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Two-Grid Discretization Scheme for Nonlinear Reaction Diffusion Equation by Mixed Finite Element Methods

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Abstract. In this paper, we study an efficient scheme for nonlinear reaction-diffusion equations discretized by mixed finite element methods. We mainly concern the case when pressure coefficients and source terms are nonlinear. To linearize the nonlinear mixed equations, we use the two-grid algorithm. We first solve the nonlinear equations on the coarse grid, then, on the fine mesh, we solve a linearized problem using Newton iteration once. It is shown that the algorithm can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H=\mathcal{O}(h^{\frac{1}{2}})$. As a result, solving such a large class of nonlinear equations will not be much more difficult than getting solutions of one linearized system.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Two-grid method, reaction-diffusion equations, mixed finite element methods.

1 Introduction

In this paper, we study the following nonlinear reaction-diffusion equations:

$$c(p)\frac{\partial p}{\partial t}-\nabla\cdot(K\nabla p)=f(p), \qquad (x,t)\in\Omega\times J, \tag{1.1}$$

with initial condition

$$p(x,0)=p^0(x), \qquad x\in\Omega, \tag{1.2}$$

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and boundary condition

$$K\nabla p\cdot\boldsymbol{\nu}=0, \qquad (x,t)\in\partial\Omega\times J, \tag{1.3}$$

where $\Omega\in\mathbb{R}^2$ is a bounded and convex domain with C^1 boundary $\partial\Omega$, $\boldsymbol{\nu}$ is the unit exterior normal direction to $\partial\Omega$, $J=(0,T]$, K: $\Omega\times\mathbb{R}\to\mathbb{R}^{2\times 2}$ is a symmetric positive definite tensor. (1.1) can be rewritten as followings

$$c(p)\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=f(p), \tag{1.4}$$

$$K^{-1}\boldsymbol{u}+\nabla p=0. \tag{1.5}$$

Two-grid method is based on the fact that the non-symmetry, indefiniteness and nonlinearity behaving like low frequencies are governed by the coarse grid and the related high frequencies are governed by some linear symmetric positive-definition operators. It was first proposed by Xu [18, 19] as a discretized method for non-symmetric, indefinite and nonlinear partial differential equations. The basic procedure of the two-grid method is to solve a complicated problem (non-symmetric indefinite or nonlinear, etc.) on the coarse grid and then solve a simple symmetric positive or linearized problem on the fine mesh. Because of its simplicity and efficiency, there are lots of investigation of two-grid method for different types of equations in the past few decades. For instance, Chen and Huang studied a multilevel iterative method for solving the finite element solutions of nonlinear singular two-point boundary value problems [4]. Xu and Zhou discussed the algorithm for eigenvalue problem [20]. Zhong [21] analyzed it for Maxwell equations and Layton [12] concerned the scheme for MHD system.

Reaction-diffusion equations have received a great deal of attention motivated by their widespread occurrence in models of hydrologic, biology and bio-geochemical phenomena [11, 13]. Classic examples include the modeling of groundwater through porous media [7]. In this case, p denotes the fluid pressure, $\boldsymbol{u}$ is the Darcy velocity of the flow and $f(p)$ models the external flow rate. Here, for brevity, we drop the dependence of variable x in $f(x,p)$.

Mixed finite element methods have been found to be very important for solving parabolic partial differential equations [10, 14]. For example, there are many applications of mixed finite element methods to miscible displacement problems that describe two-phase flow in petroleum reservoir [7]. Mixed methods have played a fundamental role in discretizing fluid dynamic problems since both the pressure and the flux, or displacements and stresses, are approximated simultaneously.

For nonlinear parabolic equations, two-grid methods were first applied to mixed finite element method by Dawson and Wheeler with f dependent on p, ∇p [8]. Later, they concerned the equations with nonlinear diffusion coefficients by two-grid difference method [9]. Moreover, Wu and Allen [17] established and analyzed a two-step two-grid algorithm for nonlinear reaction-diffusion equations discretized by expanded mixed finite element method. Based on these work, we proposed a three-step two-grid method for semi-linear parabolic problems [5] using the defect correction rationale in literature [18]. Furthermore, the continued work [6] designed a three-step and a four-step two grid algorithms for semi-linear reaction-diffusion equations in 2D. In both of the literatures, we employ the expanded mixed finite element method which was first developed by Arbogast [1]. Later on, we studied a two-grid algorithm for reaction diffusion equations with nonlinear diffusion coefficients $K(p)$ by mixed finite element discretization. We constructed a two-step two-grid algorithm and confirm its convergence both theoretically and numerically [3].

In this paper, we investigate reaction-diffusion equations with nonlinear pressure coefficients and nonlinear source term by mixed finite element methods. This is the first time that two-grid method is designed for this kind of equation based on canonical mixed finite element methods. The main algorithm in this literature involves two steps: first, solving a small scaled nonlinear system on the coarse grid, then, on the fine mesh, getting a solution of a large linear system using one Newton iteration. By utilizing two important super-convergence properties, we proved the error estimates for the two-grid algorithm theoretically. We also gave a simple numerical example to verify the efficiency of the algorithm. It is shown that the coarse grid can be coarser and the algorithm still achieves asymptotically optimal approximation as long as the mesh sizes satisfy $H=\mathcal{O}(h^{\frac{1}{2}})$.

The remainder of the paper is organized as follows: In Section 2, we introduce some notations and projection operators. The mixed formula, finite element discretization and relevant priori error estimation will be studied in Section 3. The main algorithm and its convergence analysis will be discussed in Section 4. Section 5 is devoted to the presentation of a numerical example to show the effectiveness of the proposed method.

2 Some notation and projection operators

In this paper, we assume that function $f:\bar{\Omega}\times\mathbb{R}\rightarrow\mathbb{R}$ is a triple continuously differentiable function with bounded derivatives up to the third order. $\kappa\triangleq K^{-1}$ is a square-integrable, symmetric, uniformly positive-definite tensor defined on Ω. Assume the function $c(p)$ has continuous third order derivatives with respect to t and x and bounded up and below by positive constants, namely, there exists positive constants C_* and C^*, such that

$$C_*\leq c(p)\leq C^*, \qquad \forall\, p\in L^2(\Omega).$$

Let $L^2(\Omega)$ be the set of square-integrable functions on Ω and $(L^2(\Omega))^2$, the space of vectors functions which have all components in $L^2(\Omega)$, with usual norm $\|\cdot\|^2$. Furthermore, let $(\cdot,\cdot)$ denote the L^2 inner product, scalar and vector, and $(\cdot,\cdot)_{\partial\Omega}$ present the $L^2(\partial\Omega)$ inner product with norm $\|\cdot\|_{\partial\Omega}$.

