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Convergence of Linear Multistep Methods and One-Leg Methods for Index-2 Differential-Algebraic Equations with a Variable Delay

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Abstract. Linear multistep methods and one-leg methods are applied to a class of index-2 nonlinear differential-algebraic equations with a variable delay. The corresponding convergence results are obtained and successfully confirmed by some numerical examples. The results obtained in this work extend the corresponding ones in literature.

AMS subject classifications: 65L80; 65L20

Key words: index-2 differential-algebraic equations, variable delay, linear mutistep methods, one-leg methods, convergence.

1 Introduction

Index-2 delay differential-algebraic equations (DDAEs) are a very important class of mathematical models and often arise from the fields of computer aided design, circuit analysis, mechanical system, etc. Hence, the study of numerical methods for these equations is of important theoretical and practical values. In the recent years, some researches have been devoted to numerical methods for differential algebraic equations [1–7]. Some stability and convergence results of numerical methods for linear or index-1 delay differential-algebraic equations have been presented [8–11]. Xu and Zhao [8] studied stability of Runge-Kutta methods for neutral delay integro differential-algebraic equations. Block implicit one-step methods were applied to a class of retarded differential-algebraic equations by Li [9]. Convergence of one-leg methods for index-1 delay differential-algebraic equations was proved by Xiao and Zhang [10]. Zhu and Petzold [11] discussed asymptotic stability of Hessenberg delay differential-algebraic equations. However, the researches into numerical methods

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for nonlinear high-index delay differential-algebraic equations have arised in a few references [12–14]. Ascher and Petzold [12] derived the classical convergence results of BDF methods and Runge-Kutta methods for index-2 constant-delay differential-algebraic equations. Hauber [13] applied collocation methods to retarded differential-algebraic equations. Liu and Xiao [14] obtained the convergence results of BDF methods for a class of index-2 differential-algebraic equations with a variable delay.

In this paper, we apply the linear multistep methods (LMMs) and one-leg methods to a class of index-2 nonlinear differential-algebraic equations with a variable delay. The corresponding convergence results are obtained and successfully confirmed by some numerical examples.

2 Convergence of linear multistep methods

Consider the semi-explicit index-2 DDAE

$$\begin{cases} y'(x)=f\Big(y(x),y(x-\tau(x)),z(x)\Big), & x\in[0,T],\\ 0=g(y(x)), & x\in[0,T],\\ z(0)=z_0,\quad y(x)=\varphi(x), & x\in[-\tau,0], \end{cases} \tag{2.1}$$

where delay function $\tau(x)$ is differentiable and satisfies $0<\tau(x)\le\tau,\tau'(x)<1$, $f: R^{n_1}\times R^{n_1}\times R^{n_2}\to R^{n_1}$, $g: R^{n_1}\to R^{n_2}$ are sufficiently smooth vector functions on the real Euclidean spaces and have bounded derivatives, the initial value function $\varphi:[-\tau,0]\to R^{n_1}$ is a continuous function, and $g_y(y)f_z(y,y(x-\tau(x)),z)$ is invertible and bounded in a neighbourhood of the solution. We assume that the problem (2.1) has a smooth solution $y(x),z(x)$. Throughout this paper, $\|\cdot\|$ denotes the standard Euclidean norm, and the matrix norm is subordinate to $\|\cdot\|$.

A LMM with a Lagrange interpolation polynomial of degree p applied to the system (2.1) reads

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i f(y_{n+i},y^h_{n-k+i},z_{n+i}), \tag{2.2a}$$

$$0=g(y_{n+k}), \tag{2.2b}$$

where $x_{n+i}=x_n+ih,\ n\ge 0$,

$$y^h_{n-k+i}=\begin{cases}\varphi(x_{n+i}-\tau(x_{n+i})), & x_{n+i}-\tau(x_{n+i})\le 0, i=0,1,\cdots,k,\\ \sum\limits_{j=-u}^{q}Q_j(\delta_{n_i})y_{n+i-m_{n_i}+j}, & x_{n+i}-\tau(x_{n+i})>0, i=0,1,\cdots,k,\end{cases} \tag{2.3}$$

where $\tau(x_{n+i})=(m_{n_i}-\delta_{n_i})h,\ u,q,m_{n_i}\in Z^+,\ \delta_{n_i}\in[0,1),\ q+u=p, q+1\le m_{n_i},\ Q_j(\delta_{n_i})$ is the Lagrange interpolation basic function.

