+2},\, a) + d(x_{2n+1},\, x_{2n+1},\, a)]\\
&\quad - \varphi(d(x_{2n},\, x_{2n+2},\, a),\ d(x_{2n+1},\, x_{2n+1},\, a))\\
&= kd(x_{2n},\, x_{2n+1},\, a) + ld(x_{2n},\, x_{2n+2},\, a) - \varphi(d(x_{2n},\, x_{2n+2},\, a), 0)\\
&\le kd(x_{2n},\, x_{2n+1},\, a) + ld(x_{2n},\, x_{2n+2},\, a). \qquad (2.2)
\end{aligned}
$$

Take $a = x_{2n}$ in (2.2), we obtain

$$
d(x_{2n},\, x_{2n+1},\, x_{2n+2}) = 0, \qquad n = 0, 1, 2, \cdots \qquad (2.3)
$$

By using (2.3) and Definition 1.1(iv), we obtain from (2.2) that

$$
d(x_{2n+1},\, x_{2n+2},\, a) \le kd(x_{2n},\, x_{2n+1},\, a) + l[d(x_{2n},\, x_{2n+1},\, a) + d(x_{2n+1},\, x_{2n+2},\, a)],
$$

which implies

$$
\begin{aligned}
d(x_{2n+1},\, x_{2n+2},\, a) &\le \frac{k+l}{1-l} d(x_{2n},\, x_{2n+1},\, a)\\
&\le d(x_{2n},\, x_{2n+1},\, a), \qquad n = 0, 1, 2, \cdots,\ a \in X. \qquad (2.4)
\end{aligned}
$$

Similarly, for any $n = 0, 1, 2, \cdots$ and $a \in X$, by (2.1), we have

$$
\begin{aligned}
&d(x_{2n+3},\, x_{2n+2},\, a)\\
&= d(Sx_{2n+2},\, Tx_{2n+1},\, a)\\
&\le kd(x_{2n+2},\, x_{2n+1},\, a) + l[d(x_{2n+2},\, x_{2n+2},\, a) + d(x_{2n+1},\, x_{2n+3},\, a)]\\
&\quad - \varphi(d(x_{2n+2},\, x_{2n+2},\, a),\ d(x_{2n+1},\, x_{2n+3},\, a))\\
&= kd(x_{2n+2},\, x_{2n+1},\, a) + ld(x_{2n+1},\, x_{2n+3},\, a) - \varphi(0,\, d(x_{2n+1},\, x_{2n+3},\, a))\\
&\le kd(x_{2n+2},\, x_{2n+1},\, a) + ld(x_{2n+1},\, x_{2n+3},\, a). \qquad (2.5)
\end{aligned}
$$

Take $a = x_{2n+1}$ in (2.5), we obtain

$$
d(x_{2n+1},\, x_{2n+2},\, x_{2n+3}) = 0, \qquad n = 0, 1, 2, \cdots \qquad (2.6)
$$

By using (2.6) and Definition 1.1(iv), we obtain from (2.5) that

$$
d(x_{2n+3},\, x_{2n+2},\, a) \le kd(x_{2n+2},\, x_{2n+1},\, a) + l[d(x_{2n+1},\, x_{2n+2},\, a) + d(x_{2n+2},\, x_{2n+3},\, a)],
$$

which implies

$$
\begin{aligned}
d(x_{2n+3},\, x_{2n+2},\, a) &\le \frac{k+l}{1-l} d(x_{2n+1},\, x_{2n+2},\, a)\\
&\le d(x_{2n+1},\, x_{2n+2},\, a), \qquad n = 0, 1, 2, \cdots,\ a \in X. \qquad (2.7)
\end{aligned}
$$

Combining (2.3), (2.4), (2.6) and (2.7), we have

$$
\begin{cases}
d(x_n,\, x_{n+1},\, x_{n+2}) = 0,\\
d(x_{n+1},\, x_{n+2},\, a) \le d(x_n,\, x_{n+1},\, a),
\end{cases}
\qquad n = 0, 1, 2, \cdots,\ a \in X. \qquad (2.8)
$$