We shall also use the canonical Sobolev space $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$ given by $\|\phi\|^p_{m,p}=\sum_{|\alpha|\leq m}\|D^\alpha\phi\|^p_{L^p(\Omega)}$. For $p=2$, we define $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$, and $\|\cdot\|_{0,\infty}=\|\cdot\|_{L^\infty}$.

For some integer $k\geq 0$, we assume that the solution $(p,\boldsymbol{u})$ of (1.4)-(1.5) has the following regularity:

$$p\in L^2(J;W^{k+2,4}(\Omega))\cap L^2(J;W^{k+1,\infty}(\Omega)),\quad \boldsymbol{u}\in (L^2(J;W^{k+1,4}(\Omega)))^2. \tag{2.1}$$

We also assume

$$\|p_t\|_{L^\infty(J;L^\infty)}\leq K_1,\quad \Big\|\frac{\partial c}{\partial p}\Big\|_{L^\infty(J;L^\infty)}\leq K_2, \tag{2.2}$$

$$\|p_{tt}\|_{L^\infty(J;L^2)},\qquad \Big\|\frac{\partial^2 c}{\partial p^2}\Big\|_{L^\infty(J;L^2)}\leq K_3, \tag{2.3}$$

where K_1,K_2,K_3 are the positive constants.

To analyze the discretization on a time interval (0,T), we first introduce some notations. Let $N>0$ be some integer, $\triangle t=T/N$, $t^n=n\triangle t$. Denote

$$\begin{aligned}
&\phi^n=\phi(\cdot,t^n),\quad \partial_t\phi^n=\frac{\phi^n-\phi^{n-1}}{\triangle t},\\
&\|\phi\|_{l^2((0,T);X)}=\left(\sum_{n=1}^N \triangle t\|\phi^n\|_X^2\right)^{\frac{1}{2}},\\
&\|\phi\|_{l^\infty((0,T);X)}=\max_{1\leq n\leq N}\|\phi^n\|_X,\\
&\|\phi\|_{L^2((0,T);X)}=\left(\int_0^T\|\phi(\cdot,t)\|_X^2dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Let $H(\mathrm{div},\Omega)$ be the space of vector functions in $(L^2(\Omega))^2$ which have divergence in $L^2(\Omega)$ with norm $\|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)}\equiv(\|\boldsymbol{u}\|^2+\|\nabla\cdot\boldsymbol{u}\|^2)^{\frac{1}{2}}$.

Finally, denote

$$\boldsymbol{V}=H(\mathrm{div},\Omega)\cap\{\boldsymbol{v}\cdot\boldsymbol{\nu}=0\},\quad W=L^2(\Omega).$$

Let $\mathcal{T}_h$ be a quasi-uniform partition of Ω into rectangles or triangles with mesh size h. Suppose $\boldsymbol{V}_h$ and W_h are discrete subspaces of $\boldsymbol{V}$ and W, using standard mixed finite element space such as Raviart-Thomas spaces with order k,RT_k [15], or Brazzi-Douglas-Marini [2] spaces BDM_k. Then, the following inclusion property holds for RT_k or BDM_k spaces,

$$\nabla\cdot\boldsymbol{v}_h\in W_h,\quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h.$$

Let Q_h denote the L^2 projection defined by

$$(Q_h\phi,w_h)=(\phi,w_h),\quad \forall w_h\in W_h, \tag{2.4}$$

for any $\phi\in L^2(\Omega)$, and $\boldsymbol{Q}_h$ as vector L^2 projection operator.

$$(\boldsymbol{Q}_h\boldsymbol{\phi},\boldsymbol{v}_h)=(\boldsymbol{\phi},\boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.5}$$

for any vector valued function $\boldsymbol{\phi}\in(L^2(\Omega))^2$.

In the following, assume $1<q\leq\infty$. The L^2 projection operator has the following stability and approximation properties [2]: for $\phi\in W^{k+1,q}(\Omega)(or\ \boldsymbol{\phi}\in(W^{k+1,q}(\Omega))^2)$,

$$\|Q_h\phi\|_{0,q}\leq C\|\phi\|_{0,q},\qquad 2\leq q<\infty, \tag{2.6}$$

$$\|\phi-Q_h\phi\|_{0,q}\leq Ch^r\|\phi\|_{r,q},\quad 0\leq r\leq k+1. \tag{2.7}$$

Associated with the standard mixed finite element spaces, we define the Fortin interpolation $\Pi_h:H(\text{div},\Omega)\to\boldsymbol{V}_h$ such that for $\mathbf{q}\in H(\text{div},\Omega)$,

$$(\nabla\cdot\Pi_h\boldsymbol{q},w_h)=(\nabla\cdot\boldsymbol{q},w_h),\quad \forall w_h\in W_h. \tag{2.8}$$

The following approximate properties ([2]) hold for projection Π_h:

$$\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|_{0,q}\leq Ch^r\|\boldsymbol{q}\|_{r,q},\qquad \frac{1}{q}<r\leq k+1, \tag{2.9}$$

$$\|\nabla\cdot(\boldsymbol{q}-\Pi_h\boldsymbol{q})\|_{0,q}\leq Ch^r\|\nabla\cdot\boldsymbol{q}\|_{r,q},\quad 0\leq r\leq k+1. \tag{2.10}$$

We also assume that

$$\|\boldsymbol{v}_h\|_\infty\leq Ch^{-1}\|\boldsymbol{v}_h\|,\quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{2.11}$$

Now, we recall the discrete Gronwall Lemma (see, e.g., [16]):

Lemma 2.1. *Assume that k_n is a non-negative sequence, and that the sequence ϕ_n satisfies*

$$\begin{cases}\varphi_0\leq g_0,\\ \varphi_n\leq g_0+\displaystyle\sum_{s=0}^{n-1}p_s+\sum_{s=0}^{n-1}k_s\varphi_s,\quad n\geq 1.\end{cases} \tag{2.12}$$

Then φ_n satisfies

$$\begin{cases}\varphi_1\leq g_0(1+k_0)+p_0,\\ \varphi_n\leq g_0\displaystyle\prod_{s=0}^{n-1}(1+k_s)+\sum_{s=0}^{n-2}p_s\prod_{r=s+1}^{n-1}(1+k_r)+p_{n-1},\quad n\geq 2.\end{cases} \tag{2.13}$$

Moreover, if $g_0\geq 0$ and $p_n\geq 0$ for $n\geq 0$, it follows

$$\varphi_n\leq\left(g_0+\sum_{s=0}^{n-1}p_s\right)\exp\left(\sum_{s=0}^{n-1}k_s\right),\quad n\geq 1. \tag{2.14}$$

3 Mixed finite element discretization

Fix two variables: the pressure p and the flux $\boldsymbol{u}=-K\nabla p$. Then, we have relations $\kappa\boldsymbol{u}=-\nabla p$ and $\nabla\cdot\boldsymbol{u}=-\nabla\cdot(K\nabla p)$.