The perturbed values $\hat{y}_{n+k}, \hat{z}_{n+k}$ are defined by

$$\sum_{i=0}^{k} \alpha_i \hat{y}_{n+i} = h \sum_{i=0}^{k} \beta_i f(\hat{y}_{n+i}, \hat{y}^h_{n-k+i}, \hat{z}_{n+i}) + h\delta, \tag{2.4a}$$

$$0 = g(\hat{y}_{n+k}) + \theta. \tag{2.4b}$$

Theorem 1. *If y_{n+k}, z_{n+k} are given by* (2.2)*, the perturbed values $\hat{y}_{n+k}, \hat{z}_{n+k}$ are given by* (2.4)*, the initial values satisfy*

$$\begin{aligned} &y_{n+j} - y(x_{n+j}) = O(h), \quad z_{n+j} - z(x_{n+j}) = O(h), \quad g(y_{n+j}) = O(h^2), \\ &x_{n+j} = x_n + jh, \qquad j = 0, 1, \cdots, k-1, \end{aligned} \tag{2.5}$$

and the perturbed initial values satisfy

$$\hat{y}_{n+j} - y_{n+j} = O(h^2), \quad \hat{z}_{n+j} - z_{n+j} = O(h), \quad \delta = O(h), \quad \theta = O(h^2), \tag{2.6}$$

then for any given $h < h_0$ we have the estimates

$$\|\hat{y}_{n+k} - y_{n+k}\| \le C\Big(\|\hat{Y}_n - Y_n\| + h\|\hat{Z}_n - Z_n\| + h\|\delta\| + \|\theta\|\Big), \tag{2.7a}$$

$$\begin{aligned} \|\hat{z}_{n+k} - z_{n+k}\| \le \frac{C}{h}\Big(&\sum_{j=0}^{k-1} \|g_y(\hat{y}_{n+k})(\hat{y}_{n+j} - y_{n+j})\| + h\|\hat{Y}_n - Y_n\| \\ &+ h\|\hat{Z}_n - Z_n\| + h\|\delta\| + \|\theta\|\Big), \end{aligned} \tag{2.7b}$$

where

$$\begin{aligned} &Y_n = \Big(y^T_{n+k-1}, y^T_{n+k-2}, \cdots, y^T_n, (hy^h_n)^T, (hy^h_{n-1})^T, \cdots, (hy^h_{n-k})^T\Big)^T, \\ &\hat{Y}_n = \Big(\hat{y}^T_{n+k-1}, \hat{y}^T_{n+k-2}, \cdots, \hat{y}^T_n, (h\hat{y}^h_n)^T, (h\hat{y}^h_{n-1})^T, \cdots, (h\hat{y}^h_{n-k})^T\Big)^T, \\ &Z_n = \Big(z^T_{n+k-1}, z^T_{n+k-2}, \cdots, z^T_n\Big)^T, \quad \hat{Z}_n = \Big(\hat{z}^T_{n+k-1}, \hat{z}^T_{n+k-2}, \cdots, \hat{z}^T_n\Big)^T, \\ &\|\hat{Y}_n - Y_n\| = \max\Big(\max_{0\le j\le k-1} \|\hat{y}_{n+j} - y_{n+j}\|, \max_{0\le j\le k} h\|\hat{y}^h_{n-k+j} - y^h_{n-k+j}\|\Big), \\ &\|\hat{Z}_n - Z_n\| = \max_{0\le j\le k-1} \|\hat{z}_{n+j} - z_{n+j}\|. \end{aligned}$$