For any fixed $a \in X$, let $c_n(a) = d(x_n,\, x_{n+1},\, a)$, $n = 0, 1, 2, \cdots$ Then, by (2.8), $\{c_n(a)\}_{n=0}^{\infty}$ is a non-increasing non-negative real sequence. Hence there is a real number $\xi(a) \ge 0$ such that

$$
\lim_{n\to\infty} c_n(a) = \xi(a).
$$

It is easy to obtain

$$
\xi(a) \le c_{2n+1}(a)
$$

$$\begin{aligned}
&= d(x_{2n+1},\, x_{2n+2},\, a)\\
&= d(Sx_{2n},\, Tx_{2n+1},\, a)\\
&\le kd(x_{2n},\, x_{2n+1},\, a) + ld(x_{2n},\, x_{2n+2},\, a) - \varphi[d(x_{2n},\, x_{2n+2},\, a),\; 0]\\
&\le kd(x_{2n},\, x_{2n+1},\, a) + ld(x_{2n},\, x_{2n+2},\, a)\\
&\le kd(x_{2n},\, x_{2n+1},\, a) + l[d(x_{2n},\, x_{2n+1},\, a) + d(x_{2n+1},\, x_{2n+2},\, a)]. \qquad (2.9)
\end{aligned}$$

Let $n \to \infty$. Then from the first to third line, fifth line, sixth line in (2.9), we obtain

$$(k+2l)\xi(a) \le \xi(a) \le k\xi(a) + l \lim_{n\to\infty} d(x_{2n},\, x_{2n+2},\, a) \le k\xi(a) + 2l\xi(a).$$

Hence

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}, a) = 2\,\xi(a).$$

Let $n \to \infty$ again. Then from (2.9), we obtain

$$(k+2l)\xi(a) \le \xi(a) \le k\xi(a) + 2l\xi(a) - \varphi(2\xi(a),\, 0) \le k\xi(a) + 2l\xi(a).$$

Hence we have

$$\varphi(2\xi(a),\, 0) = 0.$$

So $\xi(a) = 0$ by the property of $\varphi$, that is,

$$\lim_{n\to\infty} d(x_n,\, x_{n+1},\, a) = 0, \qquad a \in X. \qquad (2.10)$$

By Definition 1.1(ii),

$$d(x_0,\, x_1,\, x_0) = 0,$$

which implies that

$$d(x_1,\, x_2,\, x_0) = 0$$

by (2.8). Hence, by the mathematical induction,

$$d(x_n,\, x_{n+1},\, x_0) = 0, \qquad n = 0, 1, 2, \cdots \qquad (2.11)$$

And for any fixed point $m \ge 1$,

$$d(x_{m-1},\, x_m,\, x_m) = 0.$$

Hence, by (2.8) and the mathematical induction, we have

$$d(x_n,\, x_{n+1},\, x_m) = 0, \qquad n \ge m-1. \qquad (2.12)$$

For $0 \le n < m-1$, since $m-1 \ge n+1$, using (2.12), we obtain

$$d(x_{m-1},\, x_m,\, x_{n+1}) = d(x_{m-1},\, x_m,\, x_n) = 0. \qquad (2.13)$$

Hence

$$\begin{aligned}
d(x_n, x_{n+1}, x_m) &\le d(x_n,\, x_{n+1},\, x_{m-1}) + d(x_n,\, x_m,\, x_{m-1}) + d(x_{n+1},\, x_m,\, x_{m-1})\\
&= d(x_n,\, x_{n+1},\, x_{m-1}).
\end{aligned}$$

So, by the mathematical induction, we have

$$\begin{aligned}
d(x_n,\, x_{n+1},\, x_m) &\le d(x_n,\, x_{n+1},\, x_{m-1})\\
&\le d(x_n,\, x_{n+1},\, x_{m-2})\\
&\le \cdots\\
&\le d(x_n,\, x_{n+1},\, x_{n+1})\\
&= 0,
\end{aligned}$$

that is,

$$d(x_n,\, x_{n+1},\, x_m) = 0, \qquad 0 \le n < m - 1. \tag{2.14}$$

Combining (2.11), (2.12) and (2.14), we obtain

$$d(x_n,\, x_{n+1},\, x_m) = 0, \qquad n, m = 0, 1, 2, \cdots \tag{2.15}$$

For any $i, j, k = 0, 1, 2, \cdots$ (we can assume $i < j$), by (2.8) and (2.15), we have

