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Remark on the Lifespan of Solutions to 3-D Anisotropic Navier Stokes Equations

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Abstract. The goal of this article is to provide a lower bound for the lifespan of smooth solutions to 3-D anisotropic incompressible Navier-Stokes system, which in particular extends a similar type of result for the classical 3-D incompressible Navier-Stokes system.

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1 Introduction

In this article, we shall investigate the lifespan for smooth enough solutions to the following 3-D anisotropic incompressible Navier-Stokes system:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta_{\mathrm{h}} u = -\nabla p, \\ \mathrm{div}\, u = 0, \\ u\,|_{t=0} = u_0, \end{cases} \tag{1.1}$$

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where $\Delta_{\rm h}\overset{\text{déf}}{=}\partial_1^2+\partial_2^2$, u designates the velocity of the fluid and p the scalar pressure function which guarantees the divergence free condition of the velocity field.

Systems of this type appear in geophysical fluid dynamics (see for instance [5, 16]). In fact, meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\mu_{\rm h}\Delta_{\rm h}-\mu_3\partial_3^2$, where $\mu_{\rm h}$ and μ_3 are empirical constants, and μ_3 is usually much smaller than $\mu_{\rm h}$. We refer to the book of Pedlovsky [16], Chap. 4 for a complete discussion about this model.

We remark that for the classical Navier-Stokes system (NS), which corresponds to (1.1) with $\Delta_{\rm h}$ there being replaced by $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$, Leray [12] proved the global existence of weak solutions to (NS) in 1934. Yet the uniqueness and regularity to such weak solutions are still open. In [6], Chemin and Gallagher showed that: let u_0 be a regular solenoidal vector field, then the classical Navier-Stokes system (NS) has a unique regular solution on $[0,T]$. Let $T^\star(u_0)$ be the maximal time of existence of this regular solution. Then for any $\gamma\in(0,1/4)$, a positive constant C_γ exists so that

$$T^\star(u_0)\geq C_\gamma\|u_0\|_{\dot H^{\frac12+2\gamma}}^{-\frac{1}{\gamma}}. \tag{1.2}$$

In the special case when $\gamma=\frac14$, this type of result goes back to the seminal work of Leray [12]. Lately the same type of result has been proved for 3-D inhomogeneous incompressible Navier-Stokes system in [17] by the second author.

Considering that the system (1.1) has only horizontal dissipation, it is reasonable to use anisotropic Sobolev space defined as follows:

Definition 1.1. *For any (s,s') in $\mathbb{R}^2$, the anisotropic Sobolev space $\dot H^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution a such that*

$$\|a\|^2_{\dot H^{s,s'}}\overset{def}{=}\int_{\mathbb{R}^3}|\xi_{\rm h}|^{2s}|\xi_3|^{2s'}|\widehat{a}(\xi)|^2d\xi<\infty \quad with \quad \xi_{\rm h}=(\xi_1,\xi_2).$$

Mathematically, Chemin et al. [4] first studied the system (1.1). In particular, Chemin et al. [4] and Iftimie [11] proved that (1.1) is locally well-posed with initial data in $L^2\cap\dot H^{0,\frac12+\varepsilon}$ for some $\varepsilon>0$, and is globally well-posed if in addition

$$\|u_0\|^{\varepsilon}_{L^2}\|u_0\|^{1-\varepsilon}_{\dot H^{0,\frac12+\varepsilon}}\leq c \tag{1.3}$$

for some sufficiently small constant c.

Paicu [14] improved the well-posedness result in [4, 11] to be the critical case, namely, with initial data in the critical anisotropic Besov space, which basically corresponds to $\varepsilon=0$ in [4, 11]. Chemin and the second author [8] introduced an anisotropic Besov-Sobolev type space with negative index and proved the global well-posedness of (1.1) with initial data being sufficiently small in this space. Paicu and the second author [15] improved further the global well-posedness result in [8] by requiring only two components of the initial data to be small in such negative anisotropic Besov spaces. Lately Liu and the second author proved the global well-posedness of (1.1) by requiring only one directional derivative of the initial data to be sufficiently small in some critical spaces. One may check [13] and the references therein concerning the recent progresses on the well-posedness of this system (1.1).

The goal of this article is to extend similar result as (1.2) for the lifespan of solutions to the classical Navier-Stokes system to the case of (1.1). The main result states as follows:

Theorem 1.1. *Let $s=2\gamma+\frac{1}{2}$ with $\gamma\in(0,1/4)$. Let $u_0\in H^s$ be a solenoidal vector field with $\partial_3 u_0\in \dot H^{s-1,0}\cap \dot H^{-1,s}$. Then* (1.1) *has a unique solution $u\in C([0,T];H^{0,s})$ with $\nabla_{\rm h}u\in L^2((0,T);\dot H^{0,s})$ for some $T>0$. Moreover, if $T^*(u_0)$ designates the lifespan of this solution, there exists a positive constant C_γ so that*

$$T^*(u_0)\geq C_\gamma[u_0]_s^{-\frac{1}{\gamma}} \quad \textit{with} \quad [u_0]_s^2 \overset{def}{=} \|u_0\|^2_{\dot H^{0,s}}+\|\partial_3 u_0\|^2_{\dot H^{s-1,0}}+\|\partial_3 u_0\|^2_{\dot H^{-1,s}}. \tag{1.4}$$

Let us end this section with the notations that we shall use in this context.