Proof. Put

$$\begin{aligned} \eta &= -\sum_{i=0}^{k-1} \frac{\alpha_i}{\alpha_k} y_{n+i} + h \sum_{i=0}^{k-1} \frac{\beta_i}{\alpha_k} f(y_{n+i}, y^h_{n-k+i}, z_{n+i}), \\ \hat{\eta} &= -\sum_{i=0}^{k-1} \frac{\alpha_i}{\alpha_k} \hat{y}_{n+i} + h \sum_{i=0}^{k-1} \frac{\beta_i}{\alpha_k} f(\hat{y}_{n+i}, \hat{y}^h_{n-k+i}, \hat{z}_{n+i}), \end{aligned}$$

and rescale h and δ, so that (2.2) and (2.4) become

$$y_{n+k} = \eta + hf(y_{n+k}, y^h_n, z_{n+k}), \tag{2.8a}$$

$$0 = g(y_{n+k}), \tag{2.8b}$$

and

$$\hat{y}_{n+k} = \hat{\eta} + hf(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k}) + h\delta, \tag{2.9a}$$
$$0 = g(\hat{y}_{n+k}) + \theta, \tag{2.9b}$$

respectively. Rewrite (2.8b) as

$$\begin{aligned} 0 &= g(y_{n+k}) - g(\eta) + g(\eta) \\ &= \int_0^1 g_y\big(\eta + \xi(y_{n+k} - \eta)\big) d\xi (y_{n+k} - \eta) + g(\eta). \end{aligned} \tag{2.10}$$

Substituting (2.8a) into (2.10) gives

$$\int_0^1 g_y\big(\eta + \xi(y_{n+k} - \eta)\big) d\xi f(y_{n+k}, y_n^h, z_{n+k}) + \frac{1}{h} g(\eta) = 0. \tag{2.11}$$

Rewrite (2.9b) as

$$\int_0^1 g_y\big(\hat{\eta} + \xi(\hat{y}_{n+k} - \hat{\eta})\big) d\xi \big(f(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k}) + \delta\big) + \frac{1}{h} g(\hat{\eta}) + \frac{1}{h}\theta = 0. \tag{2.12}$$

Exploiting the fact that the functions f, g are smooth and the matrix $g_y f_z$ is invertible, and subtracting (2.12) from (2.11), we deduce the estimate

$$\begin{aligned} \|\hat{z}_{n+k} - z_{n+k}\| \le C_1 \Big(&\|\hat{y}_{n+k} - y_{n+k}\| + \|\hat{\eta} - \eta\| + \|\hat{y}_n^h - y_n^h\| \\ &+ \|\delta\| + \frac{1}{h}\|\theta\| + \frac{1}{h}\|g(\hat{\eta}) - g(\eta)\|\Big). \end{aligned} \tag{2.13}$$

Subtracting (2.9a) from (2.8a), we get

$$\begin{aligned} &\|\hat{y}_{n+k} - y_{n+k}\| \le \|\hat{\eta} - \eta\| + h\|f(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k}) - f(y_{n+k}, y_n^h, z_{n+k})\| \\ \le &\|\hat{\eta} - \eta\| + hL\Big(\|\hat{y}_{n+k} - y_{n+k}\| + \|\hat{y}_n^h - y_n^h\| + \|\hat{z}_{n+k} - z_{n+k}\|\Big), \end{aligned} \tag{2.14}$$

where L is classical Lipschitz constant for the function f. Substituting (2.13) into (2.14) and exploiting functional differentiability of g, yields

$$\|\hat{y}_{n+k} - y_{n+k}\| \le C_2\Big(\|\hat{\eta} - \eta\| + \|\hat{y}_n^h - y_n^h\| + h\|\delta\| + \|\theta\|\Big), \quad h \le \frac{1}{L + LC_1}. \tag{2.15}$$

Substituting (2.15) into (2.13) gives

$$\begin{aligned} &\|\hat{z}_{n+k} - z_{n+k}\| \\ \le &C_3\Big(\|\hat{\eta} - \eta\| + \|\hat{y}_n^h - y_n^h\| + \|\delta\| + \frac{1}{h}\|\theta\|\Big) + \frac{1}{h}\|g_y(\eta_0)(\hat{\eta} - \eta)\|, \end{aligned} \tag{2.16}$$