$$\begin{aligned} d(x_i,\, x_j,\, x_k) &\le d(x_i,\, x_j,\, x_{j-1}) + d(x_{j-1},\, x_j,\, x_k) + d(x_i,\, x_{j-1},\, x_k) \\ &= d(x_i,\, x_{j-1},\, x_k) \\ &\le \cdots \\ &\le d(x_i,\, x_{i+2},\, x_k) \\ &\le d(x_i,\, x_{i+1},\, x_k) + d(x_{i+1},\, x_{i+2},\, x_k) + d(x_i,\, x_{i+1},\, x_{i+2}) \\ &= 0. \end{aligned}$$

Hence

$$d(x_i,\, x_j,\, x_k) = 0, \qquad i, j, k = 0, 1, 2, \cdots \tag{2.16}$$

Suppose that $\{x_n\}$ is not Cauchy, then by Lemma 1.2, there exists a $b \in X$ and an $\epsilon > 0$ such that for any natural number $k$, there exist two natural numbers $m(k)$, $n(k)$ satisfying $m(k) > n(k) > k$ such that the following holds

$$d(x_{m(k)},\, x_{n(k)},\, b) > \epsilon, \qquad d(x_{m(k)-1},\, x_{n(k)},\, b) \le \epsilon. \tag{2.17}$$

By (2.16) and (2.17), we have

$$\begin{aligned} \epsilon &< d(x_{m(k)},\, x_{n(k)},\, b) \\ &\le d(x_{m(k)},\, x_{m(k)-1},\, b) + d(x_{m(k)-1},\, x_{n(k)},\, b) \\ &\le d(x_{m(k)},\, x_{m(k)-1},\, b) + \epsilon. \end{aligned}$$

Let $k \to \infty$. Then by (2.10) and from the above, we obtain

$$\lim_{k\to\infty} d(x_{m(k)},\, x_{n(k)},\, b) = \lim_{k\to\infty} d(x_{m(k)-1},\, x_{n(k)},\, b) = \epsilon. \tag{2.18}$$

By Definition 1.1(iv) and (2.16), we have

$$\begin{aligned} &d(x_{n(k)},\, x_{m(k)-1},\, b) \\ &\le d(x_{n(k)},\, x_{n(k)-1},\, b) + d(x_{m(k)-1},\, x_{n(k)-1},\, b) \\ &\le d(x_{n(k)},\, x_{n(k)-1},\, b) + d(x_{m(k)-1},\, x_{m(k)},\, b) + d(x_{m(k)},\, x_{n(k)-1},\, b) \end{aligned} \tag{2.19}$$

and

$$d(x_{n(k)-1},\, x_{m(k)},\, b) \le d(x_{m(k)},\, x_{n(k)},\, b) + d(x_{n(k)-1},\, x_{n(k)},\, b). \tag{2.20}$$

Letting $k \to \infty$ in (2.19) and (2.20), and using (2.10) and (2.18), we have

$$\lim_{k\to\infty} d(x_{m(k)},\, x_{n(k)},\, b) = \lim_{k\to\infty} d(x_{m(k)-1},\, x_{n(k)},\, b) = \lim_{k\to\infty} d(x_{m(k)},\, x_{n(k)-1},\, b) = \epsilon. \tag{2.21}$$

On the other hand, it is easy to know that

$$d(x_{m(k)-1},\, x_{n(k)-1},\, b) \le d(x_{m(k)-1},\, x_{m(k)},\, b) + d(x_{m(k)},\, x_{n(k)-1},\, b)$$

and

$$d(x_{m(k)-1},\, x_{n(k)},\, b) \le d(x_{n(k)},\, x_{n(k)-1},\, b) + d(x_{m(k)-1},\, x_{n(k)-1},\, b).$$

Letting $k \to \infty$ in the above two inequalities, and using (2.10) and (2.21), we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1},\, x_{n(k)-1},\, b) = \epsilon. \tag{2.22}$$