The variational formula of the nonlinear parabolic equations (1.1)-(1.3) is to find $(p,\boldsymbol{u})\in W\times\boldsymbol{V}$ such that

$$\left(c(p)\frac{\partial p}{\partial t},w\right)+(\nabla\cdot\boldsymbol{u},w)=(f(p),w),\quad \forall w\in W, \tag{3.1}$$

$$(\kappa\boldsymbol{u},\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p)=0,\qquad \forall\boldsymbol{v}\in\boldsymbol{V}. \tag{3.2}$$

Full discretization of (3.1)-(3.2) can be defined as: find $(p_h^n,\boldsymbol{u}_h^n)\in W_h\times\boldsymbol{V}_h$ $(1\le n\le N)$ such that

$$\left(c(p_h^n)\frac{p_h^n-p_h^{n-1}}{\triangle t},w_h\right)+(\nabla\cdot\boldsymbol{u}_h^n,w_h)=(f(p_h^n),w_h),\quad \forall w_h\in W_h, \tag{3.3}$$

$$(\kappa\boldsymbol{u}_h^n,\boldsymbol{v}_h)-(\nabla\cdot\boldsymbol{v}_h,p_h^n)=0,\qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.4}$$

In order to derive error estimations, we need to introduce new auxiliary mixed-method projection $(R_hp,R_h\boldsymbol{u}):W\times V\to W_h\times\boldsymbol{V}_h$ satisfying

$$(\nabla\cdot(\boldsymbol{u}-R_h\boldsymbol{u}),w_h)=0,\qquad \forall w_h\in W_h, \tag{3.5}$$

$$(\kappa(\boldsymbol{u}-R_h\boldsymbol{u}),\boldsymbol{v}_h)-(\nabla\cdot\boldsymbol{v}_h,p-R_hp)=0,\quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.6}$$

Set

$$d=Q_hp-R_hp,\quad \alpha=p-R_hp,\quad \beta=\boldsymbol{u}-R_h\boldsymbol{u}.$$

Notice that both Q_hp,R_hp are in space W_h. There might be some nice property hidden in d. We will prove in the following that d has superconvergence property. In order to prove this, we first need to present some obvious results.

Lemma 3.1. *For $t\in J$ and h sufficiently small,*

$$\|\boldsymbol{u}-R_h\boldsymbol{u}\|\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}, \tag{3.7}$$

$$\|\nabla\cdot(\boldsymbol{u}-R_h\boldsymbol{u})\|\le Ch^{k+1}\|\nabla\cdot\boldsymbol{u}\|_{k+1}. \tag{3.8}$$

Proof. The Lemma can be easily derived from Eq. (3.5) and the Fortin interpolation properties (2.9) and (2.10). □

Using Lemma 3.1, the superconvergence property of d can be easily proved.

Lemma 3.2.

$$\|d\|\le Ch(\|\beta\|+h\|\nabla\cdot\beta\|). \tag{3.9}$$

Proof. From the mixed-method projection (3.5), (3.6) and the definition of L^2 projection, we get that

$$\begin{aligned}
&(\nabla\cdot\beta,w_h)=0, && \forall w_h\in W_h,\\
&(\kappa\beta,\boldsymbol{v}_h)-(d,\nabla\cdot\boldsymbol{v}_h)=0, && \forall \boldsymbol{v}_h\in V_h.
\end{aligned}$$

Let $\xi\in L^q(\Omega)$, where $1/p+1/q=1$, and $\phi\in W_0^{2,q}(\Omega)$ satisfying homogeneous Dirichlet boundary problem

$$-\nabla\cdot(K\nabla\phi)=\xi. \tag{3.10}$$

Suppose the regularity assumption holds, namely

$$\|\phi\|_{2,q}\le C\|\xi\|_{0,q}.$$

Then

$$\begin{aligned}
(d,\xi)&=(d,-\nabla\cdot(K\nabla\phi))=(d,-\nabla\cdot\Pi_h(K\nabla\phi))\\
&=-(\kappa\beta,K\nabla\phi)+(\kappa\beta,K\nabla\phi-\Pi_h(K\nabla\phi))\\
&=(\nabla\cdot\beta,\phi-Q_h\phi)+(\kappa\beta,K\nabla\phi-\Pi_h(K\nabla\phi)),
\end{aligned}$$

we have

$$\begin{aligned}
|(d,\xi)|&\le C(h^2\|\nabla\cdot\beta\|\|\phi\|_2+h\|\beta\|_0\|\phi\|_2)\\
&\le Ch(h\|\nabla\cdot\beta\|+\|\beta\|)\|\xi\|,
\end{aligned}$$

which proves (3.9). □

Remark 3.1. Utilizing similar argument above, we can also derive the error estimation for time derivative of d and α

$$(Q_hp-R_hp)_t \quad \text{and} \quad (p-R_hp)_t,$$

by differentiating equations (3.5) and (3.6) with respect to t.

Therefore, summarize the results derived in Lemmas 3.1 and 3.2, we obtain the following results.

Lemma 3.3. *Let $(p,\boldsymbol{u})\in W\times \boldsymbol{V}$ be solution of the differential problem (3.1)-(3.2) and $(R_hp,R_h\boldsymbol{u})\in W_h\times \boldsymbol{V}_h$ be mixed-method projection of the solutions. Then, for $1\le r\le k+1$,*

$$\|Q_hp-R_hp\|+\|(Q_hp-R_hp)_t\|\le Ch^{r+1}\|p\|_{r+1}, \tag{3.11}$$
$$\|p-R_hp\|+\|(p-R_hp)_t\|\le Ch^{k+1}\|p\|_{k+1}. \tag{3.12}$$

Next, we would obtain another superconvergence result between the full discrete solution and the mixed-method projection $(R_h p, R_h \boldsymbol{u})$.

We first need to introduce an useful result which plays an important role in the following proof.