where $\|\hat{\eta}-\eta\|$ and $\|g_y(\eta_0)(\hat{\eta}-\eta)\|$ satisfy

$$
\begin{aligned}
&\|\hat{\eta}-\eta\| \\
=&\Big\|\sum_{j=0}^{k-1}\frac{\alpha_j}{\alpha_k}(\hat{y}_{n+j}-y_{n+j})\Big\|+h\Big\|\sum_{j=0}^{k-1}\frac{\beta_j}{\alpha_k}\Big(f\big(\hat{y}_{n+j},\hat{y}^h_{n-k+j},\hat{z}_{n+j}\big)-f\big(y_{n+j},y^h_{n-k+j},z_{n+j}\big)\Big)\Big\| \\
\le& C_4\big(\|\hat{Y}_n-Y_n\|+h\|\hat{Z}_n-Z_n\|\big), \qquad (2.17)\\
&\|g_y(\eta_0)(\hat{\eta}-\eta)\|\le\|g_y(\hat{y}_k)(\hat{\eta}-\eta))\|+\|\hat{\eta}-\eta\|O(h) \\
\le& C_5\big(\sum_{j=0}^{k-1}\|g_y(\hat{y}_k)(\hat{y}_{n+j}-y_{n+j})\|+h\|\hat{Y}_n-Y_n\|+h^2\|\hat{Z}_n-Z_n\|\big), \qquad (2.18)
\end{aligned}
$$

respectively. Substituting (2.18), (2.17) into (2.16), (2.15), gives (2.7).

Corollary 1. *Suppose that the LMM* (2.2) *with the interpolation procedure* (2.3) *is of order p. Then its local error satisfies*

$$y_k-y(x_k)=O(h^{p+1}),\qquad z_k-z(x_k)=O(h^p). \tag{2.19}$$

Proof. We put $n=0$, $\hat{y}_j=y(x_j)$, $\hat{z}_j=z(x_j), j=0,1,\cdots,k$. These values satisfy the conditions of Theorem 1 with $\delta=O(h^p),\theta=0$. By the interpolation formula (2.3), we have $\|\hat{y}^h_0-y^h_0\|=O(h^{p+1})$. The desired result (2.19) follows immediately from Theorem 1.

Theorem 2. *Suppose that the LMM* (2.2) *with interpolation procedure* (2.3) *is of order p, and is stable* ($\rho(\xi)=\sum_{j=0}^k\alpha_j\xi^j$ *satisfies the root condition*) *and strictly stable at infinity* (*the zeros of $\sigma(\xi)$ lie inside the unit disc $|\xi|<1$*). *If the initial values satisfy*

$$y_j-y(x_j)=O(h^{p+1}),\qquad z_j-z(x_j)=O(h^p),\quad j=0,1,\dots,k-1, \tag{2.20}$$

then this method applied to the system (2.1) *is convergent of order p, i.e.,*

$$y_n-y(x_n)=O(h^p),\ \ z_n-z(x_n)=O(h^p),\ \ x_n=nh,\ \ n\ge k. \tag{2.21}$$

Proof. We firstly study the propagation of the local errors and their accumulation over the whole interval for the y-component. We now denote the numerical solution by $\{y^0_n,z^0_n\}$, and consider the multistep solutions $\{y^l_n,z^l_n\},l=1,2,\cdots$ with starting values

$$y^l_j=y(x_j),\quad z^l_j=z(x_j),\quad j=l-1,\cdots,l+k-2.$$

Our first aim is to estimate $y^l_n-y^{l+1}_n$. For simplicity, we omit the upper index and consider two neighbouring multistep solutions $\{\hat{y}_n,\hat{z}_n\}$ and $\{\tilde{y}_n,\tilde{z}_n\}$. In order to apply Theorem 1, we fix four sufficiently large constants $\hat{C}_0,\hat{C}_1,\hat{C}_2,\hat{C}_3$ (Remark 1 describes the four constants of rationality in the end of paper) and suppose that

$$
\begin{aligned}
&\|\hat{y}_{n+j}-y(x_{n+j})\|\le\hat{C}_0h,\ \|\hat{z}_{n+j}-z(x_{n+j})\|\le\hat{C}_1h,\\
&\|\hat{y}_{n+j}-\tilde{y}_{n+j}\|\le\hat{C}_2h^2,\quad\|\hat{z}_{n+j}-\tilde{z}_{n+j}\|\le\hat{C}_3h,\qquad j=0,1,\dots,k-1. \qquad (2.22)
\end{aligned}
$$