Using (2.10), we can assume that the parity of $m(k)$ and $n(k)$ is different. Let $m(k)$ be odd and $n(k)$ be even. We obtain

$$\begin{aligned}
\epsilon &\le d(x_{m(k)},\, x_{n(k)},\, b) \\
&= d(Sx_{m(k)-1},\, Tx_{n(k)-1},\, b) \\
&\le kd(x_{m(k)-1},\, x_{n(k)-1},\, b) + l[d(x_{m(k)-1},\, x_{n(k)},\, b) + d(x_{n(k)-1},\, x_{m(k)},\, b)] \\
&\quad - \varphi(d(x_{m(k)-1},\, x_{n(k)},\, b),\ d(x_{n(k)-1},\, x_{m(k)},\, b)) \\
&\le kd(x_{m(k)-1},\, x_{n(k)-1},\, b) + l[d(x_{m(k)-1},\, x_{n(k)},\, b) + d(x_{n(k)-1},\, x_{m(k)},\, b)].
\end{aligned}$$

Let $k \to \infty$ in the above inequality. Then by (2.21) and (2.22), we have

$$(k + 2l)\epsilon \le \epsilon \le (k + 2l)\epsilon - \varphi(\epsilon,\, \epsilon) \le (k + 2l)\epsilon,$$

which implies that $\varphi(\epsilon,\, \epsilon) = 0$, i.e., $\epsilon = 0$. This is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that $SX$ is complete. Since $x_{2n+1} = Sx_{2n} \in SX$ for all $n = 0, 1, 2, \cdots$, there exists a $u \in SX$ such that $x_{2n+1} \to u$ as $n \to \infty$. And since $\{x_n\}$ is a Cauchy sequence and the following holds

$$\begin{aligned}
&d(x_{2n+2},\, u,\, a) \\
\le\ &d(x_{2n+1},\, x_{2n+2},\, a) + d(x_{2n+1},\, u,\, a) + d(x_{2n+1},\, x_{2n+2},\, u), \qquad n = 0, 1, 2, \cdots,\ a \in X,
\end{aligned}$$

so $x_{2n+2} \to u$ as $n \to \infty$.

By Lemma 1.1 and (2.1), for any $a \in X$, one has

$$\begin{aligned}
d(u,\, Tu,\, a) &= \lim_{n\to\infty} d(x_{2n+1},\, Tu,\, a) \\
&= \lim_{n\to\infty} d(Sx_{2n},\, Tu,\, a) \\
&\le \lim_{n\to\infty} \{kd(x_{2n},\, u,\, a) + l[d(x_{2n},\, Tu,\, a) + d(u,\, Sx_{2n},\, a)] \\
&\qquad\qquad - \varphi[d(x_{2n},\, Tu,\, a),\ d(u,\, Sx_{2n},\, a)]\} \\
&= ld(u,\, Tu,\, a) - \varphi[d(u,\, Tu,\, a),\ 0] \\
&\le ld(u,\, Tu,\, a).
\end{aligned}$$

Hence

$$d(u,\, Tu,\, a) = 0, \qquad a \in X,$$

so $Tu = u$.

Similarly, we have

$$\begin{aligned}
d(Su,\, u,\, a) &= \lim_{n\to\infty} d(Su,\, x_{2n+2},\, a) \\
&= \lim_{n\to\infty} d(Su,\, Tx_{2n+1},\, a) \\
&\le \lim_{n\to\infty} \{kd(u,\, x_{2n+1},\, a) + l[d(u,\, Tx_{2n+1},\, a) + d(x_{2n+1},\, Su,\, a)] \\
&\quad - \varphi[d(u,\, Tx_{2n+1},\, a),\ d(x_{2n+1},\, Su,\, a)]\} \\
&= ld(u,\, Su,\, a) - \varphi[0,\ d(u,\, Su,\, a)] \\
&\le ld(u,\, Su,\, a).
\end{aligned}$$

Hence

$$d(u,\, Su,\, a) = 0, \qquad a \in X.$$

Therefore $Su = u$. So we have $Tu = Su = u$, that is, $u$ is a common fixed point of $S$ and $T$.