Lemma 3.4. *Suppose g is a piecewise smooth function on the partition $\mathcal{T}_h$. If $\bar{g}(p)$ is the average of $g(p)$ on each element $\tau(\tau\in\mathcal{T}_h)$ of the $\mathcal{T}_h$ and $\|\nabla g\|_{0,\infty}\le K$, then*

$$|(g(p)\theta,\psi)-(\bar{g}(p)\theta,\psi)|\le CKh\|\theta\|_0\|\psi\|_0, \tag{3.13}$$

where θ,ψ are some given nice functions.

Now it is time to present one of the main results of this literature.

Lemma 3.5. *Let $(p_h^n,\boldsymbol{u}_h^n)\in W_h\times \boldsymbol{V}_h$ be the solution of the mixed finite element equations* (3.3)-(3.4)*. $(R_h p^n, R_h\boldsymbol{u}^n)$ are the mixed-method projection of the solution for $n\ge 1$. If regularity assumption* (2.1) *holds, time step size $\triangle t< C_*(2\|f\|_{1,\infty}+2K_1K_2)^{-1}$ and the initial function satisfies*

$$R_h p^0 = p_h^0, \tag{3.14}$$

then, for $1\le m\le N$, we have

$$\|R_h p^m - p_h^m\| + \|\kappa^{\frac{1}{2}}(R_h\boldsymbol{u}-\boldsymbol{u}_h)\|_{l^2(0,t^m;L^2)}\le C(h^{k+2}+\triangle t). \tag{3.15}$$

Proof. At time $t=t^n$, subtract (3.3)-(3.4) from (3.5)-(3.6) and let $\xi^n = R_h p^n - p_h^n, \eta^n = R_h\boldsymbol{u}^n - \boldsymbol{u}_h^n$, then, we get

$$(c(p_h^n)\partial_t\xi^n, w_h)+(\nabla\cdot\eta^n, w_h) = (F, w_h), \tag{3.16}$$
$$(\kappa\eta^n, \boldsymbol{v}_h)-(\xi^n,\nabla\cdot\boldsymbol{v}_h)=0, \tag{3.17}$$

where

$$F = f(p^n)-f(p_h^n)+c(p_h^n)\partial_t R_h p^n - c(p^n)p_t^n.$$

Choose $w_h=\xi^n$, $\boldsymbol{v}_h=\eta^n$ in (3.16) and (3.17) respectively, then add (3.16), (3.17) to derive

$$(c(p_h^n)\partial_t\xi^n,\xi^n)+(\kappa\eta^n,\eta^n)=(F,\xi^n). \tag{3.18}$$

First, we bound the right-hand side of (3.18). Because of the smoothness of $f(p)$ and the superconvergence property in Lemma 3.3, we have

$$f(p^n)-f(p_h^n) = f(p^n)-f(Q_h p^n)+f(Q_h p^n)-f(R_h p^n)+f(R_h p^n)-f(p_h^n), \tag{3.19}$$

$$\begin{aligned}|(f(Q_h p^n)-f(R_h p^n),\xi^n)| &\le \|f\|_{1,\infty}\|Q_h p^n - R_h p^n\|\|\xi^n\| \\ &\le Ch^{2k+4}+\epsilon\|\xi^n\|^2, \end{aligned}\tag{3.20}$$

$$|(f(R_h p^n)-f(p_h^n),\xi^n)|\le \|f\|_{1,\infty}\|\xi^n\|^2, \tag{3.21}$$

$$
\begin{aligned}
|(f(p^n)-f(Q_hp^n),\xi^n)|\le&|(f_p(p^n)(Q_hp^n-p^n),\xi^n)|\\
&+\left|\left(\frac{\|f\|_{2,\infty}}{2}(p^n-Q_hp^n)^2,|\xi^n|\right)\right|.
\end{aligned}
\tag{3.22}
$$

Setting $\theta^n=p^n-Q_hp^n$, by employing Lemma 3.4 with $g(p)=f_p(p^n)$, definition of projection (2.4) and its approximation properties (2.7), we have

$$
\begin{aligned}
|(f(p^n)-f(Q_hp^n),\xi^n)|\le&Ch\|f\|_{1,\infty}\|\theta^n\|\|\xi^n\|+C\|f\|_{2,\infty}\|\theta^n\|^2_{0,4}\|\xi^n\|\\
\le&Ch^{2k+4}+\epsilon\|\xi^n\|^2.
\end{aligned}
\tag{3.23}
$$

Hence, from (3.19)-(3.23), we conclude that

$$
|(f(p^n)-f(p_h^n),\xi^n)|\le Ch^{2k+4}+(2\epsilon+\|f\|_{1,\infty})\|\xi^n\|^2. \tag{3.24}
$$

Notice that

$$
\begin{aligned}
c(p_h^n)\partial_tR_hp^n-c(p^n)p_t^n=&c(p_h^n)\partial_t(R_hp^n-Q_hp^n)+c(p_h^n)\partial_t(Q_hp^n-p^n)\\
&+c(p_h^n)(\partial_tp^n-p_t^n)+(c(p_h^n)-c(p^n))p_t^n\\
=&T_1+T_2+T_3+T_4.
\end{aligned}
\tag{3.25}
$$

With Lemma 3.3 and the fact

$$
\|\partial_t(R_hp^n-Q_hp^n)\|_{l^2(0,T;L^2)}\le C\|[(R_hp-Q_hp)_t]^n\|_{l^2(0,T;L^2)},
$$

we can derive

$$
|(T_1,\xi^n)|\le Ch^{2k+4}+\epsilon\|\xi^n\|^2. \tag{3.26}
$$

For T_2, using Lemma 3.4 and property (2.7) to get:

$$
|(T_2,\xi^n)|\le Ch^{2k+4}+\epsilon\|\xi^n\|^2. \tag{3.27}
$$

About T_3, we have the following estimation

$$
\begin{aligned}
|(T_3,\xi^n)|=&\left|\left(\frac{1}{\triangle t}\int_{t^{n-1}}^{t^n}(s-t^{n-1})p_{ss}(\cdot,s)ds,\xi^n\right)\right|\\
\le&C\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2dt+\epsilon\|\xi^n\|^2.
\end{aligned}
\tag{3.28}
$$

For T_4, we can use similar argument as $f(p^n)-f(p_h^n)$ and derive the estimation as

$$
|(T_4,\xi^n)|\le Ch^{2k+4}+(2\epsilon+K_1K_2)\|\xi^n\|^2. \tag{3.29}
$$

Summarizing relations (3.26)-(3.29), we have

$$
|(c(p_h^n)\partial_tR_hp^n-c(p^n)p_t,\xi^n)|\le C\left(h^{2k+4}+\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2ds\right)+K_1K_2\|\xi\|^2+5\gamma\|\xi^n\|^2. \tag{3.30}
$$