Introduce the notations

$$\begin{aligned}
&\Delta z_{n+j} = \tilde{z}_{n+j} - \hat{z}_{n+j}, \quad \Delta y_{n+j} = \tilde{y}_{n+j} - \hat{y}_{n+j}, \quad \Delta y^h_{n-k+j} = \tilde{y}^h_{n-k+j} - \hat{y}^h_{n-k+j},\\
&\Delta Y_n = \Big(\Delta y^T_{n+k-1}, \Delta y^T_{n+k-2}, \ldots, \Delta y^T_n, (h\Delta y^h_n)^T, (h\Delta y^h_{n-1})^T, \quad \cdots, (h\Delta y^h_{n-k})^T\Big)^T,\\
&\Delta Z_n = (\Delta z^T_{n+k-1}, \Delta z^T_{n+k-2}, \ldots, \Delta z^T_n)^T, \qquad j = 0, 1, \cdots, k.
\end{aligned}$$

It follows from Theorem 1 with $\delta = 0$ and $\theta = 0$ that

$$\|\Delta y_{n+k}\| \leq C(\|\Delta Y_n\| + h\|\Delta Z_n\|), \tag{2.23a}$$

$$\|\Delta z_{n+k}\| \leq \frac{C}{h}\Big(\sum_{j=0}^{k-1} \|g_y(\hat{y}_{n+k})\Delta y_{n+j}\| + h\|\Delta Y_n\| + h\|\Delta Z_n\|\Big), \tag{2.23b}$$

where C does not depend on the choice of $\hat{C}_0, \hat{C}_1, \hat{C}_2$ if h is sufficiently small. Our assumption (2.22) together with (2.23) implies

$$\|\Delta y_{n+k}\| = O(h^2), \qquad \|\Delta z_{n+k}\| = O(h). \tag{2.24}$$

We substitute y_{n+i} for $\tilde{y}_{n+i}$, y^h_{n+i} for $\tilde{y}^h_{n+i}$, z_{n+i} for $\tilde{z}_{n+i}$ in the formula (2.2) respectively, and put $\delta = 0$ and $\theta = 0$ in the formula(2.4). Then subtracting (2.2a) from (2.4a) and (2.2b) from (2.4b) yields

$$\begin{aligned}
\sum_{i=0}^{k} \alpha_i \Delta y_{n+i} &= h\sum_{i=0}^{k} \beta_i \Big(f(\tilde{y}_{n+i}, \tilde{y}^h_{n-k+i}, \tilde{z}_{n+i}) - f(\hat{y}_{n+i}, \hat{y}^h_{n-k+i}, \hat{z}_{n+i})\Big)\\
&= h\sum_{i=0}^{k} \beta_i f_z(\hat{y}_{n+i}, \hat{y}^h_{n-k+i}, \hat{z}_{n+i})\Delta z_{n+i} + O(h\|\Delta Y_n\| + h^2\|\Delta Z_n\|),
\end{aligned} \tag{2.25a}$$

$$0 = g_y(\hat{y}_{n+k})\Delta y_{n+k} + O(h\|\Delta Y_n\| + h^2\|\Delta Z_n\|). \tag{2.25b}$$

We next use the projections

$$Q_{n+j} = \big(f_z(g_y f_z)^{-1} g_y\big)(\hat{y}_{n+j}, \hat{y}^h_{n-k+j}, \hat{z}_{n+j}), \quad P_{n+j} = I - Q_{n+j}, \tag{2.26}$$

for $j = 0, 1, \cdots, k$, which yields

$$\begin{aligned}
&Q^2_{n+j} = Q_{n+j}, \quad P^2_{n+j} = P_{n+j}, \quad Q_{n+j}P_{n+j} = P_{n+j}Q_{n+j} = 0,\\
&Q_{n+j+1} = Q_{n+j} + O(h), \quad P_{n+j+1} = P_{n+j} + O(h).
\end{aligned}$$

Multiplying (2.25a) by P_{n+k} and (2.25b) by $f_z(g_y f_z)^{-1}(\hat{y}_{n+k}, \hat{y}^h_n, \hat{z}_{n+k})$, we get

$$\sum_{i=0}^{k} \alpha_i P_{n+i}\Delta y_{n+i} = O\Big(h\|\Delta Y_n\| + h^2\|\Delta Z_n\|\Big), \tag{2.27a}$$

$$Q_{n+k}\Delta y_{n+k} = O\Big(h\|\Delta Y_n\| + h^2\|\Delta Z_n\|\Big). \tag{2.27b}$$