If $v$ is also a common fixed point of $S$ and $T$ and $u \neq v$, then there exists an $a^* \in X$ such that $d(u,\, v,\, a^*) > 0$. By (2.1), we have

$$\begin{aligned} d(u,\, v,\, a^*) &= d(Su,\, Tv,\, a^*) \\ &\leq kd(u,\, v,\, a^*) + l[d(u,\, Tv,\, a^*) + d(v,\, Su,\, a^*)] - \varphi[d(u,\, Tv,\, a^*),\ d(v,\, Su,\, a^*)] \\ &= (k + 2l)d(u,\, v,\, a^*) - \varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)] \\ &\leq d(u,\, v,\, a^*) - \varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)] \\ &\leq d(u,\, v,\, a^*). \end{aligned}$$

Hence

$$\varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)] = 0,$$

which implies that

$$d(u,\, v,\, a^*) = 0$$

by the property of $\varphi$. This is a contradiction to the choice of $a^*$. So $u$ is the unique common fixed point of $S$ and $T$.

Similarly, we can prove the same result for $TX$ being complete. The proof is completed.

**Remark 2.1** If $l = 0$ and $\varphi(x,\, y) = 0$ for any $x, y \in [0, +\infty)$, then Theorem 2.1 becomes Banach type common fixed point theorem; if $k = 0$ and $\varphi(x, y) = 0$ for any $x, y \in [0, +\infty)$, then Theorem 2.1 is Kannan type common fixed point theorem; if $k = 0$ and $l = \dfrac{1}{2}$, then Theorem 2.1 is the variant result of Theorem 2.3 in [10]. Hence Theorem 2.1 greatly generalizes and improves some (common) fixed point theorems.

From now, we discuss the existence problems of common fixed points for two mappings on non-complete ordered 2-metric spaces.

**Theorem 2.2** *Let $(X,\, \preceq,\, d)$ be an ordered 2-metric space and $S, T\colon X \to X$ be two maps. Suppose that for each comparable elements $x, y \in X$,*

$$\begin{aligned} d(Sx,\, Ty,\, a) \leq kd(x,\, y,\, a) &+ l[d(x,\, Ty,\, a) + d(y,\, Sx,\, a)] \\ &- \varphi[d(x,\, Ty,\, a),\ d(y,\, Sx,\, a)], \qquad a \in X, \end{aligned} \tag{2.23}$$

*where $k$, $l$ are two real numbers satisfying $l > 0$ and $0 < k + l \leq 1 - l$. If $S$ and $T$ satisfy the following conditions*:

(i) *for each $x \in X$, $x \preceq Sx$ and $x \preceq Tx$*;

(ii) *$S$ and $T$ are both continuous*;

(iii) *$S(X)$ or $T(X)$ is complete*,

*then $S$ and $T$ have a common fixed point.*

*Proof.* Take an element $x_0 \in X$. Using (i), we have

$$x_0 \preceq Sx_0 =: x_1, \quad x_1 \preceq Tx_1 =: x_2, \quad x_2 \preceq Sx_2 =: x_3, \quad x_3 \preceq Tx_3 =: x_4, \quad \cdots$$

Hence we obtain a sequence $\{x_n\}$ satisfying

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad x_n \preceq x_{n+1}, \qquad n = 0, 1, 2, \cdots \tag{2.24}$$

For each $m, n = 0, 1, 2, \cdots$, $x_n$ and $x_m$ are comparable by (2.24), hence modifying the derivation process of Theorem 2.1, we can prove that $\{x_n\}$ is a Cauchy sequence.

Suppose that $SX$ is complete. Since $x_{2n+1} = Sx_{2n} \in SX$ for all $n = 0, 1, 2, \cdots$, there exists a $u \in SX$ such that $x_{2n+1} \to u$ as $n \to \infty$. And since $\{x_n\}$ is Cauchy and

$$\begin{aligned} & d(x_{2n+2},\, u,\, a) \\ \leq\ & d(x_{2n+1},\, x_{2n+2},\, a) + d(x_{2n+1},\, u,\, a) + d(x_{2n+1},\, x_{2n+2},\, u), \qquad n = 0, 1, 2, \cdots,\ a \in X, \end{aligned}$$

so $x_{2n+2} \to u$ as $n \to \infty$. Hence, by (ii), we have

$$u = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S \lim_{n\to\infty} x_{2n} = Su,$$

$$u = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = T \lim_{n\to\infty} x_{2n+1} = Tu.$$

Therefore, $u$ is a common fixed point of $S$ and $T$.