Notations: Let A,B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B, for $a\lesssim b$, we means that there is a uniform constant C, which may be different in each occurrence, such that $a\leq Cb$. We shall denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^3)$ inner product of a and b. We always denote $(c_\ell)_{\ell\in\mathbb{Z}}$ to be a nonnegative generic element of $\ell^2(\mathbb{Z})$ so that $\sum c_\ell^2=1$. Finally, we denote $L^r_T(L^p_{\rm h}(L^q_{\rm v}))$ the space $L^r([0,T];L^p(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2};L^q(\mathbb{R}_{x_3})))$, and $\nabla_{\rm h}\overset{\rm def}{=}(\partial_{x_1},\partial_{x_2})$, ${\rm div}_{\rm h}=\partial_{x_1}+\partial_{x_2}$.

2 Littlewood-Paley analysis and product laws

In the rest of this paper, we shall frequently use Littlewood-Paley decomposition in the vertical variable. For the convenience of the readers, we collect some basic facts on anisotropic Littlewood-Paley theory in this section. Let us first recall from [1] that

$$\begin{aligned} &\Delta_k^{\rm h}a=\mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\rm h}|)\widehat{a}), \qquad S_k^{\rm h}a=\mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\rm h}|)\widehat{a}),\\ &\Delta_\ell^{\rm v}a=\mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \qquad S_\ell^{\rm v}a=\mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}), \end{aligned} \tag{2.1}$$

where $\xi=(\xi_{\rm h},\xi_3)$, $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

$$\begin{aligned}
&\text{Supp }\varphi\subset\Big\{\tau\in\mathbb{R}\,/\ \frac{3}{4}\le|\tau|\le\frac{8}{3}\Big\}\quad\text{and}\quad \forall\tau>0,\ \sum_{k\in\mathbb{Z}}\varphi(2^{-k}\tau)=1,\\
&\text{Supp }\chi\subset\Big\{\tau\in\mathbb{R}\,/\ |\tau|\le\frac{4}{3}\Big\}\qquad\ \ \text{and}\quad \chi(\tau)+\sum_{k\ge 0}\varphi(2^{-k}\tau)=1.
\end{aligned}$$

Next we recall the following anisotropic Bernstein inequalities from [8,14]:

Lemma 2.1. *Let* $\mathbf{B}_{\rm h}$ *(resp.* $\mathbf{B}_{\rm v}$*) a ball of* $\mathbb{R}^2_{\rm h}$ *(resp.* $\mathbb{R}_{\rm v}$*), and* $\mathcal{C}_{\rm h}$ *(resp.* $\mathcal{C}_{\rm v}$*) a ring of* $\mathbb{R}^2_{\rm h}$ *(resp.* $\mathbb{R}_{\rm v}$*); let* $1\le p_2\le p_1\le\infty$ *and* $1\le q_2\le q_1\le\infty$*. Then there holds*

$$\begin{aligned}
&\textit{if}\ \ \text{Supp }\widehat{a}\subset 2^k\mathbf{B}_{\rm h}\ \Rightarrow\ \|\partial^\alpha_{x_{\rm h}}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{k\left(|\alpha|+\frac{2}{p_2}-\frac{2}{p_1}\right)}\|a\|_{L^{p_2}_{\rm h}(L^{q_1}_{\rm v})};\\
&\textit{if}\ \ \text{Supp }\widehat{a}\subset 2^\ell\mathbf{B}_{\rm v}\ \Rightarrow\ \|\partial^\beta_{x_3}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{\ell\left(\beta+\frac{1}{q_2}-\frac{1}{q_1}\right)}\|a\|_{L^{p_1}_{\rm h}(L^{q_2}_{\rm v})};\\
&\textit{if}\ \ \text{Supp }\widehat{a}\subset 2^k\mathcal{C}_{\rm h}\ \Rightarrow\ \|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{-kN}\sup_{|\alpha|=N}\|\partial^\alpha_{x_{\rm h}}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})};\\
&\textit{if}\ \ \text{Supp }\widehat{a}\subset 2^\ell\mathcal{C}_{\rm v}\ \Rightarrow\ \|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{-\ell N}\|\partial^N_{x_3}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}.
\end{aligned}$$