Thus, the right hand side for (3.16) is

$$|(F,\xi^n)|\leq C\left(h^{2k+4}+\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2ds\right)+(\|f\|_{1,\infty}+K_1K_2)\|\xi^n\|^2+7\gamma\|\xi^n\|^2. \tag{3.31}$$

While for the left hand side of (3.16), it is easy to obtain that

$$|(c(p_h^n)\partial_t\xi^n,\xi^n)|\geq C_*\frac{1}{2\triangle t}(\|\xi^n\|^2-\|\xi^{n-1}\|^2), \tag{3.32}$$

$$|(\kappa\eta^n,\eta^n)|=\|\kappa^{\frac{1}{2}}\eta^n\|^2. \tag{3.33}$$

From (3.31), (3.32) and (3.33), choose γ sufficiently small, then, we have

$$\begin{aligned}&C_*\frac{1}{2\triangle t}(\|\xi^n\|^2-\|\xi^{n-1}\|^2)+\|\kappa^{\frac{1}{2}}\eta^n\|^2\\ \leq&C\left(h^{2k+4}+\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2dt\right)+(\|f\|_{1,\infty}+K_1K_2)\|\xi^n\|^2.\end{aligned}$$

Multiply $2\triangle t$ on both side of the last inequality and sum over from $n=1$ to m. Choosing $\triangle t<C_*(2\|f\|_{1,\infty}+2K_1K_2)^{-1}$ and noticing the assumption $R_hp^0=p_h^0$, we use the discrete Gronwall inequality to get the result

$$\|\xi^m\|^2+\|\kappa^{\frac{1}{2}}\eta\|^2_{l^2(0,t^m;L^2)}\leq C(h^{2k+4}+(\triangle t)^2).$$

The proof is complete. □

Now, it is time to demonstrate one of the main results of this literature.

Theorem 3.1. *Let* $(p^n,\boldsymbol{u}^n)$ *be the solution to* (3.1) *and* (3.2) *at* $t=t^n$. $(p_h^n,\boldsymbol{u}_h^n)$ *be the mixed finite element solution. If we choose*

$$R_hp^0=p_h^0,\quad \triangle t<C_*(2\|f\|_{1,\infty}+2K_1K_2)^{-1}, \tag{3.34}$$

then, for $1\leq m\leq N$ *we have*

$$\|p^m-p_h^m\|+\|\kappa^{\frac{1}{2}}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{l^2(0,t^m;L^2)}\leq C(h^{k+1}+\triangle t). \tag{3.35}$$

Proof. From Lemma 3.3, 3.5, It is easy to derive the conclusion by applying triangle inequality. □

4 Two-grid method and its convergence analysis

In this section, we will construct the main algorithm of this paper. The fundamental ingredient of this method is another mixed finite element space $\boldsymbol{V}_H\times W_H(\subset\boldsymbol{V}_h\times W_h)$ defined on the related coarse mesh. This scheme has two steps:

Step 1: On the coarse grid, compute $(p_H^n, u_H^n) \in W_H \times V_H$ to satisfy the nonlinear equations

$$
\begin{aligned}
&(c(p_H^n)\partial_t p_H^n, w_H) + (\nabla \cdot \boldsymbol{u}_H^n, w_H) = (f(p_H^n), w_H), \quad && w_H \in W_H, && (4.1)\\
&(\kappa \boldsymbol{u}_H^n, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, p_H^n) = 0, && \boldsymbol{v}_H \in \boldsymbol{V}_H. && (4.2)
\end{aligned}
$$

Step 2: On the fine grid, solve the linearized systems to get $(P_h^n, \boldsymbol{U}_h^n) \in W_h \times \boldsymbol{V}_h$

$$
\begin{aligned}
&(c'(p_H^n)\partial_t p_H^n P_h^n + c(p_H^n)\partial_t P_h^n, w_h) + (\nabla \cdot \boldsymbol{U}_h^n, w_h) = (G, w_h), \quad && w_h \in W_h, && (4.3)\\
&(\kappa \boldsymbol{U}_h^n, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, P_h^n) = 0, && \boldsymbol{v}_h \in \boldsymbol{V}_h, && (4.4)
\end{aligned}
$$

where

$$G = f(p_H^n) + f'(p_H^n)(P_h^n - p_H^n) + c'(p_H^n)\partial_t p_H^n p_H^n.$$

In order to prove the convergence of the proposed two-grid method, we need to analyze $\|p^n - p_H^n\|_{0,p}$ with $2 \le p \le \infty$.

Lemma 4.1. *p^n is the solution of Eqs.* (3.1) *and* (3.2) *at $t = t^n$. p_H^n is the mixed finite element approximation in W_H. If we choose*

$$R_H p^0 = p_H^0, \quad \triangle t < C_*(2\|f\|_{1,\infty} + 2K_1K_2)^{-1}, \tag{4.5}$$

then, for $1 \le m \le N$ and $2 \le p \le \infty$, we have

$$\|p^m - p_H^m\|_{0,p} \le C(H^{k+1} + \triangle t). \tag{4.6}$$

Proof. Since the domain is quasi-uniformly partitioned, we can apply inverse inequality, Theorem 3.1 and Lemma 4.1 to obtain the error estimation. $\square$

In the following, we will demonstrate another main result of this paper.

Theorem 4.1. *$(p^n, \boldsymbol{u}^n)$ be the solution satisfy* (3.1)-(3.2) *at $t = t^n$. $(P_h^n, \boldsymbol{U}_h^n)$ be the two-grid solution satisfy* (4.3)-(4.4)*, then, If we choose*

$$R_h p^0 = p_h^0, \quad R_H p^0 = p_H^0, \quad \triangle t < C_*(2\|f\|_{1,\infty} + 2K_1K_2)^{-1}, \tag{4.7}$$

then, for $1 \le m \le N$, we have

$$\|p^m - P_h^m\| + \|\kappa^{\frac{1}{2}}(\boldsymbol{u} - \boldsymbol{U}_h)\|_{l^2(0,t^m;L^2)} \le C(h^{k+1} + H^{2k+2} + \triangle t). \tag{4.8}$$

