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A New Post-Processing Technique for Finite Element Methods with L^2-Superconvergence

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Abstract. A simple post-processing technique for finite element methods with L^2-superconvergence is proposed. It provides more accurate approximations for solutions of two- and three-dimensional systems of partial differential equations. Approximate solutions can be constructed locally by using finite element approximations u_h provided that u_h is superconvergent for a locally defined projection $\widetilde{P}_h u$. The construction is based on the least-squares fitting algorithm and local L^2-projections. Error estimates are derived and numerical examples illustrate the effectiveness of this approach for finite element methods.

AMS subject classifications: 65N30, 65N15

Key words: Finite element method, post-processing, least-square fitting, L^2-superconvergence.

1. Introduction

The post-processing of approximate solutions is a commonly used procedure to obtain more accurate approximations for important quantities in numerical methods for partial differential equations [4–6, 22, 23]. Post-processing or/and recovery techniques have been developed for plenty of finite element methods with superconvergence [1, 7, 8, 10, 12, 13, 15, 18, 20]. In particular, for the Raviart-Thomas and Brezzi-Douglas-Marini mixed elements methods for second order elliptic problems, the post-processed approximations with improved accuracy are constructed via element-by-element solution of local problems with respect to the finite element solutions of the scalar variable and the Lagrange multiplier [1, 8]. In contrast to the post-processing methods [1, 8], Stenberg [18] proposed an approach based on solving local problems with respect to the mixed finite element approximations of the scalar variable and its gradient. Following ideas of [18], Cockburn *et al.* [12, 13] developed an element-by-element post-processing of the scalar variable for the elliptic problems and velocity variable in the Stokes problem for HDG methods.

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Bramble and Xu [7] proposed a general post-processing technique for various mixed finite element methods with the superconvergence estimate

$$\|\widetilde{P}_h u - u_h\|_{L^p(\Omega)} \leq Ch^{k+2}|\log h|^{\mu_1} \tag{1.1}$$

and the gradient approximation estimate

$$\|\nabla u - (\nabla u)_h\|_{L^p(\Omega)} \leq Ch^{k+1}|\log h|^{\mu_2},$$

where u is the exact solution of a system of partial differential equations on a domain $\Omega \subset \Re^2$, $C > 0$ a generic constant, which depends on u but not on the mesh size h; μ_1, μ_2 are nonnegative constants and $u_h \in W_h$ and $(\nabla u)_h \in V_h$ are finite element approximations of u and ∇u, respectively. Moreover, W_h and V_h are finite-dimensional subspaces of $L^p(\Omega)$, $p \geq 1$, W_h consists of discontinuous piecewise polynomials of degree at most $k \geq 0$, and $\widetilde{P}_h$ is a locally defined operator, which is invariant on polynomials of degree k. Under a regularity condition for u, the post-processed approximation u_h^* obtained from u_h and $(\nabla u)_h$, satisfies the estimate

$$\|u - u_h^*\|_{L^p(\Omega)} \leq C\left(\|\widetilde{P}_h u - u_h\|_{L^p(\Omega)} + h\|\nabla u - (\nabla u)_h\|_{L^p(\Omega)} + h^{k+2}\right).$$

Further, Zienkiewicz and Zhu [22, 23] used the well-known gradient recovery technique, usually referred to as superconvergence patch recovery (SPR), to post-process the gradient ∇u_h of the finite element solution u_h. They constructed an SPR-recovered gradient by a local discrete least-squares fitting of polynomials of degree k to the gradient values at sampling points on element patches. The superconvergence properties of this technique was discussed in Refs. [14,19,21]. Zhang and Naga [20] introduced a different gradient recovery method called the polynomial preserving recovery (PPR). To determine a recovered gradient, the method uses the least-squares algorithm to assign a polynomial of degree $k+1$ to the solution at chosen nodal points and computes the corresponding partial derivatives. Under certain conditions, the PPR post-processed gradient $G_h u_h$ satisfies the superconvergence estimate

$$\|\nabla u - G_h u_h\|_{L^\infty(\Omega_0)} \leq C\left(h^{k+1}|\log h|^{\bar{r}} + h^{k+\sigma}\right),$$

where σ is a positive constant, $\Omega_0 \subset\subset \Omega$, $\bar{r} = 1$ if $k = 1$ and $\bar{r} = 0$ if $k \geq 2$.

However, to the best of authors' knowledge, there is no post-processing technique, which uses only u_h to construct a superconvergent post-processed approximation u_h^*. Here, we present a general post-processing technique for direct construction of the improved approximation of u. The method is based on the least-squares algorithm and the local L^2-projection to determine a fitting polynomial from the finite element solution u_h. Our analysis depends only on a superconvergence result similar to (1.1) and the main result is proved in general approximation-theoretic settings. Therefore, its application is not restricted to the above mentioned finite element methods.

The rest of the paper is organised as follows. Section 2 contains necessary notations. Section 3 is devoted to the construction of the post-processed approximation, the error estimation, and the verification of assumptions. Finally, numerical results in Section 4 are aimed to verify the performance of the post-processing method proposed.

2. Notations

Let Ω be a bounded domain in $\mathbb{R}^n$, $n=2,3$. For any bounded domain $D \subset \mathbb{R}^n$, $n=2,3$ and a nonnegative integer m, we denote by $H^m(D)$ the usual m-order Sobolev space on D and let $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ refer to the corresponding norm and semi-norm, respectively. In particular, $H^0(D)$ is the space of square integrable functions $L^2(D)$ with the inner product $(\cdot,\cdot)_D$ and the norm $\|\cdot\|_{0,D}$. If $D=\Omega$, we write $\|\cdot\|_m := \|\cdot\|_{m,\Omega}$ and $|\cdot|_m := |\cdot|_{m,\Omega}$. By $\mathscr{P}_m(D)$ we denote the set of all polynomials on D of degree at most m.

Let $\mathscr{T}_h$ be a shape regular partition of the domain Ω, which consists of closed polygons T— cf. [11], with the mesh size $h = \max_{T\in\mathscr{T}_h} h_T$, where h_T is the diameter of T. The partition $\mathscr{T}_h$ can be conforming or nonconforming, which allows hanging nodes. Let $\mathscr{N}_h = \{\boldsymbol{x_i} : i=1,2,\ldots,n_h\}$ be the set of all nodes of the partition $\mathscr{T}_h$. For any $\boldsymbol{x_i} \in \mathscr{N}_h$, we denote by h_i the length of the longest edge attached to $\boldsymbol{x_i}$ and let M_i be the patch defined by

$$M_i = M_i(\alpha) := \bigcup_{T\in\mathscr{T}_h, T\subseteq B_{\alpha h_i}(\boldsymbol{x_i})} T,$$

where $B_{\alpha h_i}(\boldsymbol{x_i})$, $\alpha>0$ is the ball

$$B_{\alpha h_i}(\boldsymbol{x_i}) := \{\boldsymbol{x} \in \Omega : |\, \boldsymbol{x} - \boldsymbol{x_i} \,| \le \alpha h_i\}.$$

If $n_i = n_i(\alpha)$ is the number of elements in the patch M_i, we set

$$\mathbb{M}_h := \{M_i : \ i = 1,2,\ldots,n_h\}.$$

For any $T \in \mathscr{T}_h$ and for any integer $j \ge 0$, let $P_T^j : L^2(T) \longrightarrow \mathscr{P}_j(T)$ be the usual L^2-orthogonal projection. Define $\mathrm{P}_{M_i}^j : L^2(M_i) \longrightarrow L^2(M_i)$, such that for any $v \in L^2(M_i)$ the relation

$$\left(\mathrm{P}_{M_i}^j v\right)\Big|_T = P_T^j(v|_T) \quad \text{for all} \quad T \in M_i \tag{2.1}$$

holds.