Before preceding, we recall Bony's decomposition for the vertical variable from [2]:

$$ab=T^{\rm v}_a b+R^{\rm v}(a,b)\quad\text{with}\quad T^{\rm v}_a b=\sum_{\ell\in\mathbb{Z}}S^{\rm v}_{\ell-1}a\Delta^{\rm v}_\ell b,\quad R^{\rm v}(a,b)=\sum_{\ell\in\mathbb{Z}}\Delta^{\rm v}_\ell a S^{\rm v}_{\ell+2}b. \tag{2.2}$$

Let us now apply the above basic facts on Littlewood-Paley theory to prove the following laws of product:

Lemma 2.2. *Let* $s\in\left(\frac{1}{2},1\right)$*, one has*

$$\begin{aligned}
\|\Delta^{\rm v}_\ell(ab)\|_{L^2_{\rm v}(L^{\frac{4}{3}}_{\rm h})}\lesssim c_\ell 2^{-\ell s}\Big(&\|a\|^{\frac{1}{2}}_{\dot H^{0,s}}\|\nabla_{\rm h}a\|^{\frac{1}{2}}_{\dot H^{0,s}}\|b\|^{1-\frac{1}{2s}}_{L^2}\|b\|^{\frac{1}{2s}}_{\dot H^{0,s}}\\
&+\left(\|a\|_{L^2}\|\nabla_{\rm h}a\|_{L^2}\right)^{\frac{1}{2}-\frac{1}{4s}}\left(\|a\|_{\dot H^{0,s}}\|\nabla_{\rm h}a\|_{\dot H^{0,s}}\right)^{\frac{1}{4s}}\|b\|_{\dot H^{0,s}}\Big).
\end{aligned}\tag{2.3}$$

Proof. By applying Bony's decomposition (2.2) in the vertical variable to ab, we find

$$ab=T^{\rm v}_a b+R^{\rm v}(a,b).$$

Considering the support properties to the Fourier transform of the terms in $T^{\rm v}_a b$, we write

$$\begin{aligned}\|\Delta^{\rm v}_\ell(T^{\rm v}_a b)\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})} &\lesssim \sum_{|j-\ell|\le 5}\|S^{\rm v}_{j-1}a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta^{\rm v}_j b\|_{L^2}\\ &\lesssim \|a\|_{L^\infty_{\rm v}(L^4_{\rm h})}\Big(\sum_{|j-\ell|\le 5}\|\Delta^{\rm v}_j b\|_{L^2}\Big)\\ &\lesssim \|a\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})}\|\nabla_{\rm h}a\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})}\Big(\sum_{|j-\ell|\le 5}\|\Delta^{\rm v}_j b\|_{L^2}\Big),\end{aligned}$$

from which, and

$$\|f\|_{L^\infty_{\rm v}(L^2_{\rm h})}\lesssim\|f\|_{L^2_{\rm h}(L^\infty_{\rm v})}\lesssim\|f\|_{L^2_{\rm h}(B^{\frac12}_{2,1})_{\rm v}}\lesssim\|f\|^{1-\frac{1}{2s}}_{L^2}\|f\|^{\frac{1}{2s}}_{\dot H^{0,s}}. \tag{2.4}$$

We deduce that

$$\|\Delta^{\rm v}_\ell(T^{\rm v}_a b)\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\lesssim c_\ell 2^{-\ell s}\big(\|a\|_{L^2}\|\nabla_{\rm h}a\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|a\|_{\dot H^{0,s}}\|\nabla_{\rm h}a\|_{\dot H^{0,s}}\big)^{\frac{1}{4s}}\|b\|_{\dot H^{0,s}}. \tag{2.5}$$

Along the same line to the proof of (2.5), we infer

$$\begin{aligned}\|\Delta^{\rm v}_\ell(R^{\rm v}(a,b))\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})} &\lesssim \sum_{j\ge\ell-N_0}\|S^{\rm v}_{j+2}b\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta^{\rm v}_j a\|_{L^2_{\rm v}(L^4_{\rm h})}\\ &\lesssim \|b\|_{L^\infty_{\rm v}(L^2_{\rm h})}\Big(\sum_{j\ge\ell-N_0}\|\Delta^{\rm v}_j a\|_{L^2_{\rm v}(L^4_{\rm h})}\Big)\\ &\lesssim \|b\|^{1-\frac{1}{2s}}_{L^2}\|b\|^{\frac{1}{2s}}_{\dot H^{0,s}}\Big(\sum_{j\ge\ell-N_0}\|\Delta^{\rm v}_j a\|^{\frac12}_{L^2}\|\Delta^{\rm v}_j\nabla_{\rm h}a\|^{\frac12}_{L^2}\Big)\\ &\lesssim \|b\|^{1-\frac{1}{2s}}_{L^2}\|b\|^{\frac{1}{2s}}_{\dot H^{0,s}}\|a\|^{\frac12}_{\dot H^{0,s}}\|\nabla_{\rm h}a\|^{\frac12}_{\dot H^{0,s}}\sum_{j\ge\ell-N_0}c_j 2^{-js}.\end{aligned}$$