Proof. Subtract Eqs. (4.3)-(4.4) from (3.1)-(3.2) at $t = t^n$ to get the error equations

$$
\begin{aligned}
&(H_1, w_h) + (\nabla \cdot (\boldsymbol{u}^n - \boldsymbol{U}_h^n), w_h) = (H_2, w_h),\\
&(\kappa(\boldsymbol{u}^n - \boldsymbol{U}_h^n), \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, p^n - P_h^n) = 0,
\end{aligned}
$$

where

$$\begin{aligned}
H_1 &= c(p^n)(p_t)^n - c'(p_H^n)\partial_t p_H^n P_h^n + c'(p_H^n)p_H^n\partial_t p_H^n - c(p_H^n)\partial_t P_h^n \\
&= c(p^n)[(p_t)^n - \partial_t p^n] + c(p^n)\partial_t p^n - c'(p_H^n)\partial_t p_H^n(P_h^n - p_H^n) - c(p_H^n)\partial_t P_h^n, \\
H_2 &= f(p^n) - f(p_H^n) - f'(p_H^n)(P_h^n - p_H^n).
\end{aligned}$$

Denote $\rho^n = Q_h p^n - P_h^n$, $\sigma^n = \Pi_h \boldsymbol{u}^n - \boldsymbol{U}_h^n$. Applying the definition of the L^2 projection (2.4) and Fortin interpolation (2.8), we have the following relation

$$(H_1, w_h) + (\nabla\cdot\sigma^n, w_h) = (H_2, w_h), \qquad w_h \in W_h, \tag{4.9}$$
$$(\kappa\sigma^n, \boldsymbol{v}_h) - (\nabla\cdot \boldsymbol{v}_h, \rho^n) = (\kappa(\Pi_h \boldsymbol{u}^n - \boldsymbol{u}^n), \boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.10}$$

Choosing $w_h = \rho^n$, $\boldsymbol{v}_h = \sigma^n$ and adding Eqs. (4.9) and (4.10) to get:

$$(H_1, \rho^n) + (\kappa\sigma^n, \sigma^n) = (H_2, \rho^n) + (\kappa(\Pi_h \boldsymbol{u}^n - \boldsymbol{u}^n), \sigma^n). \tag{4.11}$$

In order to bound H_2, we use the Taylor expansion

$$f(p^n) = f(p_H^n) + f'(p_H^n)(p^n - p_H^n) + \frac{1}{2}f''(p^*)(p^n - p_H^n)^2, \tag{4.12}$$

then, replace $f(p^n)$ in H_2 with relation (4.12) to get

$$H_2 = f'(p_H^n)(p^n - P_h^n) + \frac{1}{2}f''(p^*)(p^n - p_H^n)^2.$$

Apply the Young's inequality to (H_2, ρ^n) with parameter τ, we get

$$|(H_2, \rho^n)| \le C(h^{2k+2} + H^{4k+4}) + (2\tau + \|f\|_{1,\infty})\|\rho^n\|^2. \tag{4.13}$$

For H_1, we have the following relation for $c(p^n)p_t^n$

$$\begin{aligned}
c(p^n)\partial_t p^n = {} & c(p_H^n)\partial_t p_H^n + c'(p_H^n)\partial_t p_H^n(p^n - p_H^n) + c(p_H^n)(\partial_t p^n - \partial_t p_H^n) \\
& + \frac{1}{2}c''(p^*)(p^n - p_H^n)^2\partial_t(p^*)^n + c'(p^*)(p^n - p_H^n)(\partial_t p^n - \partial_t p_H^n).
\end{aligned} \tag{4.14}$$

Substitute relation (4.14) in H_1 to get

$$\begin{aligned}
H_1 = {} & c(p^n)(p_t^n - \partial_t p^n) + c'(p_H^n)\partial_t p_H^n(p^n - P_h^n) + c(p_H^n)\partial_t(p^n - P_h^n) \\
& + \frac{1}{2}c''(p^*)(p^n - p_H^n)^2\partial_t(p^*)^n + c'(p^*)(p^n - p_H^n)\partial_t(p^n - p_H^n).
\end{aligned}$$

Computing H_1 directly, we derive the error equation as follows

$$\begin{aligned}
& (c(p_H^n)\partial_t\rho^n, \rho^n) + (\kappa\sigma^n, \sigma^n) \\
= {} & (H_2, \rho^n) + (\kappa(\Pi_h \boldsymbol{u}^n - \boldsymbol{u}^n), \sigma^n) - (c(p^n)(p_t^n - \partial_t p^n), \rho^n) \\
& - (c'(p_H^n)\partial_t p_H^n(p^n - P_h^n), \rho^n) - (c(p_H^n)\partial_t(p^n - R_h p^n), \rho^n) \\
& - \Big(\frac{1}{2}c''(p^*)(p^n - p_H^n)^2\partial_t(p^*)^n, \rho^n\Big) - (c'(p^*)(p^n - p_H^n)\partial_t(p^n - p_H^n), \rho^n).
\end{aligned} \tag{4.15}$$

We need to bound each term on the right hand side of (4.15) except H_2. Employing the Young's inequality, it is easy to get

$$\|(\kappa(\Pi_h \boldsymbol{u}^n - \boldsymbol{u}^n),\sigma^n)\| \le Ch^{2k+2} + \epsilon\|\sigma^n\|^2. \tag{4.16}$$

For the third term of the right hand side of (4.15), using Lemma 4.1 and the same arguments as (3.26), we get:

$$|(c(p^n)(p_t^n - \partial_t p^n),\rho^n)| \le C\triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2 dt + \tau\|\rho^n\|^2. \tag{4.17}$$

About the fourth term, triangle inequality results in

$$\begin{aligned}|(c'(p_H^n)\partial_t p_H^n(p^n - P_h^n),\rho^n)| \le& |(c'(p_H^n)\partial_t p_H^n(p^n - Q_h p^n),\rho^n)| \\ &+ |(c'(p_H^n)\partial_t p_H^n(Q_h p^n - P_h^n),\rho^n)| \\ \le& Ch^{2k+2} + \tau\|\rho^n\|^2 + K_1K_2\|\rho^n\|^2. \end{aligned} \tag{4.18}$$