Similarly, we can prove the same result for $TX$ being complete. The proof is completed.

The following result is the non-continuous version of Theorem 2.2.

**Theorem 2.3** *Let $(X, \preceq, d)$ be a ordered 2-metric space and $S$, $T\colon X \to X$ be two maps. Suppose that (2.23) holds. If $S$ and $T$ satisfy:*

(i) *for each $x \in X$, $x \preceq Sx$ and $x \preceq Tx$;*

(ii) *if $\{x_n\}$ is non-decreasing sequence and $\lim\limits_{n\to\infty} x_n = x$, then for each $n$, $x_n \preceq x$;*

(iii) *$S(X)$ or $T(X)$ is complete,*

*then $S$ and $T$ have a common fixed point.*

*Proof.* By the derivation process of Theorem 2.2, we can construct a non-decreasing sequence $\{x_n\}$ satisfying (2.24). Suppose that $SX$ is complete. Then there exists a $u \in SX$ such that $x_{2n+1} \to u$ as $n \to \infty$ and $x_{2n+2} \to u$ as $n \to \infty$ (see the proof of Theorem 2.2), hence $\lim\limits_{n\to\infty} x_n = u$. Therefore, $x_n \preceq u$ for all $n = 0, 1, 2, \cdots$ by (ii). Since $x_{2n}$ and $u$ are comparable, by (2.23), for any $a \in X$,

$$\begin{aligned} d(u,\, Tu,\, a) &= \lim_{n\to\infty} d(x_{2n+1},\, Tu,\, a) \\ &= \lim_{n\to\infty} d(Sx_{2n},\, Tu,\, a) \\ &\leq \lim_{n\to\infty} \{kd(x_{2n},\, u,\, a) + l[d(x_{2n},\, Tu,\, a) + d(u,\, Sx_{2n},\, a)] \\ &\qquad - \varphi[d(x_{2n},\, Tu,\, a),\ d(u,\, Sx_{2n},\, a)]\} \\ &= ld(u,\, Tu,\, a) - \varphi[d(u,\, Tu,\, a),\ 0] \\ &\leq ld(u,\, Tu,\, a). \end{aligned}$$

Hence

$$d(u,\, Tu,\, a) = 0, \qquad a \in X.$$

So $Tu = u$.

Similarly, Since $u$ and $x_{2n+1}$ are comparable, we have

$$\begin{aligned}
d(Su,\, u,\, a) &= \lim_{n\to\infty} d(Su,\, x_{2n+2},\, a)\\
&= \lim_{n\to\infty} d(Su,\, Tx_{2n+1},\, a)\\
&\leq \lim_{n\to\infty} \{kd(u,\, x_{2n+1},\, a) + l[d(u,\, Tx_{2n+1},\, a) + d(x_{2n+1},\, Su,\, a)]\\
&\qquad - \varphi[d(u,\, Tx_{2n+1},\, a),\ d(x_{2n+1},\, Su,\, a)]\}\\
&= ld(u,\, Su,\, a) - \varphi[0,\ d(u,\, Su,\, a)]\\
&\leq ld(u,\, Su,\, a).
\end{aligned}$$

Hence

$$d(u,\, Su,\, a) = 0, \qquad a \in X.$$

So $Su = u$. Therefore $Tu = Su = u$, i.e., $u$ is a common fixed point of $S$ and $T$. The proof is completed.

Now, we give a sufficient condition under which there exists a unique common fixed point for two mappings in Theorems 2.2 and 2.3.