Notice that

$$\Big\|\Big(2^{\ell s}\sum_{j\ge\ell-5}c_j 2^{-js}\Big)_{\ell\in\mathbb{Z}}\Big\|_{\ell^2}\lesssim\|1_{(-\infty,5]}2^{s\cdot}\|_{\ell^1}\|c_j\|_{\ell^2}\lesssim 1, \tag{2.6}$$

we conclude that

$$\|\Delta^{\rm v}_\ell(R^{\rm v}(a,b))\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\lesssim c_\ell 2^{-\ell s}\|b\|^{1-\frac{1}{2s}}_{L^2}\|b\|^{\frac{1}{2s}}_{\dot H^{0,s}}\|a\|^{\frac12}_{\dot H^{0,s}}\|\nabla_{\rm h}a\|^{\frac12}_{\dot H^{0,s}}. \tag{2.7}$$

Combining (2.5) with (2.7) leads to (2.3). $\square$

Remark 2.1. The proof of Lemma 2.2 also implies that

$$\|\Delta_\ell^{\rm v}(ab)\|_{L^2_{\rm v}(L^{\frac{4}{3}}_{\rm h})}\lesssim c_\ell 2^{-\ell s}\Big(\|a\|_{L^\infty_{\rm v}(L^4_h)}\|b\|_{\dot H^{0,s}}+\|a\|^{\frac12}_{\dot H^{0,s}}\|\nabla_{\rm h}a\|^{\frac12}_{\dot H^{0,s}}\|b\|^{1-\frac{1}{2s}}_{L^2}\|b\|^{\frac{1}{2s}}_{\dot H^{0,s}}\Big), \tag{2.8}$$

and

$$\|\Delta_\ell^{\rm v}(ab)\|_{L^2_{\rm v}(L^{\frac{4}{3}}_{\rm h})}\lesssim c_\ell 2^{-\ell s}\Big(\|a\|_{L^\infty_{\rm v}(L^4_h)}\|b\|_{\dot H^{0,s}}+\|a\|^{\frac12}_{\dot H^{0,s}}\|\nabla_{\rm h}a\|^{\frac12}_{\dot H^{0,s}}\|b\|_{L^\infty_{\rm v}(L^2_{\rm h})}\Big). \tag{2.9}$$

Lemma 2.3. *Let $a=(a^{\rm h},a^3)$ be a solenoidal vector field. Then for any $s\in\left(\frac12,1\right)$, one has*

$$\begin{aligned}&\big|\big(\Delta_\ell^{\rm v}(a^3\partial_3 b)\,|\,\Delta_\ell^{\rm v}b\big)_{L^2}\big|\\ \lesssim\,& c_\ell^2 2^{-2\ell s}\Big(\|\nabla_{\rm h}a^{\rm h}\|^{1-\frac{1}{2s}}_{L^2}\|\nabla_{\rm h}a^{\rm h}\|^{\frac{1}{2s}}_{\dot H^{0,s}}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\\ &+\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}\big(\|b\|_{L^2}\|\nabla_{\rm h}b\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\Big).\end{aligned} \tag{2.10}$$

Proof. By applying Bony's decomposition (2.2) in the vertical variable to $a^3\partial_3 b$, we write

$$a^3\partial_3 b=T^{\rm v}_{a^3}\partial_3 b+R^{\rm v}(a^3,\partial_3 b).$$

We first deduce from the support properties to the Fourier transform of the terms in $R^{\rm v}(a^3,\partial_3 b)$ and Lemma 2.1 that

$$\begin{aligned}&\big|\big(\Delta_\ell^{\rm v}(R^{\rm v}(a^3,\partial_3 b))\,|\,\Delta_\ell^{\rm v}b\big)_{L^2}\big|\\ \lesssim&\sum_{j\geq\ell-N_0}2^j\|\Delta_j^{\rm v}a^3\|_{L^2}\|S^v_{j+2}b\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v}b\|_{L^2_{\rm v}(L^4_{\rm h})}\\ \lesssim&\sum_{j\geq\ell-N_0}\|\Delta_j^{\rm v}\partial_3 a^3\|_{L^2}\|b\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})}\|\nabla_{\rm h}b\|^{\frac12}_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}b\|^{\frac12}_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h}b\|^{\frac12}_{L^2},\end{aligned}$$

which together with (2.4) and $\partial_3 a^3=-{\rm div_h}\,a^{\rm h}$ ensures that

$$\begin{aligned}&\big|\big(\Delta_\ell^{\rm v}(R^{\rm v}(a^3,\partial_3 b))\,|\,\Delta_\ell^{\rm v}b\big)_{L^2}\big|\\ \lesssim\,& c_\ell 2^{-\ell s}\big(\|b\|_{L^2}\|\nabla_{\rm h}b\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}\sum_{j\geq\ell-N_0}c_j2^{-js}\\ \lesssim\,& c_\ell^2 2^{-2\ell s}\big(\|b\|_{L^2}\|\nabla_{\rm h}b\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\big)^{\frac{1}{4s}+\frac12}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}},\end{aligned} \tag{2.11}$$

where in the last step we used (2.6) once again.