Since p is a smooth function with respect to t, all the other terms on the right hand side of (4.15) can be bounded easily by using Lemma 4.1, relation (2.7) and the assumptions (2.2), (2.3)

$$|(c(p_H^n)\partial_t(p^n - Q_h p^n),\rho^n)| \le Ch^{2k+2} + \tau\|\rho^n\|^2, \tag{4.19}$$

$$|(c(p_H^n)\partial_t(Q_h p^n - R_h p^n),\rho^n)| \le Ch^{2k+2} + \tau\|\rho^n\|^2, \tag{4.20}$$

$$\left|\left(\frac{1}{2}c''(p^*)\partial_t(p^*)^n(p^n - p_H^n)^2,\rho^n\right)\right| \le C(H^{4k+4} + (\triangle t)^2) + \tau\|\rho^n\|^2, \tag{4.21}$$

$$|(c'(p^*)(p^n - p_H^n)\partial_t(p^n - p_H^n),\rho^n)| \le CH^{4k+4} + \tau\|\rho^n\|^2. \tag{4.22}$$

About the left hand side of (4.15), we have

$$|(c(p_H^n)\partial_t\rho^n,\rho^n)| \ge \frac{C_*}{2\triangle t}(\|\rho^n\|^2 - \|\rho^{n-1}\|^2), \tag{4.23}$$

$$|(\kappa\sigma^n,\sigma^n)| = \|\kappa^{\frac{1}{2}}\sigma^n\|^2. \tag{4.24}$$

From (4.16)-(4.22) and (4.23)-(4.24), choose ϵ, τ to be sufficiently small, we finally get

$$\begin{aligned}\frac{C_*}{2\triangle t}(\|\rho^n\|^2 - \|\rho^{n-1}\|^2) + \|\kappa^{\frac{1}{2}}\sigma^n\|^2 \le& C\left(h^{2k+2} + H^{4k+4} + \triangle t\int_{t^{n-1}}^{t^n}\|p_{tt}\|^2 dt\right) \\ &+ (\|f\|_{1,\infty} + K_1K_2)\|\rho^n\|^2. \end{aligned} \tag{4.25}$$

Multiply $2\triangle t$ on both side of inequality (4.25) and sum over from $n=1$ to m, choose time step size $\triangle t$ to satisfy $\triangle t < C_*/(2\|f\|_{1,\infty} + 2K_1K_2)$. Notice the assumption that $\rho^0 = 0$, we employ the discrete Gronwall inequality to get

$$\|\rho^m\|^2 + \|\kappa^{\frac{1}{2}}\sigma\|^2_{l^2(0,t^m;L^2)} \le C(h^{2k+2} + H^{4k+4} + (\triangle t)^2). \tag{4.26}$$

The theorem can be easily derived from (4.26) and the triangle inequality. $\square$

5 Numerical example

In this section, we will demonstrate the efficiency of our algorithm constructed in Section 4 with a simple numerical example. Domain Ω is uniformly partitioned into triangulations with mesh size H (the coarse grid) and h (the fine mesh), respectively. $\boldsymbol{V}_h$ denotes the lowest Raviart-Thomas space RT_0. J is also uniformly partitioned so that $\triangle t$ is a constant. We fixed the fine mesh size $h=\frac{1}{64}$ and choose different coarse mesh size H to see the performance of the algorithm. We also selected $\triangle t = h$ to satisfy the conditions required in Theorems 3.1 and 4.1 associated with this example. For the discrete nonlinear system, we use the simple iterative method as a solver, e.t. using p^{l-1} (l is iteration number) in nonlinear term $c(p^{l-1})$, $f(p^{l-1})$ in the lth iteration. Since the discrete linear system is symmetric, we use the MinRes or GMRes iterative method as the solver. All the numerical results are computed on a laptop with 1G memory, 2.13 GHz Intel CPU processor.

Example 5.1. We consider the following initial-boundary valued reaction-diffusion equation

$$\begin{aligned}
&c(p)\frac{\partial p}{\partial t} - \nabla\cdot(K\nabla p) = f(p), && \forall(\boldsymbol{x},t)\in\Omega\times J,\\
&p(\boldsymbol{x},0)=0, && \forall(\boldsymbol{x},t)\in\Omega\times\{t=0\},\\
&K\nabla p\cdot\nu = 0, && \forall(\boldsymbol{x},t)\in\partial\Omega\times J,
\end{aligned}$$

where

$$\Omega=[0,1]^2,\quad J=\Big[0,\frac{1}{16}\Big],\quad K=\begin{pmatrix} x_1^2+1 & 0\\ 0 & x_2^2+1\end{pmatrix},\quad f(p)=p^3+g(\boldsymbol{x},t),\quad c(p)=e^{0.1p}$$

and $g(\boldsymbol{x},t)$ is chosen so that $p(\boldsymbol{x},t)=t^2(x_1^2(x_1-1)^2+x_2^2(x_2-1)^2)$ is exact solution.

The Mixed Finite Element Method (MFEM) solutions p_h are computed and the error estimate $\|p^n-p_h^n\|_{L^2(\Omega)}$ at different time level $t^n=n\triangle t$ are listed in the second column of Table 1. Two-Grid (T-G) solutions $(P_h,\boldsymbol{U}_h)$ with respect to different coarse mesh sizes are worked out by the algorithm discussed in Section 4. $\|p^n-P_h^n\|_{L^2(\Omega)}$ relative to $H=\frac{1}{2},\frac{1}{4}$ and $\frac{1}{8}$ are demonstrated in the third, fourth and fifth columns in Table 1. Fig. 1 represents

Table 1: Error of MFEM ($\|p^n-p_h^n\|$), Two-Grid method ($\|p^n-P_h^n\|$).

n	$\|p^n-p_h^n\|_{L^2}$	$H=\frac{1}{2},h=\frac{1}{64}$	$H=\frac{1}{4},h=\frac{1}{64}$	$H=\frac{1}{8},h=\frac{1}{64}$
1	1.0949748e-003	1.0949747e-003	1.0949748e-003	1.0949748e-003
2	3.8577307e-003	3.8577298e-003	3.8577306e-003	3.8577307e-003
3	8.3395689e-003	8.3395657e-003	8.3395687e-003	8.3395689e-003
4	1.4564119e-002	1.4564111e-002	1.4564119e-002	1.4564119e-002

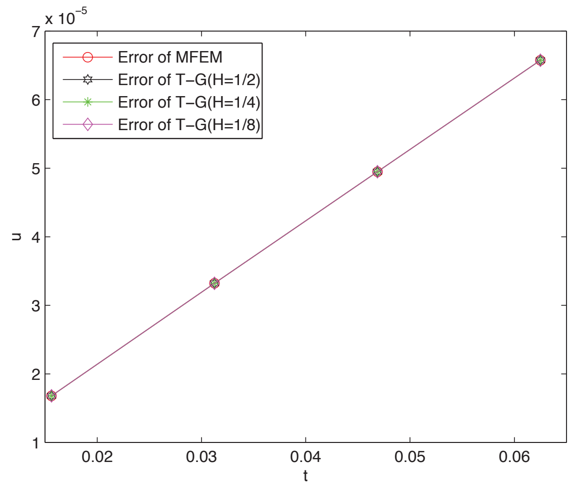

Figure 1: $\|u-u_h\|_V$, $\|u-U_h\|_V$ at different time levels.