Throughout this paper, the notation $a \lesssim b$ ($a \gtrsim b$) means that $a \le Cb$ ($a \ge Cb$) with a constant C, which depends on u but not on the mesh size h.

3. Main Results

Let us assume that every patch $M_i \in \mathbb{M}_h$ satisfies the following conditions.

Condition 3.1. For a given integer $k \ge 0$, there exists a nonnegative integer $j \le k$, such that for any $q \in \mathscr{P}_{k+1}(M_i)$, $M_i \in \mathbb{M}_h$, the inequality

$$\|\, q \,\|_{0,M_i} \lesssim \left\|\mathrm{P}_{M_i}^j q\right\|_{0,M_i} \tag{3.1}$$

holds.

Condition 3.2. Any $v \in H^{k+2}(M_i)$, $i = 1, 2, \ldots, n_h$ satisfies the inequality

$$\inf_{w \in \mathscr{P}_{k+1}(M_i)} \|v - w\|_{0,M_i} \lesssim h^{k+2} |v|_{k+2,M_i}.$$

Condition 3.1 yields that M_i has enough elements. We note that if $\alpha_1 > \alpha_2$, then $n_i(\alpha_1) > n_i(\alpha_2)$, where $n_i(\alpha_i)$ is the number of element patches in $M_i(\alpha_j)$, $j = 1, 2$. Thus it is natural to choose the smallest α satisfying Condition 3.1. On the other hand, Condition 3.2 represents the standard approximation property.

Condition 3.1 allows us to equip $\mathscr{P}_{k+1}(M_i)$ with an inner product and a norm.

Lemma 3.1. *For any $M_i \in \mathbb{M}_h$, the inner product and the norm on the space $\mathscr{P}_{k+1}(M_i)$ can be, respectively, defined as $(P^j_{M_i} \cdot, P^j_{M_i} \cdot)_{M_i}$ and $\|P^j_{M_i} \cdot \|_{0,M_i}$.*

Proof. Let us show that $(\mathrm{P}^j_{M_i} \cdot, \mathrm{P}^j_{M_i} \cdot)_{M_i}$ is an inner product on $\mathscr{P}_{k+1}(M_i)$. Recalling the relation (2.1), we only have to show that if $q \in \mathscr{P}_{k+1}(M_i)$ and

$$\left(\mathrm{P}^j_{M_i} q, \mathrm{P}^j_{M_i} q\right)_{M_i} = 0,$$

then $q = 0$. However, the condition (3.1) yields

$$\| q \|^2_{0,M_i} \lesssim \left\|\mathrm{P}^j_{M_i} q\right\|^2_{0,M_i} = \left(\mathrm{P}^j_{M_i} q, \mathrm{P}^j_{M_i} q\right)_{M_i} = 0,$$

and the proof is completed. □

3.1. Recovery operator

Definition 3.1. For any $M_i \in \mathbb{M}_h$, the local recovery operator $R_{M_i} : L^2(M_i) \to \mathscr{P}_{k+1}(M_i)$ is defined by the relations

$$\left(\mathrm{P}^j_{M_i} R_{M_i} v, \mathrm{P}^j_{M_i} q\right)_{M_i} = \left(\mathrm{P}^j_{M_i} v, \mathrm{P}^j_{M_i} q\right)_{M_i} \quad \text{for all} \quad v \in L^2(M_i),\ q \in \mathscr{P}_{k+1}(M_i).$$

According to Lemma 3.1, the operator R_{M_i} is well-defined. Moreover, the following result holds.

Lemma 3.2. *For any $M_i \in \mathbb{M}_h$, the recovery operator R_{M_i} is an orthogonal projection onto $\mathscr{P}_{k+1}(M_i)$ with respect to the inner product $(P^j_{M_i} \cdot, P^j_{M_i} \cdot)_{M_i}$ and if $v \in L^2(M_i)$, then*

$$R_{M_i} v = \underset{q \in \mathscr{P}_{k+1}(M_i)}{\arg\min} \left\|P^j_{M_i}(v - q)\right\|_{0,M_i}.$$

Other consequences of Conditions 3.1 and the inequality 3.1 are presented in Lemmas 3.2-3.5.

Lemma 3.3. *For any $v \in L^2(M_i)$ and $M_i \in \mathbb{M}_h$ the inequalities*

$$\left\|R_{M_i} v\right\|_{0,M_i} \lesssim \left\|P^j_{M_i} v\right\|_{0,M_i} \lesssim \|v\|_{0,M_i} \tag{3.2}$$

hold.

Lemma 3.4. *For any $v \in H^{k+2}(M_i)$ and $M_i \in \mathbb{M}_h$, the inequality*

$$\|v - R_{M_i} v\|_{0,M_i} \lesssim h^{k+2} |v|_{k+2,M_i}$$

holds.

Proof. Since $w = R_{M_i} w$ for any $w \in \mathscr{P}_{k+1}(M_i)$, Condition 3.2 and the inequality (3.2) lead to the estimate

$$\begin{aligned}\|v - R_{M_i} v\|_{0,M_i} &= \inf_{w \in \mathscr{P}_{k+1}(M_i)} \|v - w - R_{M_i}(v - w)\|_{0,M_i} \\ &\lesssim \inf_{w \in \mathscr{P}_{k+1}(M_i)} \|v - w\|_{0,M_i} \lesssim h^{k+2} |v|_{k+2,M_i}\end{aligned}$$

as required. □

Let $\widetilde{P}_h$ be an operator defined on $L^2(\Omega)$, such that its restriction $\widetilde{P}_h|_T : L^2(T) \longrightarrow L^2(T)$, $T \in M_i$ satisfies the conditions

$$(\widetilde{P}_h w, v)_T = (w, v)_T \tag{3.3}$$

valid for all $w \in L^2(T)$ and all $v \in \mathscr{P}_j(T)$.

Lemma 3.5. *If $v \in L^2(M_i)$ and $M_i \in \mathbb{M}_h$, then*

$$R_{M_i} v = R_{M_i} \widetilde{P}_h v. \tag{3.4}$$

Proof. It follows from the definitions of R_{M_i}, $\mathrm{P}^j_{M_i}$, P^j_T and the Eq. (3.3) that for $v \in L^2(M_i)$ and $q \in \mathscr{P}_{k+1}(M_i)$ one has

$$\begin{aligned}\left(\mathrm{P}^j_{M_i} R_{M_i} \widetilde{P}_h v, \mathrm{P}^j_{M_i} q\right)_{M_i} &= \left(\mathrm{P}^j_{M_i} \widetilde{P}_h v, \mathrm{P}^j_{M_i} q\right)_{M_i} = \sum_{T \in M_i} \left(P^j_T \widetilde{P}_h v, P^j_T q\right)_{M_i} \\ &= \sum_{T \in M_i} \left(\widetilde{P}_h v, P^j_T q\right)_{M_i} = \sum_{T \in M_i} \left(v, P^j_T q\right)_{M_i} \\ &= \sum_{T \in M_i} \left(P^j_T v, P^j_T q\right)_{M_i} = \left(\mathrm{P}^j_{M_i} v, \mathrm{P}^j_{M_i} q\right)_{M_i} = \left(\mathrm{P}^j_{M_i} R_{M_i} v, \mathrm{P}^j_{M_i} q\right)_{M_i},\end{aligned}$$

which yields the representation (3.4). □

3.2. Post-processed approximation

For any $T \in \mathcal{T}_h$, we set

$$\mathbb{M}_T := \{M_i \in \mathbb{M}_h : T \in M_i\},$$

where n_T is the number of element patches in $\mathbb{M}_T$.