To handle the term $\big(\Delta_\ell^{\rm v}(T^{\rm v}_{a^3}\partial_3 b)\,|\,\Delta_\ell^{\rm v} b\big)_{L^2}$, we get, by using a standard commutator's process (see for instance [4]), that

$$\begin{aligned}\Delta_\ell^{\rm v}(T_{a^3}\partial_3 b)=&S^{\rm v}_{\ell-1}a^3\partial_3\Delta_\ell^{\rm v}b+\sum_{|j-\ell|\le 5}[\Delta_\ell^{\rm v};S^{\rm v}_{j-1}a^3]\partial_3\Delta_j^{\rm v}b\\&+\sum_{|j-\ell|\le 5}(S^{\rm v}_{j-1}a^3-S^{\rm v}_{\ell-1}a^3)\partial_3\Delta_\ell^{\rm v}\Delta_j^{\rm v}b.\end{aligned}\tag{2.12}$$

Corresponding to the first term in (2.12), we get, by applying (2.4) and $\operatorname{div} a=0$, that

$$\begin{aligned}\big|(S^{\rm v}_{\ell-1}a^3\partial_3\Delta_\ell^{\rm v}b\,|\,\Delta_\ell^{\rm v}b)_{L^2}\big|&=\frac12\big|(S^{\rm v}_{\ell-1}\partial_3a^3\Delta_\ell^{\rm v}b\,|\,\Delta_\ell^{\rm v}b)_{L^2}\big|\\&\lesssim\|\partial_3a^3\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}b\|^2_{L^2_{\rm v}(L^4_{\rm h})}\\&\lesssim\|\nabla_{\rm h}a^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}b\|_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h}b\|_{L^2}\\&\lesssim c_\ell^2 2^{-2\ell s}\|\nabla_{\rm h}a^{\rm h}\|_{L^2}^{1-\frac{1}{2s}}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}^{\frac{1}{2s}}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}.\end{aligned}$$

Similarly, corresponding to the last term in (2.12), we get, by applying Lemma 2.1, that

$$\begin{aligned}&\Big|\sum_{|j-\ell|\le 5}(S^{\rm v}_{j-1}a^3-S^{\rm v}_{\ell-1}a^3)\partial_3\Delta_\ell^{\rm v}\Delta_j^{\rm v}b\big|\Delta_\ell^{\rm v}b\big)_{L^2}\Big|\\&\lesssim\sum_{|j-\ell|\le 5}\|S^{\rm v}_{j-1}a^3-S^{\rm v}_{\ell-1}a^3\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\partial_3\Delta_\ell^{\rm v}b\|_{L^2_{\rm v}(L^4_{\rm h})}\|\Delta_\ell^{\rm v}b\|_{L^2_{\rm v}(L^4_{\rm h})}\\&\lesssim\sum_{|j-\ell|\le 5}\|\Delta_j^{\rm v}a^3\|_{L^\infty_{\rm v}(L^2_{\rm h})}2^\ell\|\Delta_\ell^{\rm v}b\|^2_{L^2_{\rm v}(L^4_{\rm h})}\\&\lesssim\sum_{|j-\ell|\le 5}\|\Delta_j^{\rm v}\partial_3a^3\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_\ell^{\rm v}b\|_{L^2}\|\Delta_\ell^{\rm v}\nabla_{\rm h}b\|_{L^2}\\&\lesssim c_\ell^2 2^{-2\ell s}\|\nabla_{\rm h}a^{\rm h}\|_{L^2}^{1-\frac{1}{2s}}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}^{\frac{1}{2s}}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}.\end{aligned}$$

Finally let us deal with the commutator term. We observe that

$$\begin{aligned}&\sum_{|j-\ell|\le 5}\big|\big([\Delta_\ell^{\rm v};S^{\rm v}_{j-1}a^3]\partial_3\Delta_j^{\rm v}b\big|\Delta_\ell^{\rm v}b\big)_{L^2}\big|\\&\lesssim\sum_{|j-\ell|\le 5}\|[\Delta_\ell^{\rm v};S^{\rm v}_{j-1}a^3]\partial_3\Delta_j^{\rm v}b\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\|\Delta_\ell^{\rm v}b\|_{L^2_{\rm v}(L^4_{\rm h})}.\end{aligned}$$