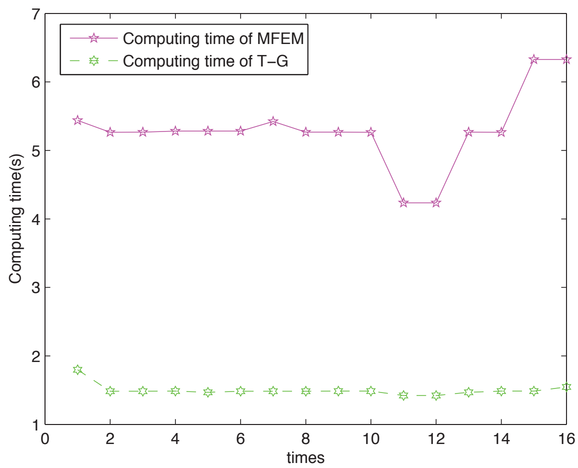

Figure 2: Computational complexity of MFEM and T-G.

error of flux $\boldsymbol{u}$ by MFEM and T-G algorithm. In Fig. 2, we also compared the computational complexity of the MFEM and the T-G method. From the numerical results, we can demonstrate that the two-grid method achieves asymptotically the same accuracy as the mixed finite element method when $h=H^2$. Furthermore, the computational complexity for two-grid algorithm is much less than that of the mixed finite element method. As we have seen from the performance, the numerical results is much better than what we expected since the coarse grid can be much coarser, such as $H=\frac{1}{2}$ or $\frac{1}{4}$. However, the theoretical error estimation is the best we can obtain.

6 Conclusion

In this paper, we investigate a two-grid algorithm for nonlinear reaction-diffusion equations discretized by mixed finite element methods. The nonlinear property primarily appears in the pressure coefficients and the source terms. It is the first time to implement two-grid algorithm for this equation by mixed finite element discretization. The prime ingredient of the two-grid method in this literature is that we use coarse grid solution as the initial guess of Newton iteration on the fine mesh. We proved that when the coarse mesh size and the fine mesh size satisfy $H=\mathcal{O}(h^{\frac{1}{2}})$, the two-grid algorithm achieves the same accuracy as the mixed finite element method. The two-grid method studied in this paper provides a new approach to take advantage of some nice properties hidden in a complex problem. In our future work, we will consider more complicated two-grid algorithms for (1.1)-(1.3).

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References

[1] Arbogast, T., Wheeler, M.F., Yotov, I.: Mixed finite elements for elliptic problems with tensor cofficients as cell-centered finite differences. SIAM J. Numer. Anal. **32**, 828-852 (1997).

[2] Brezzi, F., Douglas, JR.J., Marini, L.D.: Two families of mixed finite elemetns for second order elliptic problems. Numer. Math. **47**, 217-235 (1985).

[3] Chen, L., Chen, Y.: Two-grid method for nonlinear reaction diffusion equations by mixed finite element methods. J. Sci. Comp. **49**(3), 383-401(2011).

[4] Chen, Y., Huang, Y.: A multilevel method for finite element solutions for singular two-point boundary value problems. Natural Science Journal of Xiangtan University, **16**, 23-26 (1994) (in chinese).

[5] Chen, Y., Huang, Y., Yu, D.: A two-grid method for expanded mixed finite-element solution of semilinear reaction-diffusion equations. Int. J. Numer. Meth. Engng. **57**, 193-209 (2003).

[6] Chen, Y., Yu, D., Liu, S.: Analysis of two-grid methods for reaction-diffusion equations by expanded mixed finite element methods. Int. J. Numer. Meth. Engng. **69**, 408-422 (2007).

[7] Douglas, Jr.J., Ewing, R.E., Wheeler, M.F.: The approximation of the pressure by a mixed method in the simulation of miscible displacement. RAIRO: Analyse Numérique. **17**, 17-33(1983).

[8] Dawson, C.N., Wheeler, M.F.: Two-grid method for mixed finite element approximations of non-linear parabolic equations. Contemporary mathematics. **180**, 191-203 (1994).

[9] Dawson, C.N., Wheeler, M.F., Woodward, C.S.: A two-grid finite difference sceme for nonlinear parabolic euqations. SIAM J. Numer. Anal. **35**, 435-452 (1998).

[10] Garcia, M.F.: Improved error estimates for mixed finite element approximations for nonlinear parabolic equations: the continuously-time case. Numerical methods for partial diferential equations, **10**, 129-149 (1994).
[11] Huyakorn, P.S., Pinder, G.F.: Computational methods in Surface Flow. Academic Press: New York(1983).
[12] Layton W., Meir A.J., Schmidt P.G. A two-level discretization method for the stationary MHD equations, Electronic Transactions on Numerical Analysis. 6(12),198–210(1997).
[13] Murray, J.: Mathmatical Biology, 2nd edn. Springer: New York(1993).
[14] Squeff, M.C.J.: Superconvergence of mixed finite elements for parabolic equations. Math. Model. Numer. Anal. **21**, 327-352 (1987).
[15] Raviart, P.A., Thomas, J.M.: A mixed finite element method for 2nd order elliptic problems, Math Aspects of the Finite Element Method. Lecture Notes in Math, 606. Springer, Berlin, 292-315 (1977).
[16] Thomée, V.: Galerkin finite element methods for parabolic problems, Springer-Verlag(1997).
[17] Wu, L., Allen, M.B.: A two-grid method for mixed finite element solution of reaction-diffusion equations. Numerical Methods for Partial differential Equations. **15**, 317-332 (1999).
[18] Xu, J.: Two-grid discretization techniques for linear and non-linear PDEs, SIAM J. Numer. Anal. **33**, 1759-1777 (1996).
[19] Xu, J.: A novel two-grid method for semi-linear equations. SIAM J. Sci. Comput. **15**, 231-237 (1994).
[20] Xu. J, Zhou A.: A two-grid discretization scheme for eigenvalue problems, Math. Comp. 70(233),17-25(1999).
[21] Zhong L., Shu S., Wang J., J. Xu, Two-grid methods for time-harmonic Maxwell equations,Numerical Linear Algebra with Applications. 1,93-111(2012).