**Theorem 2.4** *Suppose that all of the conditions in Theorem* 2.2 *or Theorem* 2.3 *hold. Furthermore, if*

(I) *for each $x, y \in X$, there exists a $z \in X$ such that $z$ and $x$ are comparable, $z$ and $y$ are comparable*;

(II) *$u \prec v$ implies that $S^n u \preceq v$ and $T^n u \preceq v$ for all $n = 1, 2, \cdots$,*

*then $S$ and $T$ have a unique common fixed point.*

*Proof.* From Theorems 2.2 and 2.3 we know that $S$ and $T$ have a common fixed point $u$. Suppose that $v$ is another common fixed point of $S$. Then $u \neq v$.

Case 1. $u$ and $v$ are comparable.

Since $u \neq v$, there exists an $a^* \in X$ such that $d(u,\, v,\, a^*) > 0$. By (2.23), we have

$$\begin{aligned}
d(u,\, v,\, a^*) &= d(Su,\, Tv,\, a^*)\\
&\leq kd(u,\, v,\, a^*) + l[d(u,\, Tv,\, a^*) + d(v,\, Su,\, a^*)]\\
&\quad - \varphi[d(u,\, Tv,\, a^*),\ d(v,\, Su,\, a^*)]\\
&= (k+2l)d(u,\, v,\, a^*) - \varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)]\\
&\leq d(u,\, v,\, a^*) - \varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)]\\
&\leq d(u,\, v,\, a^*).
\end{aligned}$$

Hence

$$\varphi[d(u,\, v,\, a^*),\ d(u,\, v,\, a^*)] = 0,$$

which implies $d(u,\, v,\, a^*) = 0$ by the property of $\varphi$. This is a contradiction to the choice of $a^*$. Therefore, $u$ is the unique common fixed point of $S$ and $T$.

Case 2. $u$ and $v$ are not comparable.

By (I), there exists a $w \in X$ such that $w$ and $u$ are comparable and $w$ and $v$ are also comparable. Hence $w \neq u$ and $w \neq v$. Assume that $u \prec w$. Then by (II) and the condition

(i) in Theorem 2.2 or Theorem 2.3, we obtain that for each $n = 1, 2, \cdots$,

$$S^n u \preceq w \preceq Tw \preceq T^2 w \preceq \cdots \preceq T^n w,$$

which means that $S^n u$ and $T^n w$ are comparable. By (2.23), for each fixed $a \in X$, we have

$$\begin{aligned} & d(u,\, T^n w,\, a) \\ = {} & d(SS^{n-1}u,\, TT^{n-1}w,\, a) \\ \le {} & kd(S^{n-1}u,\, T^{n-1}w,\, a) + l[d(S^{n-1}u,\, TT^{n-1}w,\, a) + d(SS^{n-1}u,\, T^{n-1}w,\, a)] \\ & - \varphi[d(S^{n-1}u,\, TT^{n-1}w,\, a),\ d(SS^{n-1}u,\, T^{n-1}w,\, a)] \\ = {} & kd(u,\, T^{n-1}w,\, a) + l[d(u,\, T^n w,\, a) + d(u,\, T^{n-1}w,\, a)] \\ & - \varphi[d(u,\, T^n w,\, a),\ d(u,\, T^{n-1}w,\, a)] \\ \le {} & kd(u,\, T^{n-1}w,\, a) + l[d(u,\, T^n w,\, a) + d(u,\, T^{n-1}w,\, a)]. \end{aligned} \tag{2.25}$$

Hence

$$d(u,\, T^n w,\, a) \le \frac{k+l}{1-l} d(u,\, T^{n-1}w,\, a) \le d(u,\, T^{n-1}w,\, a), \qquad n = 1, 2, \cdots,\ a \in X.$$

This shows that $\{d(u,\, T^n w,\, a)\}_{n=1}^{\infty}$ is a non-increasing non-negative real number sequence. Hence there exists $M(a) \ge 0$ such that

$$\lim_{n\to\infty} d(u,\, T^n w,\, a) = M(a). \tag{2.26}$$

Letting $n \to \infty$ in (2.25) and using (2.26), we obtain

$$M(a) \le (k+2l)M(a) - \varphi(M(a),\, M(a)) \le M(a) - \varphi(M(a),\, M(a)) \le M(a).$$

Hence

$$\varphi(M(a),\, M(a)) = 0,$$

which implies $M(a) = 0$, i.e.,

$$\lim_{n\to\infty} d(u,\, T^n w,\, a) = 0, \qquad a \in X.$$

Therefore,

$$\lim_{n\to\infty} T^n w = u.$$

If $u$ in the above derivation process is replaced by $v$, then we similarly obtain

$$\lim_{n\to\infty} T^n w = v.$$

Hence $u = v$ by Lemma 1.5, which is a contradiction. So $u$ is the unique common fixed point of $S$ and $T$.