Let $h\stackrel{\rm def}{=}\mathcal{F}^{-1}\varphi$ and $h_1(z)\stackrel{\rm def}{=}zh(z)$. Then again due to $\partial_3 a^3=-{\rm div}_{\rm h}a^{\rm h}$, we find

$$
\begin{aligned}
&\big|[\Delta_\ell^{\rm v};S_{j-1}^{\rm v}a^3]\partial_3\Delta_j^{\rm v}b(x_{\rm h},x_3)\big|\\
=&2^\ell\Big|\int_{\mathbb{R}}h(2^\ell y_3)\big(S_{j-1}^{\rm v}a^3(x_{\rm h},x_3-y_3)-S_{j-1}^{\rm v}a^3(x_{\rm h},x_3)\big)\partial_3\Delta_j^{\rm v}b(x_{\rm h},x_3-y_3)dy_3\Big|\\
=&2^\ell\Big|\int_{\mathbb{R}\times[0,1]}y_3h(2^\ell y_3)S_{j-1}^{\rm v}\partial_3a^3(x_{\rm h},x_3-\tau y_3)\partial_3\Delta_j^{\rm v}b(x_{\rm h},x_3-y_3)d\tau dy_3\Big|\\
=&\Big|\int_{\mathbb{R}\times[0,1]}h_1(2^\ell y_3)S_{j-1}^{\rm v}{\rm div}_{\rm h}a^h(x_{\rm h},x_3-\tau y_3)\partial_3\Delta_j^{\rm v}b(x_{\rm h},x_3-y_3)d\tau dy_3\Big|.
\end{aligned}
$$

When $|j-\ell|\le 5$, taking the $L^2_{\rm v}(L^{\frac43}_{\rm h})$ norm to the above quantity gives rise to

$$
\begin{aligned}
&\|[\Delta_\ell^{\rm v};S_{j-1}^{\rm v}a^3]\partial_3\Delta_j^{\rm v}b\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\\
\lesssim&\int_{\mathbb{R}\times[0,1]}|\,h_1(2^\ell y_3)|\,\|S_{j-1}^{\rm v}{\rm div}_{\rm h}a^h(\cdot,\cdot-\tau y_3)\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\partial_3\Delta_j^{\rm v}b(\cdot,\cdot-y_3)\|_{L^2_{\rm v}(L^4_{\rm h})}d\tau dy_3\\
=&2^\ell\|S_{j-1}^{\rm v}{\rm div}_{\rm h}a^h(\cdot,\cdot)\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_j^{\rm v}b\|_{L^2_{\rm v}(L^4_{\rm h})}\int_{\mathbb{R}}|\,h_1(2^\ell y_3)|\,dy_3\\
\lesssim&\|\nabla_{\rm h}a^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\Delta_j^{\rm v}b\|_{L^2}^{\frac12}\|\Delta_j^{\rm v}\nabla_{\rm h}b\|_{L^2}^{\frac12}.
\end{aligned}
$$

Then by virtue of (2.4), we deduce that

$$
\begin{aligned}
&\sum_{|\,j-\ell|\,\le 5}\big|\big([\Delta_\ell^{\rm v};S_{j-1}^{\rm v}a^3]\partial_3\Delta_j^{\rm v}b\big|\Delta_\ell^{\rm v}b\big)_{L^2}\big|\\
\lesssim&c_\ell^2 2^{-2\ell s}\|\nabla_{\rm h}a^{\rm h}\|_{L^2}^{1-\frac1{2s}}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}^{\frac1{2s}}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}.
\end{aligned}
$$

As a result, it comes out

$$
\big|\big(\Delta_\ell^{\rm v}(T^{\rm v}_{a^3}\partial_3b)|\,\Delta_\ell^{\rm v}b\big)_{L^2}\big|\lesssim c_\ell^2 2^{-2\ell s}\|\nabla_{\rm h}a^{\rm h}\|_{L^2}^{1-\frac1{2s}}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}^{\frac1{2s}}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}.
$$

Along with (2.11), we complete the proof of (2.10). □

Remark 2.2. The proof of Lemma 2.3 implies that

$$
\begin{aligned}
&\big|\big(\Delta_\ell^{\rm v}(a^3\partial_3b)\;|\;\Delta_\ell^{\rm v}b\big)_{\dot H^{0,s}}\big|\\
\lesssim&c_\ell^2 2^{-2\ell s}\Big(\|\nabla_{\rm h}a^{\rm h}\|_{L^\infty_{\rm v}(L^2)}\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\\
&+\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}\big(\|b\|_{L^2}\|\nabla_{\rm h}b\|_{L^2}\big)^{\frac12-\frac1{4s}}\big(\|b\|_{\dot H^{0,s}}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}\big)^{\frac12+\frac1{4s}}\Big).
\end{aligned}
\tag{2.13}
$$

3 Scaled L^2 energy estimate

Inspired by the lecture notes of Chemin [3], we are going to present a scaled energy estimate for smooth enough solutions of (1.1). We remark that estimate of this type was first proposed by Chemin and Plamchon in [7] for the classical 3-D Navier-Stokes system (see also [6]). The main result of this section states as follows:

Proposition 3.1. *Let $s\in\left(\frac{1}{2},1\right)$ and $u_L\overset{def}{=}e^{t\Delta_{\rm h}}u_0$. Let $u\overset{def}{=}u_L+w$ be a smooth enough solution of* (1.1) *on* $[0,T^*)$. *Then for any $t<T^*$, one has*

$$\begin{aligned}&\frac{\|w(t)\|_{L^2}^2}{t^s}+\int_0^t\Big(\frac{s}{4}\frac{\|w(t')\|_{L^2}^2}{(t')^{1+s}}+\frac{\|\nabla_{\rm h}w(t')\|_{L^2}^2}{(t')^s}\Big)dt' \\ \le& C_st^{s-\frac{1}{2}}\big(\|u_0\|_{H^s}^2+\|\partial_3u_0\|_{\dot H^{s-1,0}}^2\big)\|u_0\|_{H^s}^2\exp\Big(C_st^{\frac{2s-1}{3}}\big(\|u_0\|_{\dot H^s}^{\frac{4}{3}}+\|\partial_3u_0\|_{\dot H^{s-1,0}}^{\frac{4}{3}}\big)\Big).\end{aligned}\tag{3.1}$$

Proof. Due to $u\overset{\rm def}{=}w+u_L$, by virtue of (1.1), w verifies

$$\begin{cases}\partial_tw+(u_L+w)\cdot\nabla w+w\cdot\nabla u_L-\Delta_{\rm h}w=-u_L\cdot\nabla u_L-\nabla p,\\ {\rm div}\,w=0,\\ w|_{t=0}=0.\end{cases}\tag{3.2}$$

By taking L^2 inner product of (3.2) with w and dividing the resulting equality by t^s, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\frac{\|w(t)\|_{L^2}^2}{t^s}\Big)+\frac{s}{2}\frac{\|w(t)\|_{L^2}^2}{t^{1+s}}+\frac{\|\nabla_{\rm h}w(t)\|_{L^2}^2}{t^s}\\ =&-\frac{(w\cdot\nabla u_L|w)_{L^2}}{t^s}-\frac{(u_L\cdot\nabla u_L|w)_{L^2}}{t^s}.\end{aligned}\tag{3.3}$$

In what follows, we shall separate the estimate of the terms on the right-hand side of (3.3) with vertical derivative and without vertical derivative.

Notice that for any $p>2$, we have

$$\|f\|_{L_{\rm h}^{\frac{2p}{p-1}}}\le C\|f\|_{L_{\rm h}^2}^{1-\frac{1}{p}}\|\nabla_{\rm h}f\|_{L_{\rm h}^2}^{\frac{1}{p}},\tag{3.4}$$

from which, we infer

$$\begin{aligned}&\big|(w^{\rm h}\cdot\nabla_{\rm h}u_L|\,w)_{L^2}\big|\le\|\nabla_{\rm h}u_L\|_{L_{\rm v}^\infty(L_{\rm h}^p)}\|w\|_{L_{\rm v}^2(L_{\rm h}^{\frac{2p}{p-1}})}^2\\ \le& C\|\nabla_{\rm h}u_L\|_{L_{\rm v}^\infty(L_{\rm h}^p)}\|w\|_{L^2}^{2(1-\frac{1}{p})}\|\nabla_{\rm h}w\|_{L^2}^{\frac{2}{p}}\le C\|\nabla_{\rm h}u_L\|_{L_{\rm v}^\infty(L_{\rm h}^p)}^{\frac{p}{p-1}}\|w\|_{L^2}^2+\frac{1}{8}\|\nabla_{\rm h}w\|_{L^2}^2.\end{aligned}$$

In particular, taking $p=4$ in the above inequality gives rise to

$$\big|(w^{\rm h}\cdot\nabla_{\rm h}u_L\,|\,w)_{L^2}\big|\leq C\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}^{\frac{4}{3}}\|w\|_{L^2}^2+\frac{1}{8}\|\nabla_{\rm h}w\|_{L^2}^2. \tag{3.5}$$

While observing that ${\rm div}\,w=0$ and

$$\|f\|_{L^2_{\rm h}(L^\infty_{\rm v})}\leq\|f\|_{L^2}^{\frac{1}{2}}\|\partial_3 f\|_{L^2}^{\frac{1}{2}},$$

we deduce that

$$\begin{aligned}\big|(w^3\partial_3u_L\,|\,w)_{L^2}\big|&\leq\|w^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|w\|_{L^2}\\&\leq\|w^3\|_{L^2}^{\frac{1}{2}}\|\partial_3w^3\|_{L^2}^{\frac{1}{2}}\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|w\|_{L^2}\\&\leq C\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}^{\frac{4}{3}}\|w\|_{L^2}^2+\frac{1}{8}\|\nabla_{\rm h}w\|_{L^2}^2.\end{aligned} \tag{3.6}$$