Let $u_h \in L^2(\Omega)$ be a finite element approximation of u, such that

$$\|u - u_h\|_0 \lesssim h^r, \quad r \leq k+1.$$

Considering a post-processed approximation u_h^* defined by

$$u_h^*|_T = \sum_{M_i \in \mathbb{M}_T} \frac{1}{n_T} (R_{M_i} u_h)|_T, \quad T \in \mathcal{T}_h, \tag{3.5}$$

and using Lemmas 3.3, 3.4 and 3.5, we obtain the following results.

Theorem 3.1. *If $\widetilde{P}_h$ satisfies the projection property* (3.3) *and $u \in H^{k+2}(\Omega)$, then*

$$\|u - u_h^*\|_0 \lesssim \|\widetilde{P}_h u - u_h\|_0 + h^{k+2}|u|_{k+2}. \tag{3.6}$$

Moreover, if

$$\|\widetilde{P}_h u - u_h\|_0 \lesssim h^{k+2}|u|_{k+2}, \tag{3.7}$$

then the superconvergence estimate

$$\|u - u_h^*\|_0 \lesssim h^{k+2}|u|_{k+2} \tag{3.8}$$

holds.

Proof. It follows from (3.5) that

$$\begin{aligned}
\left\|u - u_h^*\right\|_0^2 &= \sum_{T \in \mathcal{T}_h} \left\|u - u_h^*\right\|_{0,T}^2 \leq \sum_{M_i \in \mathbb{M}_h} \sum_{T \in M_i} \left\| u - \sum_{M_j \in \mathbb{M}_T} \frac{1}{n_T} R_{M_j} u_h \right\|_{0,T}^2 \\
&= \sum_{M_i \in \mathbb{M}_h} \sum_{T \in M_i} \left\| \sum_{M_j \in \mathbb{M}_T} \frac{1}{n_T} \left(u - R_{M_j} u_h\right) \right\|_{0,T}^2 \lesssim \sum_{M_i \in \mathbb{M}_h} \left\|u - R_{M_i} u_h\right\|_{0,M_i}^2.
\end{aligned} \tag{3.9}$$

Using triangle inequality and Lemmas 3.3 , 3.4 and 3.5, we obtain

$$\begin{aligned}
\|u - R_{M_i} u_h\|_{0,M_i} &\lesssim \|u - R_{M_i} u\|_{0,M_i} + \|R_{M_i} u - R_{M_i} u_h\|_{0,M_i} \\
&\lesssim \|u - R_{M_i} u\|_{0,M_i} + \|R_{M_i}(\widetilde{P}_h u - u_h)\|_{0,M_i} \\
&\lesssim h^{k+2}|u|_{k+2,M_i} + \|\widetilde{P}_h u - u_h\|_{0,M_i}.
\end{aligned}$$

This and (3.9) first yield (3.6) and consequently (3.8). □

Remark 3.1. Theorem 3.1 can be applied to various finite element methods, including the Raviart-Thomas triangular elements $\mathbf{RT}_k$ and rectangular elements $\mathbf{RT}_{[k]}$ with $k \geq 0$, the Brezzi-Douglas-Marini triangular elements $\mathbf{BDM}_k$ and rectangular elements $\mathbf{BDM}_{[k]}$ with $k \geq 2$, the **PEERS** elements, the mixed elements by Stenberg, the hybridised Discontinuous Galerkin triangular elements $\mathbf{HDG}_k$ and rectangular elements $\mathbf{HDG}_{[k]}$ with $k \geq 1$ — cf. Refs. [1, 2, 8, 9, 12, 17]. Let us note the following properties of the above listed $2D$-elements:

(1) $\mathbf{RT}_k$ elements ($k \geq 0$): $u_h|_T \in \mathscr{P}_k(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow \mathscr{P}_k(T)$ is the L^2-orthogonal projection satisfying the projection property (3.3) for any $j \leq k$ and the superconvergence estimate (3.7).

(2) $\mathbf{RT}_{[k]}$ elements ($k \geq 0$): $u_h|_T \in \mathscr{Q}_k(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow Q_k(T)$ is the L^2-orthogonal projection satisfying (3.3) (with any $j \leq k$) and (3.7). Here $Q_k(T)$ denotes the set of all polynomials on T of degree at most k in each variable.

(3) $\mathbf{BDM}_k$ and $\mathbf{BDM}_{[k]}$ elements ($k \geq 2$): $u_h|_T \in \mathscr{P}_{k-1}(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow \mathscr{P}_{k-1}(T)$ is the L^2-orthogonal projection satisfying (3.3) for any $j \leq k-1$ and (3.7).

(4) **PEERS** elements: $u_h|_T \in \mathscr{P}_0(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow \mathscr{P}_0(T)$ is the L^2-orthogonal projection satisfying (3.3) for $j = 0$ and (3.7) with $k = 0$.

(5) The mixed elements by Stenberg ($k \geq 1$): $u_h|_T \in \mathscr{P}_{k-1}(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow \mathscr{P}_{k-1}(T)$ is the L^2-orthogonal projection satisfying (3.3) for any $j \leq k-1$ and (3.7).

(6) $\mathbf{HDG}_k$ and $\mathbf{HDG}_{[k]}$ elements ($k \geq 1$): $u_h|_T \in \mathscr{P}_k(T)$ and $\widetilde{P}_h|_T : L^2(T) \longrightarrow \mathscr{P}_k(T)$ is an operator satisfying (3.3) for $j = k-1$ and (3.7).

3.3. Discussion on Condition 3.1

As shown in Subsection 3.2, Condition 3.1 is crucial for the construction and evaluation of the post-processed approximation u_h^*. However, for a given j, Condition 3.1 requires the availability of sufficiently large number of elements n_i in M_i.

Theorem 3.2. *If the inequality* (3.1) *holds for any $q \in \mathscr{P}_{k+1}(M_i)$, $M_i \in \mathbb{M}_h$, then*

$$n_i \geq \frac{C_{k+1+n}^n}{C_{j+n}^n},$$

where $n = 2$ or $n = 3$ is the space dimension and $C_{l+n}^n = (l+n)!/(l!n!)$.