Multiplying (2.25a) by $(g_y f_z)^{-1} g_y(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k})$, we get

$$h \sum_{i=0}^{k} \beta_i \Delta z_{n+i} = \sum_{i=0}^{k} \alpha_i (g_y f_z)^{-1} g_y(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k}) \Delta y_{n+j} + O\Big(h\|\Delta Y_n\| + h^2 \|\Delta Z_n\|\Big). \tag{2.28}$$

Using the conditions (2.3), (2.22), (2.23) and the delay function $\tau'(x) < 1$ yields

$$\Delta y_{n+1}^h = O\Big(\|\Delta Y_n\| + h\|\Delta Z_n\|\Big). \tag{2.29}$$

Introducing the vectors

$$U_n = \Big((P_{n+k-1}\Delta y_{n+k-1})^T, \cdots, (P_n \Delta y_n)^T, \frac{1}{2}h(\Delta y_n^h)^T, \cdots, \frac{1}{2}h(\Delta y_{n-k}^h)^T\Big)^T, \tag{2.30a}$$

$$V_n = \Big((Q_{n+k-1}\Delta y_{n+k-1})^T, \cdots, (Q_n \Delta y_n)^T, \frac{1}{2}h(\Delta y_n^h)^T, \cdots, \frac{1}{2}h(\Delta y_{n-k}^h)^T\Big)^T. \tag{2.30b}$$

We have $\Delta Y_n = U_n + V_n$. The relations (2.27) and (2.28) become

$$U_{n+1} = (A \otimes I)U_n + O\Big(h\|U_n\| + h\|V_n\| + h^2\|\Delta Z_n\|\Big), \tag{2.31a}$$

$$V_{n+1} = (N \otimes I)V_n + O\Big(h\|U_n\| + h\|V_n\| + h^2\|\Delta Z_n\|\Big), \tag{2.31b}$$

$$h\Delta z_{n+1} = (B \otimes I)h\Delta z_n + O\Big(h\|U_n\| + \|V_n\| + h^2\|\Delta Z_n\|\Big), \tag{2.31c}$$

where $\alpha_j' = \alpha_j/\alpha_k, \beta_j' = \beta_j/\alpha_k$,

$$A = \begin{pmatrix} \tilde{A} & O \\ O & \tilde{N}_k \end{pmatrix}, \quad N = \begin{pmatrix} \tilde{N}_{k-1} & O \\ O & \tilde{N}_k \end{pmatrix}, \quad B = \begin{pmatrix} -\beta_{k-1}' & \cdots & -\beta_1' & -\beta_0' \\ 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.32}$$

$\tilde{A}$ and $\tilde{N}_m$ can be expressed as

$$\tilde{A} = \begin{pmatrix} -\alpha_{k-1}' & \cdots & -\alpha_1' & -\alpha_0' \\ 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \tilde{N}_m = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}_{m\times m}. \tag{2.33}$$

Since $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ satisfies the root condition and all the roots of $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$ lie inside the unit disc, we now choose a norm $\|U\|$ such that $\|A \otimes I\| \leq 1$, a norm $\|V\|$ such that $\|N \otimes I\| \leq \rho$ and a norm $\|W\|$ such that $\|B \otimes I\| \leq \kappa < 1$ [15], consequently it follows from (2.31) that

$$\begin{pmatrix} \|U_{n+1}\| \\ \|V_{n+1}\| \\ h\|\Delta Z_{n+1}\| \end{pmatrix} \leq \begin{pmatrix} 1+O(h) & O(h) & O(h) \\ O(h) & \rho + O(h) & O(h) \\ O(h) & O(1) & \kappa + O(h) \end{pmatrix} \begin{pmatrix} \|U_n\| \\ \|V_n\| \\ h\|\Delta Z_n\| \end{pmatrix}. \tag{2.34}$$