## References


[1] Gähler V S. 2-metrische Räume und ihre topolodische struktur. *Math Nachr.*, 1963/1964, **26**: 115–118.

[2] Gähler V S. Linera 2-normierte Räume. *Math Nachr.*, 1965, **28**: 1–43.

[3] Gähler V S. Über die uniformisierbarkeit 2-metrischer Räume. *Math Nachr.*, 1965, **28**: 235–244.

[4] Aliouche A, Simpson C. Fixe points and lines in 2-metric spaces. *Adv. Math.*, 2012, **229**: 668–690.

[5] Deshpande B, Chouhan X. Common fxie point theorems for hybird pairs of mappings with some weaker conditions in 2-metric spaces. *Fasc. Math.*, 2011, **46**: 37–55.

[6] Lahiri B K, Das P, Dey L K. Cantor's theorem in 2-metric spaces ad its applications to fixed point problems. *Taiwanese J. Math.*, 2011, **15**: 337–352.

[7] Singh S L, Mishra S N, Stofile S. Suzuki contraction theorem on a 2-metric space. *J. Adv. Math. Stud.*, 2012, **5**(1): 71–76.

[8] Liu Z Q, Zhang F R. Characterizations of common fixed point in 2-metric spaces. *Rostock. Math. Kolloq.*, 2001, **55**: 49-64.

[9] Singh S L, Mishra S N, Stofile S. Erratum to "Suzuki contraction theorem on a 2-metric space". *J. Adv. Math. Stud.*, 2012, **5**(2): 138.

[10] Dung N V, Hang V T L. Fixed point theorems for weak $c$-contractions in partial ordered 2-metric spaces. *Fixed Point Theory Appl.*, Doi: 1186/1687-1812-2013-161.

[11] Nguyen V Dung, Nguyen T Hieu, Nguyen T Thanh Ly, Vo D thinh. Remarks on fixed point problems of 2-metric spaces. *Fixed Point Theory Appl.*, Doi: 1186/1687-1812-2013-167.

[12] Piao Y J, Jin Y F. New unique common fixed point results for four mappings with $\varPhi$-contractive type in 2-metric spaces. *Appl. Math.*, 2012, **3**(7): 734–737.

[13] Piao Y J. Uniqueness of common gixed points for a family of maps with $\phi_j$-quasi contractive type in 2-metric spaces (in Chinese). *Acta Math. Sci.*, 2012, **A32**(6): 1079–1085.

[14] Jin H L, Piao Y J. Four mappings satisfying $\varPsi$-contractive type condition and having unique common fixed point on 2-metric spaces. *Adv. Pure Math.*, 2013, **3**: 277–281.

[15] Piao Y J. Common fixed point for two mappings satisfying some expansive conditions on 2-metric spaces (in Chinese). *J. Systems Sci. Math. Sci.*, 2013, **33**(11): 1370–1379.

[16] Piao Y J. Fixed point theorems for contractive and expansive mappings of Geraghty type on 2-metric spaces. *Adv. Fixed Point Theory*, 2016, **6**(2): 123–135.

[17] Choudhury B S. Unique fixed point theoem for weakly $C$-contractive mappings. *Kathmandu Univ J. Sci. Engi. Tech.*, 2009, **5**: 6–13.

[18] Harjani J, López B, Sadarangani K. Fixed point theorems for weakly $C$-contractive mappings in ordered metric spaces. *Comput. Math. Appl.*, 2011, **61**: 790–796.

[19] Zhang D, Gu F. The common fixed point theorems for a class of $\varPhi$-contraction conditions mappings in 2-metric spaces (in Chinese). *J. Jiangxi Norm. Univ. Nat. Sci. Ed.*, 2011, **35**(6): 595–600.