Proof. Since

$$\mathscr{P}_l(M_i) = \operatorname{span}\left\{1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, x_1^l, x_1^{l-1}x_2, \ldots, x_n^l\right\},$$

we can represent any $q \in \mathscr{P}_{k+1}(M_i)$ in the form

$$q = \mathbf{Pa},$$

where

$$\begin{aligned}&\mathbf{P} := \big(1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, x_1^{k+1}, x_1^k x_2, \ldots, x_n^{k+1}\big),\\&\mathbf{a} = (a_1, a_2, \ldots, a_\gamma)^T, \quad \gamma = \mathbf{dim}(P_{k+1}) = C_{k+1+n}^n.\end{aligned}$$

It follows from (3.1) that if $P_T^j q = 0$ for all $T \in M_i$, then $q = 0$. Along with the definition of the projection P_T^j, this means that if the relation

$$(q, v)_T = \left(P_T^j q, v\right)_T = 0 \tag{3.10}$$

holds for any $v \in P_j(T)$, $T \in M_i$, then $q = 0$. Setting $M_i = \{T_l : l = 1, 2, \ldots, n_i\}$, we obtain that if (3.10) holds, then the condition

$$A\mathbf{a} = 0,$$

where $A = (A_1, A_2, \ldots, A_{n_i})^T$ and

$$A_l = \begin{pmatrix} (1,1)_{T_l} & (1,x_1)_{T_l} & \cdots & (1,x_n)_{T_l} & (1,x_1^2)_{T_l} & \cdots & (1,x_n^{k+1})_{T_l} \\ (x_1,1)_{T_l} & (x_1,x_1)_{T_l} & \cdots & (x_1,x_n)_{T_l} & (x_1,x_1^2)_{T_l} & \cdots & (x_1,x_n^{k+1})_{T_l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (x_n,1)_{T_l} & (x_n,x_1)_{T_l} & \cdots & (x_n,x_n)_{T_l} & (x_n,x_1^2)_{T_l} & \cdots & (x_n,x_n^{k+1})_{T_l} \\ (x_1^2,1)_{T_l} & (x_1^2,x_1)_{T_l} & \cdots & (x_1^2,x_n)_{T_l} & (x_1^2,x_1^2)_{T_l} & \cdots & (x_1^2,x_n^{k+1})_{T_l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ (x_n^j,1)_{T_l} & (x_n^j,x_1)_{T_l} & \cdots & (x_n^j,x_n)_{T_l} & (x_n^j,x_1^2)_{T_l} & \cdots & (x_n^j,x_n^{k+1})_{T_l} \end{pmatrix}_{C_{j+n}^j \times C_{k+1+n}^n}$$

yields $\mathbf{a} = 0$.

It is easily seen that a necessary condition for this claim is

$$n_i \times \; C_{j+n}^n \geq C_{k+1+n}^n.$$

In other words, the number of equations in the system $A\mathbf{a} = 0$ is greater than or equal to the number of variables. $\square$

Remark 3.2. This theorem states that for a given j, each patch M_i has to contain at least C_{k+1+n}^n / C_{j+n}^n elements. On the other hand, for larger j, the number C_{k+1+n}^n / C_{j+n}^n becomes smaller. Therefore, it would be natural to choose the largest j such that Condition 3.1 and the projection property (3.3) hold. We refer the reader to Remark 3.1 for the range of j in the case of specific elements.

Since Condition 3.1 depends on the choice of $M_i \in \mathbb{M}_h$, it is not easy to provide general recommendations for its verification. However, for certain structured meshes with all patches $M_1, M_2, \ldots, M_{n_h}$ having the same number of elements, the verification of this condition on each M_i can be done on a reference patch $\hat{M}$. To this end, we assume that

$$M_i = \bigcup_{l=1}^{n_i} T_l, \quad \hat{M} = \bigcup_{l=1}^{n_i} \hat{T}_l, \tag{3.11}$$

where $T_l = \Psi_l(\hat{T}_l)$, the function $\Psi_l : \mathbf{R}^n \to \mathbf{R}^n$ is defined by

$$\Psi(\hat{\mathbf{x}}) := \mathbf{B}_l \hat{\mathbf{x}} + \mathbf{b}_l,$$

the matrix $\mathbf{B}_l \in \mathbf{R}^{n\times n}$ is invertible and $\mathbf{b} \in \mathbf{R}^n$. For any $v \in L^2(M_i)$, we define $\hat{v}(\hat{\mathbf{x}}) := v(\Psi(\hat{\mathbf{x}}))$.

Theorem 3.3. *Assume that the conditions* (3.11) *hold and*

$$\|\hat{q}\|_{0,\hat{M}}^2 \lesssim \left\|P_{\hat{M}}^j \hat{q}\right\|_{0,\hat{M}}^2 \quad \textit{for any} \quad \hat{q} \in \mathscr{P}_{k+1}(\hat{M}). \tag{3.12}$$

Then

$$\|q\|_{0,M_i}^2 \lesssim \left\|P_{M_i}^j q\right\|_{0,M_i}^2 \quad \textit{for any} \quad q \in \mathscr{P}_{k+1}(M_i). \tag{3.13}$$

Proof. It follows from the definitions of the projections $P_{\hat{T}}^j$ and $\widehat{P_T^j}$ that for any $T = T_l, \hat{T} = \hat{T}_l$, $v \in L^2(T)$ we have

$$\left(P_{\hat{T}}^j \hat{v}, \hat{w}\right)_{\hat{T}} = (\hat{v}, \hat{w})_{\hat{T}} = (v, w)_T = \left(P_T^j v, w\right)_T = \left(\widehat{P_T^j v}, \hat{w}\right)_{\hat{T}} \quad \text{for all} \quad w \in \mathscr{P}_j(T).$$

Therefore,

$$P_{\hat{T}}^j \hat{v} = \widehat{P_T^j v}.$$

This and the condition (3.12) yield that for any $q \in \mathscr{P}_{k+1}(M_i)$, we have

$$\begin{aligned}
\|q\|_{0,M_i}^2 &= \sum_{l=1}^{n_i} \|q\|_{0,T_l}^2 = \sum_{l=1}^{n_i} \|\hat{q}\|_{0,\hat{T}_l}^2 = \|\hat{q}\|_{0,\hat{M}}^2 \\
&\lesssim \left\|P_{\hat{M}}^j \hat{q}\right\|_{0,\hat{M}}^2 = \sum_{l=1}^{n_i} \left\|P_{\hat{T}_l}^j \hat{q}\right\|_{0,\hat{T}_l}^2 = \sum_{l=1}^{n_i} \left\|\widehat{P_{T_l}^j q}\right\|_{0,\hat{T}_l}^2 \\
&= \sum_{l=1}^{n_i} \left\|P_{T_l}^j q\right\|_{0,T_l}^2 = \left\|P_{M_i}^j q\right\|_{0,M_i}^2,
\end{aligned}$$

and (3.13) is proved. □

Remark 3.3. Since $T_l = \Psi_l(\hat{T}_l)$, $l = 1, 2, \ldots, n_i$, this theorem can be used in the case of structured simplicial meshes or parallelogram/parallelepiped meshes.

As an example, we verify the condition (3.12) with $k = 1$ and $j = 1$ for rectangular meshes — i.e.

$$\|\hat{q}\|_{0,\hat{M}}^2 \lesssim \left\|P_{\hat{M}}^1 \hat{q}\right\|_{0,\hat{M}}^2 \quad \text{for all} \quad \hat{q} \in \mathscr{P}_2(\hat{M}),$$

where the reference patch $\hat{M}$ is the square $[-1,1] \times [-1,1]$ — cf. Fig. 1, which consists of four reference rectangles $\hat{T}_i$, $i = 1, 2, 3, 4$ — cf. Fig. 2. For $\hat{q} \in \mathscr{P}_2(\hat{M})$, we assume that

$$\begin{aligned}
\hat{q} &= a_1 \hat{x}^2 + a_2 \hat{y}^2 + a_3 \hat{x} + a_4 \hat{y} + a_5 \hat{x}\hat{y} + a_6, \\
P_{\hat{T}_i}^1 \hat{q} &= \mathscr{A}_i \hat{x} + \mathscr{B}_i \hat{y} + \mathscr{C}_i,
\end{aligned}$$

Figure 1: Rectangular mesh, $k=1$, $j=1$. Left: patch M with respect to node z. Right: reference patch $\hat{M}$.