We diagonalize the matrix in (2.34) and obtain

$$\begin{pmatrix} \|U_{n+1}\| \\ \|V_{n+1}\| \\ h\|\Delta Z_{n+1}\| \end{pmatrix} \le T^{-1} \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix} T \begin{pmatrix} \|U_0\| \\ \|V_0\| \\ h\|\Delta Z_0\| \end{pmatrix}, \tag{2.35}$$

where $\lambda_1 = 1 + O(h), \lambda_2 = \rho + O(h), \lambda_3 = \kappa + O(h)$, the transformation matrix T consists of the corresponding eigenvectors and satisfies

$$T = \begin{pmatrix} 1 & O(h) & O(h) \\ O(h) & 1 & O(h) \\ O(h) & O(h) & 1 \end{pmatrix}.$$

The vectors U_0, V_0, Z_0 are composed of local errors or errors in the starting values, which are of size $O(h^{p+1})$. Hence, it follows from (2.19) and (2.21) that

$$\|U_0\| \le H_0 h^{p+1}, \qquad \|V_0\| \le H_1 h^{p+1}, \qquad \|\Delta Z_0\| \le H_2 h^p. \tag{2.36}$$

Using (2.19), (2.20) and (2.35), we obtain

$$\begin{aligned} &\|\Delta y_n\| \le C_6 h^{p+1}, \qquad \|\Delta z_n\| \le C_7(\rho^n + \kappa^n + h)h^p, \\ &\|g_y(\hat{y}_{n+k})\Delta y_{n+j}\| \le C_8(\rho^n + h)h^{p+1}. \end{aligned} \tag{2.37}$$

Hence

$$\|y_n - y(x_n)\| \le \sum_{l=0}^{n-k+1} \|y_n^l - y_n^{l+1}\| \le C_9 h^p. \tag{2.38}$$

We can similarly prove the second part of (2.21). This completes the proof of the theorem.

Remark 1. *In general, the constants C_6, C_7, C_9 and C_{10} depend on $\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{C}_3$ of the assumption* (2.22)*, and we can restrict the step size h so that*

$$C_9 h^{p-1} \le \hat{C}_0, \quad C_{10} h^{p-1} \le \hat{C}_1, \quad C_6 h^{p-1} \le \hat{C}_2, \quad C_7 h^{p-1} \le \hat{C}_3, \tag{2.39}$$

and the numerical solutions will never violate the conditions (2.22) *on the considered interval.*

3 Convergence of one-leg methods

A one-leg method (ρ, σ) with the generating polynomials

$$\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i, \qquad \sigma(\xi) = \sum_{i=0}^{k} \beta_i \xi^i, \tag{3.1}$$

together with Lagrange interpolation polynomials of degree p applied to the problem (2.1) reads

$$\rho y_n = hf(\sigma y_n, \sigma y^h_{n-k}, \sigma z_n), \tag{3.2a}$$
$$0 = g(y_{n+k}), \tag{3.2b}$$

where

$$\rho y_n = \sum_{i=0}^{k} \alpha_i y_{n+i}, \qquad \sigma y_n = \sum_{i=0}^{k} \beta_i y_{n+i},$$
$$\sigma y^h_{n-k} = \sum_{i=0}^{k} \beta_i y^h_{n-k+i}, \quad \sigma z_n = \sum_{i=0}^{k} \beta_i z_{n+i},$$

and y^h_{n-k+i} is given by (2.3).

Theorem 3. *Suppose that the one-leg method* (3.2) *with the interpolation procedure* (2.3) *is of order p, and is stable* ($\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ *satisfies the root condition*) *and strictly stable at infinity* (*the zeros of* $\sigma(\xi)$ *lie inside the unit disc* $|\xi| < 1$), *and the initial values satisfy*

$$y_j - y(x_j) = O(h^{p+1}), \qquad z_j - z(x_j) = O(h^p), \quad j = 0, 1, \ldots, k-1, \tag{3.3}$$

then this method applied to the system (2.1) *is convergent of order p, i.e.,*

$$y_n - y(x_n) = O(h^p), \quad z_n - z(x_n) = O(h^p), \quad x_n = nh, \quad n \geq k. \tag{3.4}$$

Proof. The proof process is similar to that of Theorem 2.