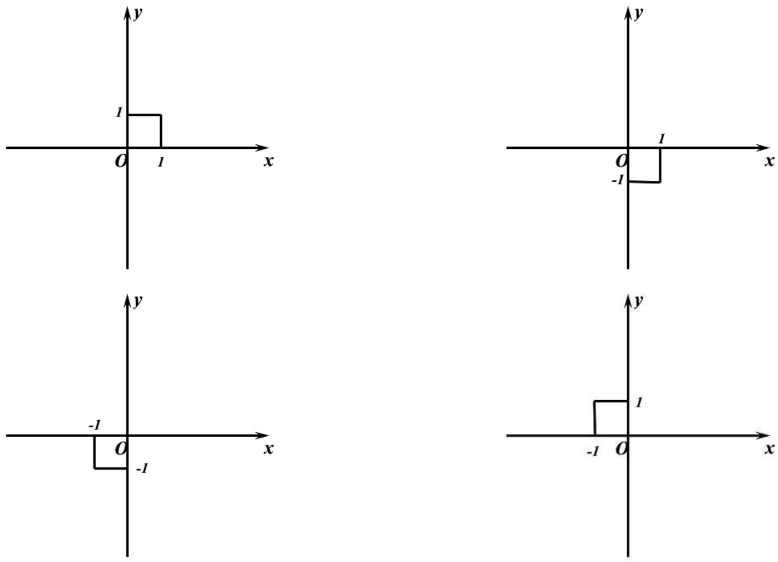

Figure 2: Subelements $\hat{T}_i$, $i=1,2,3,4$ in reference patch $\hat{M}$.

where a_l, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, $l=1,2,\ldots,6$ and $i=1,2,\ldots,4$ are constants.

According to the definition of the projection $P^1_{\hat{M}}$, we have

$$(\hat{q},\hat{x})_{\hat{T}_i}=\big(P^1_{\hat{T}_i}\hat{q},\hat{x}\big)_{\hat{T}_i},\quad (\hat{q},\hat{y})_{\hat{T}_i}=\big(P^1_{\hat{T}_i}\hat{q},\hat{y}\big)_{\hat{T}_i},\quad (\hat{q},1)_{\hat{T}_i}=\big(P^1_{\hat{T}_i}\hat{q},1\big)_{\hat{T}_i},\quad i=1,2,\ldots,4,$$

and simple calculations show that

$$\begin{aligned}
\mathscr{A}_1&=a_1+a_3+\frac{1}{2}a_5, & \mathscr{B}_1&=a_2+a_4+\frac{1}{2}a_5, & \mathscr{C}_1&=-\frac{1}{6}a_1-\frac{1}{6}a_2-\frac{1}{4}a_5+a_6,\\
\mathscr{A}_2&=a_1+a_3-\frac{1}{2}a_5, & \mathscr{B}_2&=-a_2+a_4+\frac{1}{2}a_5, & \mathscr{C}_2&=-\frac{1}{6}a_1-\frac{1}{6}a_2-\frac{1}{4}a_5+a_6,\\
\mathscr{A}_3&=-a_1+a_3+\frac{1}{2}a_5, & \mathscr{B}_3&=a_2+a_4-\frac{1}{2}a_5, & \mathscr{C}_3&=-\frac{1}{6}a_1-\frac{1}{6}a_2-\frac{1}{4}a_5+a_6,\\
\mathscr{A}_4&=-a_1+a_3-\frac{1}{2}a_5, & \mathscr{B}_4&=-a_2+a_4-\frac{1}{2}a_5, & \mathscr{C}_4&=-\frac{1}{6}a_1-\frac{1}{6}a_2-\frac{1}{4}a_5+a_6.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\left\|P_{\hat{M}}^1\hat{q}\right\|_{0,\hat{M}}^2 =&\sum_{i=1}^4 \|P_{\hat{T}_i}^1\hat{q}\|_{0,\hat{T}_i}^2 = \frac{1}{3}\mathscr{A}_1^2+\frac{1}{3}\mathscr{B}_1^2+\mathscr{C}_1^2+\frac{1}{2}\mathscr{A}_1\mathscr{B}_1+\mathscr{B}_1\mathscr{C}_1+\mathscr{A}_1\mathscr{C}_1\\
&+\frac{1}{3}\mathscr{A}_2^2+\frac{1}{3}\mathscr{B}_2^2+\mathscr{C}_2^2-\frac{1}{2}\mathscr{A}_2\mathscr{B}_2-\mathscr{B}_2\mathscr{C}_2+\mathscr{A}_2\mathscr{C}_2\\
&+\frac{1}{3}\mathscr{A}_3^2+\frac{1}{3}\mathscr{B}_3^2+\mathscr{C}_3^2-\frac{1}{2}\mathscr{A}_3\mathscr{B}_3+\mathscr{B}_3\mathscr{C}_3-\mathscr{A}_3\mathscr{C}_3\\
&+\frac{1}{3}\mathscr{A}_4^2+\frac{1}{3}\mathscr{B}_4^2+\mathscr{C}_4^2+\frac{1}{2}\mathscr{A}_4\mathscr{B}_4-\mathscr{B}_4\mathscr{C}_4-\mathscr{A}_4\mathscr{C}_4\\
=&\frac{1}{3}a_1^2+\frac{1}{3}a_2^2+\frac{4}{3}a_3^2+\frac{4}{3}a_4^2+\frac{5}{12}a_5^2+\left(\frac{1}{3}a_1+\frac{1}{3}a_2-a_5+a_6\right)^2\\
&+\frac{1}{3}(a_1+a_2+3a_6)^2.
\end{aligned}
$$

On the other hand,

$$
\|\hat{w}\|_{0,\hat{M}}^2=\frac{16}{45}a_1^2+\frac{16}{45}a_2^2+\frac{4}{3}a_3^2+\frac{4}{3}a_4^2+\frac{4}{9}a_5^2+\frac{4}{9}(a_1+a_2+3a_6)^2,
$$

so that

$$
\|\hat{q}\|_{\hat{M}}^2 \le 2\left\|P_T^j\hat{q}\right\|_{\hat{M}}^2.
$$

4. Numerical Results

In this section, we apply the proposed post-processing method to the triangular elements $\mathbf{RT}_k$, $\mathbf{BDM}_k$, $\mathbf{HDG}_k$ and to the rectangular elements $\mathbf{RT}_{[k]}$, $\mathbf{BDM}_{[k]}$, $\mathbf{HDG}_{[k]}$. To this end, we consider the following second order elliptic equations:

$$
\begin{aligned}
&\mathbf{q}+\nabla u=0 && \text{in} \quad \Omega,\\
&\nabla\cdot\mathbf{q}=f && \text{in} \quad \Omega, \qquad\qquad (4.1)\\
&u=g && \text{on} \quad \partial\Omega,
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^2$ is a bounded polyhedral domain, $f\in L^2(\Omega)$ and $g\in H^{1/2}(\partial\Omega)$.