4 Numerical examples

Example 1 Consider the semi-explicit index-2 DDAEs

$$\begin{cases} y_1'(x) = -2y_1(\frac{x}{2})y_2(x), & 0 \leq x \leq 2, \\ y_2'(x) = -3\sqrt{y_1(\frac{x}{2})y_2^2(\frac{x}{2})} + 2z(x), & 0 \leq x \leq 2, \\ 0 = y_1(x) - y_2^2(x), & 0 \leq x \leq 2, \\ y_1(0) = 1, \quad y_2(0) = -1, \quad z(0) = 1, \end{cases} \tag{4.1}$$

the exact solution of the system (4.1) is

$$y_1(x) = e^{-2x}, \quad y_2(x) = -e^{-x}, \quad z(x) = e^{-x}.$$

θ-method ($\theta = 3/4$) is applied to (4.1). Let $yerr1(h)$, $yerr2(h)$, $zerr(h)$ denote the global errors of the components y_1, y_2, z at $x = 2$ for the stepsize h. We estimate the corresponding orders $py1(h)$, $py2(h)$ and $pz(h)$ by

$$py(h) = \frac{\ln \frac{yerr1(h)}{yerr1(0.5h)}}{\ln 2}, \quad py2(h) = \frac{\ln \frac{yerr2(h)}{yerr1(0.5h)}}{\ln 2}, \quad pz(h) = \frac{\ln \frac{zerr(h)}{zerr(0.5h)}}{\ln 2}.$$

Table 1: Numerical results (θ-method, $\theta = 3/4$).

h	yerr1	yerr2	zerr	py1	py2	pz
0.1	$0.3700E-2$	$0.4798E-2$	$0.3989E-2$	1.99	1.99	1.99
0.05	$0.9310E-3$	$0.1200E-2$	$0.1001E-2$	2.01	1.97	1.99
0.025	$0.2309E-3$	$0.3044E-3$	$0.2510E-3$	2.03	2.02	2.01
0.0125	$0.5640E-4$	$0.7465E-4$	$0.6201E-4$			

Example 2 Consider the semi-explicit index-2 DDAEs

$$\begin{cases} y_1'(x) = y_1 y_2^2(\frac{x}{2})z(x), & 0 \le x \le 2, \\ y_2'(x) = y_1^4(\frac{x}{2})y_2^2(x) - 3y_2^2(x)z^2(x), & 0 \le x \le 2, \\ 0 = 1 - y_1^2(x)y_2(x), & 0 \le x \le 2, \\ y_1(0) = 1, \quad y_2(0) = 1, \quad z(0) = 1, \end{cases} \tag{4.2}$$

the exact solution of the system (4.2) is

$$y_1(x) = e^x, \quad y_2(x) = e^{-2x}, \quad z(x) = e^x.$$

To show the relevance of our theoretical results, we have applied the following two-step one-leg method

$$\rho(\xi) = \frac{5}{4}\xi^2 - \frac{3}{2}\xi + \frac{1}{4}, \qquad \sigma(\xi) = \frac{21}{32}\xi^2 + \frac{6}{17}\xi - \frac{3}{32}, \tag{4.3}$$

with linear Lagrange interpolation. The symbols $yerr1(h)$, $yerr2(h)$, $zerr(h)$, $py1(h)$, $py2(h)$ and $pz(h)$ in Table 2 are defined in Example 1.

Table 2: Numerical results.

h	yerr1	yerr2	zerr	py1	py2	pz
0.1	$0.3601E-2$	$0.4898E-2$	$0.7203E-2$	1.99	2.02	1.96
0.05	$0.9001E-3$	$0.1203E-2$	$0.1695E-2$	2.00	1.97	1.98
0.025	$0.2254E-3$	$0.3065E-3$	$0.4271E-3$	2.00	1.99	2.01
0.0125	$0.0556E-3$	$0.0768E-3$	$0.1054E-3$			

Tables 1 and 2 list the computing results of numerical examples, and the numerical results confirm the corresponding theoretical results.

5 Conclusion

In this paper, we obtain some convergence results for linear multistep methods and one-leg methods applied to a class of index-2 nonlinear variable-delay differential-algebraic equations by using some proof techniques given by Hairer and Wanner [7]. When the delay is a constant and the used methods are BDF methods, these results are consistent with those of BDF methods in the literature [12]. Therefore, the obtained results extend the corresponding results in some references.

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