For simplicity, we follow the HDG framework of [12] to describe the finite element schemes considered. Let $\mathscr{T}_h:=\bigcup\{T\}$ be a conforming and shape regular partition of Ω, where each T is a polyhedral element. Denote by $\mathcal{F}_h:=\bigcup\{F\}$ the set of all edges/faces of all $T\in\mathscr{T}_h$, and let $\partial\mathscr{T}_h:=\{\partial T : T\in\mathscr{T}_h\}$. We consider the local finite dimensional spaces $\mathbf{V}(T)$, $W(T)$ and $\widetilde{W}(F)$ and set

$$
\begin{aligned}
\mathbf{V}_h &:= \left\{\mathbf{v}\in\mathbf{L}^2(\mathscr{T}_h):\mathbf{v}|_T\in\mathbf{V}(T) \quad \text{for any} \quad T\in\mathscr{T}_h\right\},\\
W_h &:= \left\{w\in L^2(\mathscr{T}_h):w|_T\in W(T) \quad \text{for any} \quad T\in\mathscr{T}_h\right\},\\
\widetilde{W}_h(g) &:= \Big\{\mu\in L^2(\mathscr{T}_h):\mu|_F\in\widetilde{W}(F),(\mu,\tilde{\mu})_{F\cap\partial\Omega}=(g,\tilde{\mu})_{F\cap\partial\Omega}\\
&\qquad \text{for any} \quad F\in\mathcal{F}_h,\forall\tilde{\mu}\in\widetilde{W}(F)\Big\}.
\end{aligned}
$$

Notice that

$$\widetilde{W}_h(0) = \left\{\mu \in L^2(\mathscr{T}_h) : \mu|_F \in \widetilde{W}(F) \quad \text{for all} \quad F \in \mathcal{E}_h \quad \text{and} \quad \mu|_{\partial\Omega} = 0\right\}.$$

The HDG method for the problem (4.1) consists in finding $(u_h, \mathbf{q}_h, \hat{u}_h) \in W_h \times \mathbf{V}_h \times \widetilde{W}_h(g)$ such that

$$(\mathbf{q}_h, \mathbf{v})_{\mathscr{T}_h} - (u_h, \nabla\cdot\mathbf{v})_{\mathscr{T}_h} + \langle \hat{u}_h, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\mathscr{T}_h} = 0 \qquad \text{for any} \quad \mathbf{v} \in \mathbf{V}_h, \tag{4.2a}$$

$$(w, \nabla\cdot\mathbf{q}_h)_{\mathscr{T}_h} + \langle \alpha(u_h - \hat{u}_h), w\rangle_{\partial\mathscr{T}_h} = (f, w)_{\partial\mathscr{T}_h} \qquad \text{for any} \quad w \in W_h, \tag{4.2b}$$

$$\langle \mathbf{q}_h\cdot\mathbf{n} + \alpha(u_h - \hat{u}_h), \mu\rangle_{\partial\mathscr{T}_h} = 0 \qquad \text{for any} \quad \mu \in \widetilde{W}_h(0), \tag{4.2c}$$

where

$$(\cdot,\cdot)_{\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h} (\cdot,\cdot)_T, \quad \langle\cdot,\cdot\rangle_{\partial\mathscr{T}_h} := \sum_{T\in\mathscr{T}_h} (\cdot,\cdot)_{\partial T}$$

and α is a nonnegative penalty function defined on $\partial\mathscr{T}_h$.

Within the framework (4.2), there are elements of six types — viz. the hybridised versions of $\mathbf{RT}_k$, $\mathbf{RT}_{[k]}$, $\mathbf{BDM}_k$, $\mathbf{BDM}_{[k]}$ and the hybridised discontinuous Galerkin elements $\mathbf{HDG}_k$, $\mathbf{HDG}_{[k]}$, which respectively correspond to the following choices of local spaces $\mathbf{V}(T)$, $W(T)$ and $\widetilde{W}(F)$ and penalty functions α:

- Hybridised $\mathbf{RT}_k$ triangular elements: $k \geq 0$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2 + \mathscr{P}_k(T)\boldsymbol{x}$, $W(T) = \mathscr{P}_k(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$ and $\alpha = 0$. Here and in the following $\boldsymbol{x} = (x, y)^t$.

- Hybridised $\mathbf{RT}_{[k]}$ rectangular elements: $k \geq 0$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2 + \mathscr{P}_k(T)\boldsymbol{x}$, $W(T) = \mathscr{Q}_k(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$ and $\alpha = 0$.

- Hybridised $\mathbf{BDM}_k$ triangular elements: $k \geq 2$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2$, $W(T) = \mathscr{P}_{k-1}(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$ and $\alpha = 0$.

- Hybridised $\mathbf{BDM}_{[k]}$ rectangular elements: $k \geq 2$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2 + \nabla\times(xyx^k) + \nabla\times(xyy^k)$, $W(T) = \mathscr{P}_{k-1}(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$ and $\alpha = 0$.

- $\mathbf{HDG}_k$ triangular elements: $k \geq 1$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2$, $W(T) = \mathscr{P}_k(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$ and $\alpha = 1/h_T$.

- $\mathbf{HDG}_{[k]}$ rectangular elements: $k \geq 1$, $\mathbf{V}(T) = \mathscr{P}_k(T)^2 + \nabla\times(xy\bar{\mathscr{P}}_k(T))$, $W(T) = \mathscr{P}_k(T)$, $\widetilde{W}(F) = \mathscr{P}_k(F)$, and $\alpha = 1/h_T$. Here $\bar{\mathscr{P}}_k(T)$ is the set of all homogeneous polynomials on T of the degree at most k.

We recall that the hybridised **RT** elements and hybridised **BDM** elements are equivalent to the corresponding **RT** and **BDM** mixed elements, respectively [1,8].

Let $\Omega = (0,1)\times(0,1)$ and f and g be functions such that the function

$$u = \sin(\pi x)\cdot\sin(\pi y)$$

is the solution of the model problem (4.1).

We compute the hybridised $\mathbf{RT}_k$ and $\mathbf{RT}_{[k]}$ elements for $k=0,1,2$, the hybridised $\mathbf{BDM}_k$ and $\mathbf{BDM}_{[k]}$ elements for $k=2,3$, and the $\mathbf{HDG}_k$ and $\mathbf{HDG}_{[k]}$ elements for $k=1,2$ on the $N\times N$ uniform meshes with $N=4,8,16,32$ — cf. Fig. 3.

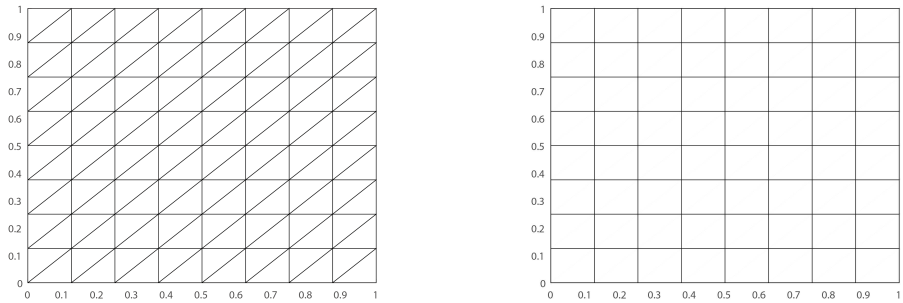

Figure 3: 8×8 uniform triangular and rectangular meshes.

Fig. 4 demonstrates the patch choice for an interior or a boundary node $z=\boldsymbol{x}_i$ with the corresponding M_i consisting of shadow elements. If $j=k$ and $j=k-1$, we choose M_i as in Figs. 4(a)-4(c) for triangular meshes and as in Figs. 4(g)-4(h) for rectangular meshes. Although it was recommended in Theorem 3.2 and Remark 3.2 to choose the largest j such that for a given k Condition 3.1 and the projection property (3.3) hold, a smaller j also works well in the post-processing method proposed. To show this, we also consider $\mathbf{RT}_k$ for $k=2, j=0$. Figs. 4(d)-4(e) show possible choices of M_i in this situation.

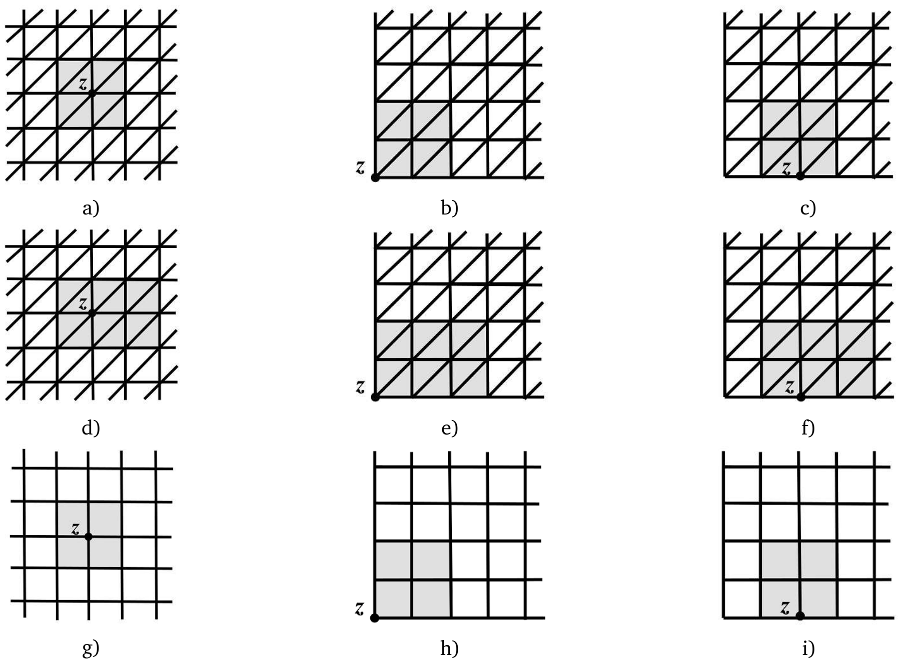

Figure 4: Element selection for M_i.

Tables 1-7 provide numerical relative errors $\|u-u_h\|_0$, $\|\widetilde{P}_h u-u_h\|_0$, and $\|u-u_h^*\|_0$ for the elements $\mathbf{RT}_k$, $\mathbf{RT}_{[k]}$, $\mathbf{BDM}_k$, $\mathbf{BDM}_{[k]}$, $\mathbf{HDG}_k$, and $\mathbf{HDG}_{[k]}$. In particular, we want to point out the following features:

- $\|\widetilde{P}_h u-u_h\|_0$ has the convergence order $k+2$ for all elements, which satisfy the superconvergence estimate (3.7).
- The corresponding post-processing solution u_h^* is of higher accuracy than the finite element solution u_h. More precisely, $\|u-u_h^*\|_0$ has the same convergence order $k+2$ as $\|\widetilde{P}_h u-u_h\|_0$, consistent with Theorem 3.1.

Table 1: Convergence history for $\mathbf{RT}_k$ triangular elements, $j=k$.

Degree k	Mesh	$\frac{\\|u-u_h\\|_0}{\\|u\\|_0}$		$\frac{\\|\widetilde{P}_h u-u_h\\|_0}{\\|u\\|_0}$		$\frac{\\|u-u_h^*\\|_0}{\\|u\\|_0}$	
		Error	Order	Error	Order	Error	Order
0	4×4	2.57E-01	-	1.66E-02	-	8.24E-02	-
	8×8	1.30E-01	0.98	4.50E-03	1.88	1.92E-02	2.10
	16×16	6.54E-02	0.99	1.18E-03	1.93	4.00E-03	2.26
	32×32	3.27E-02	1.00	2.85E-04	2.04	8.75E-04	2.19
1	4×4	3.91E-02	-	1.80E-03	-	2.08E-02	-
	8×8	9.90E-03	1.98	2.12E-04	3.08	2.40E-03	3.11
	16×16	2.50E-03	1.99	2.61E-05	3.02	3.19E-04	2.90
	32×32	6.21E-04	2.00	3.26E-06	3.00	3.77E-05	2.94
2	4×4	4.30E-03	-	7.01E-04	-	3.50E-03	-
	8×8	5.49E-04	2.97	4.53E-05	3.94	2.56E-04	3.77
	16×16	6.89E-05	2.99	2.87E-06	3.98	1.72E-05	3.90
	32×32	8.63E-06	3.00	1.85E-07	3.96	1.01E-06	3.95

Table 2: Convergence history for $\mathbf{RT}_k$ triangular elements, $j\le k-1$.

Degree k	Mesh	$\frac{\\|u-u_h\\|_0}{\\|u\\|_0}$		$\frac{\\|u-u_h^*\\|_0}{\\|u\\|_0}(j=0)$		$\frac{\\|u-u_h^*\\|_0}{\\|u\\|_0}(j=1)$	
		Error	Order	Error	Order	Error	Order
1	4×4	3.91E-02	-	4.27E-02	-	-	
	8×8	9.90E-03	1.98	5.81E-03	2.91	-	
	16×16	2.50E-03	1.99	7.48E-04	2.96	-	
	32×32	6.21E-04	2.00	9.56E-05	2.97	-	
2	4×4	4.30E-03	-	4.48E-03	-	3.80E-03	-
	8×8	5.49E-04	2.97	3.25E-04	3.78	2.70E-04	3.81
	16×16	6.89E-05	2.99	2.27E-05	3.84	1.83E-05	3.89
	32×32	8.63E-06	3.00	1.54E-06	3.88	1.18E-06	3.95

Table 3: Convergence history for $\mathbf{BDM}_k$ triangular elements, $j = k-1$.

Degree k	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$		$\frac{\|\widetilde{P}_h u-u_h\|_0}{\|u\|_0}$		$\frac{\|u-u_h^*\|_0}{\|u\|_0}$	
		Error	Order	Error	Order	Error	Order
2	4×4	3.91E-02	-	1.70E-03	-	1.44E-02	-
	8×8	9.90E-03	1.98	1.10E-04	3.95	8.52E-04	4.07
	16×16	2.50E-03	1.99	6.94E-06	3.98	4.00E-05	4.41
	32×32	6.21E-04	2.00	4.13E-07	4.07	1.92E-06	4.38
3	4×4	4.30E-03	-	4.68E-05	-	7.70E-03	-
	8×8	5.49E-04	2.97	1.67E-06	4.80	3.21E-04	4.58
	16×16	6.89E-05	2.99	5.57E-08	4.91	1.15E-05	4.81
	32×32	8.63E-06	3.00	1.79E-09	4.96	3.70E-07	4.95

Table 4: Convergence history for $\mathbf{HDG}_k$ triangular elements, $j = k-1$.

Degree k	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$		$\frac{\|\widetilde{P}_h u-u_h\|_0}{\|u\|_0}$		$\frac{\|u-u_h^*\|_0}{\|u\|_0}$	
		Error	Order	Error	Order	Error	Order
1	4×4	4.55E-02	-	2.30E-03	-	3.84E-02	-
	8×8	1.09E-02	2.06	3.10E-04	2.89	4.93E-03	2.96
	16×16	2.70E-03	2.01	3.86E-05	3.00	6.18E-04	2.99
	32×32	6.70E-04	2.01	4.83E-06	3.00	7.76E-05	3.00
2	4×4	5.39E-03	-	8.36E-04	-	4.76E-03	-
	8×8	6.81E-04	2.98	5.37E-05	3.96	3.12E-04	3.93
	16×16	8.60E-05	2.98	3.42E-06	3.97	2.02E-05	3.94
	32×32	1.08E-06	2.99	2.13E-07	4.00	1.24E-06	4.02

Table 5: Convergence history for $\mathbf{RT}_{[k]}$ rectangular elements, $j = k$.

Degree k	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$		$\frac{\|\widetilde{P}_h u-u_h\|_0}{\|u\|_0}$		$\frac{\|u-u_h^*\|_0}{\|u\|_0}$	
		Error	Order	Error	Order	Error	Order
0	4×4	3.17E-01	-	4.73E-02	-	1.06E-01	-
	8×8	1.59E-01	0.99	1.26E-02	1.90	1.79E-02	2.57
	16×16	8.01E-02	0.99	3.20E-03	1.97	3.90E-03	2.19
	32×32	4.01E-02	1.00	8.02E-04	1.99	9.57E-04	2.02
1	4×4	3.22E-02	-	5.33E-04	-	1.49E-02	-
	8×8	8.10E-03	1.99	5.61E-05	3.25	1.9E-03	2.97
	16×16	2.00E-03	2.02	6.30E-06	3.15	2.30E-04	3.05
	32×32	5.08E-04	1.98	7.60E-07	3.06	2.75E-05	3.06
2	4×4	2.10E-03	-	1.41E-05	-	1.70E-03	-
	8×8	2.69E-04	2.96	7.42E-07	4.24	9.68E-05	4.13
	16×16	3.41E-05	2.98	4.12E-08	4.17	5.98E-06	4.02
	32×32	4.20E-06	3.02	2.50E-9	4.04	3.74E-07	4.00

Table 6: Convergence history for $\mathbf{BDM}_{[k]}$ rectangular elements, $j = k-1$.

Degree k	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$		$\frac{\|\widetilde{P}_h u-u_h\|_0}{\|u\|_0}$		$\frac{\|u-u_h^*\|_0}{\|u\|_0}$	
		Error	Order	Error	Order	Error	Order
2	4×4	5.94E-02	-	1.70E-03	-	6.30E-03	-
	8×8	1.51E-02	1.97	1.10E-04	3.95	4.32E-04	3.86
	16×16	3.80E-03	1.99	6.94E-06	3.98	2.57E-05	4.07
	32×32	9.50E-04	2.00	4.13E-07	4.07	1.53E-06	4.07
3	4×4	7.50E-03	-	6.68E-05	-	1.30E-03	-
	8×8	9.55E-04	2.97	1.85E-06	5.16	2.88E-05	5.49
	16×16	1.20E-04	2.99	5.14E-08	5.17	7.45E-07	5.27
	32×32	1.50E-05	3.00	1.56E-09	5.04	2.33E-08	5.00

Table 7: Convergence history for $\mathbf{HDG}_{[k]}$ rectangular elements, $j = k-1$.

Degree k	Mesh	$\frac{\|u-u_h\|_0}{\|u\|_0}$		$\frac{\|\widetilde{P}_h u-u_h\|_0}{\|u\|_0}$		$\frac{\|u-u_h^*\|_0}{\|u\|_0}$	
		Error	Order	Error	Order	Error	Order
1	4×4	6.29E-02	-	2.52E-03	-	3.94E-02	-
	8×8	1.59E-02	1.98	3.23E-04	2.96	5.03E-03	2.97
	16×16	4.01E-03	1.99	4.08E-05	2.98	6.31E-04	2.99
	32×32	1.01E-04	1.99	5.13E-06	2.99	7.89E-05	3.00
2	4×4	8.67E-03	-	9.53E-04	-	6.76E-03	-
	8×8	1.10E-03	2.97	6.25E-05	3.93	4.43E-04	3.93
	16×16	1.39E-04	2.99	3.92E-06	3.99	2.83E-05	3.97
	32×32	1.74E-05	2.99	2.45E-07	4.00	1.78E-06	3.99

References

[1] D.N. Arnold, F. Brezzi., *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, ESAIM Math. Model. Numer. Anal. **19(1)**, 7-32 (1985).

[2] D.N. Arnold, F. Brezzi and J. Douglas, *PEERS: A new mixed finite element for plane elasticity*, Japan J. Appl.Math. **1**, 347-367 (1984).

[3] D.N. Arnold and R.S. Falk, *A new mixed formulation for elasticity*, Numer. Math. **53**, 13-30 (1988).

[4] I. Babuska and A. Miller, *The post-processing approach in the finite element method. Part 1. Calculation of displacements, stresses, and other higher derivatives of the displacements*, Internat. J. Numer. Methods Engrg. **20**, 1085-1109 (2010).

[5] I. Babuska and A. Miller, *The post-processing approach in the finite element method. Part 2: The calculation of stress intensity factors*, Internat. J. Numer. Methods Engrg. **20**, 1111-1129 (2010).

[6] I. Babuska and A. Miller, *The post-processing approach in the finite element method. Part 3: A posteriori error estimates and adaptive mesh selection*, Internat. J. Numer. Methods Engrg. **20**, 2311-2324 (2010).

[7] J. H. Bramble and J. Xu, *A local post-processing technique for improving the accuracy in mixed finite-element approximations*, SIAM J. Numer. Anal. **26**, 1267-1275 (1989).

[8] F. Brezzi, J. Douglas and L.D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. **47**, 19-34 (1985).

[9] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Series in Computational Mathematics **15**, Springer-Verlag (1991).

[10] C. Chen and Y. Huang, *High accuracy theory of finite element methods (in Chinese)*, Hunan Science Press (1995).

[11] G. Chen and X. Xie., *A robust Weak Galerkin finite element method for linear elasticity with strong symmetric stresses*, Comput. Methods Appl. Math **16**, 389-408 (2016).

[12] B. Cockburn, W. Qiu and K. Shi, *Conditions for superconvergence of HDG methods for second-order elliptic problems*, Math. Comp. **81**, 1327-1353 (2012).

[13] B. Cockburn and K. Shi, *Conditions for superconvergence of HDG methods for Stokes flow*, Math. Comp. **82(282)**, 651-671 (2012).

[14] B. Li and Z. Zhang, *Analysis of a class of superconvergence patch recovery techniques for linear and bilinear finite elements*, Numer. Methods Partial Differential Equations **15**, 151-167 (1999).

[15] Q. Lin and N. Yan, *Construction and analysis of hign efficient finite elements (in Chinese)*, Hebei University Press (1996).

[16] Z.C. Shi and M. Wang, *Finite element methods*, Science Press (2013).

[17] R. Stenberg, *A family of mixed finite elements for the elasticity problem*, Numer. Math. **53**, 513-538 (1988).

[18] R. Stenberg, *Postprocessing schemes for some mixed finite elements*, ESAIM Math. Model. Numer. Anal. **25**, 151-167 (1991).

[19] J. Xu and Z. Zhang, *Analysis of recovery type a posteriori error estimators for mildly structured grids*, Math. Comp. **73**, 1139-1152 (2003).

[20] Z. Zhang and A. Naga, *A new finite element gradient recovery method: superconvergence property*, SIAM J. Sci. Comput. **26**, 1192-1213 (2005).

[21] Z. Zhang, *Ultraconvergence of the patch recovery technique II*, Math. Comp. **69**, 141-158 (2000).

[22] O.C. Zienkiewicz and J. Zhu, *The superconvergence patch recovery and a posteriori error estimates. Part 1: The recovery technique*, Internat. J. Numer. Methods Engrg. **33**, 1331-1364 (1992).

[23] O.C. Zienkiewicz and J. Zhu, *The superconvergence patch recovery and a posteriori error estmates. Part 2: Error estmates and adaptivity*, Internat. J. Numer. Methods Engrg. **33**, 1365-1382 (1992).