Finally applying Young's inequality gives rise to

$$\begin{aligned}\frac{\big|(u_L\cdot\nabla u_L|w)_{L^2}\big|}{t^s}&\leq t^{\frac{1-s}{2}}\|u_L\cdot\nabla u_L\|_{L^2}\frac{\|w\|_{L^2}}{t^{\frac{1+s}{2}}}\\&\leq\frac{t^{1-s}}{s}\|u_L\cdot\nabla u_L\|_{L^2}^2+\frac{s\|w\|_{L^2}^2}{4t^{1+s}}.\end{aligned} \tag{3.7}$$

Inserting the estimates (3.5), (3.6) and (3.7) into (3.3), we achieve

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\|w(t)\|_{L^2}^2}{t^s}\Big)+\frac{s}{4}\frac{\|w(t)\|_{L^2}^2}{t^{1+s}}+\frac{\|\nabla_{\rm h}w(t)\|_{L^2}^2}{t^s}\\&\leq C\big(\|\nabla_{\rm h}u_L\|_{L^4_{\rm h}(L^\infty_{\rm v})}^{\frac{4}{3}}+\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}^{\frac{4}{3}}\big)\frac{\|w\|_{L^2}^2}{t^s}+\frac{2}{s}t^{1-s}\|u_L\cdot\nabla u_L\|_{L^2}^2.\end{aligned}$$

Applying Gronwall's inequality gives rise to

$$\begin{aligned}&\frac{\|w(t)\|_{L^2}^2}{t^s}+\int_0^t\Big(\frac{s}{4}\frac{\|w(t')\|_{L^2}^2}{(t')^{1+s}}+\frac{\|\nabla_{\rm h}w(t')\|_{L^2}^2}{(t')^s}\Big)dt'\\&\leq\frac{2}{s}\int_0^t(t')^{1-s}\|u_L\cdot\nabla u_L(t')\|_{L^2}^2dt'\\&\quad\times\exp\Big(C\int_0^t\big(\|\nabla_{\rm h}u_L(t')\|_{L^4_{\rm h}(L^\infty_{\rm v})}^{\frac{4}{3}}+\|\partial_3u_L(t')\|_{L^\infty_{\rm h}(L^2_{\rm v})}^{\frac{4}{3}}\big)dt'\Big).\end{aligned} \tag{3.8}$$

Let us now handle term by term in (3.8).

Estimate of $\int_0^t \|\nabla_{\rm h} u_L(t')\|^{\frac43}_{L^4_{\rm h}(L^\infty_{\rm v})}dt'$

We first observe from Lemma 2.4 of [1] and Lemma 2.1 that

$$\int_0^t \|\nabla_{\rm h} u_L(t')\|^{\frac43}_{L^4_{\rm h}(L^\infty_{\rm v})}dt' \leq C\int_0^t \Big(\sum_{k\in\mathbb{Z}} e^{-ct'2^{2k}} 2^{\frac{3k}{2}} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} u_0\|_{L^2}\Big)^{\frac43} dt'.$$

Yet it follows from Lemma 4.3 of [9] that $\dot{H}^s(\mathbb{R}^3)\hookrightarrow \dot{H}^{s-\frac12}_{\rm h}\big(B^{\frac12}_{2,1}\big)_{\rm v}$, so that

$$\sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} u_0\|_{L^2} \lesssim c_k 2^{-k(s-\frac12)} \|u_0\|_{\dot{H}^s}, \tag{3.9}$$

where $(c_k)_{k\in\mathbb{Z}}$ is a generic element of $\ell^2(\mathbb{Z})$ so that $\sum_{k\in\mathbb{Z}} c_k^2=1$. As a result, we obtain

$$\int_0^t \|\nabla_{\rm h} u_L(t')\|^{\frac43}_{L^4_{\rm h}(L^\infty_{\rm v})}dt' \leq C\int_0^t \Big(\sum_{k\in\mathbb{Z}} e^{-ct'2^{2k}} 2^{k(2-s)}\Big)^{\frac43} dt' \|u_0\|^{\frac43}_{\dot{H}^s}.$$

On the other hand, it follows from Lemma 2.35 of [1] that

$$\sup_{t>0}\Big(\sum_{k\in\mathbb{Z}} t^{1-\frac{s}{2}} 2^{k(2-s)} e^{-ct2^{2k}}\Big) \stackrel{\rm def}{=} M_s<\infty.$$

Hence we obtain

$$\begin{aligned}\int_0^t \|\nabla_{\rm h} u_L(t')\|^{\frac43}_{L^4_{\rm h}(L^\infty_{\rm v})}dt' &\leq C M_s^{\frac43}\int_0^t (t')^{-\frac43+\frac{2s}{3}} dt' \|u_0\|^{\frac43}_{\dot{H}^s}\\ &\leq C_s t^{\frac{2s-1}{3}} \|u_0\|^{\frac43}_{\dot{H}^s}.\end{aligned} \tag{3.10}$$

Estimate of $\int_0^t \|\partial_3 u_L(t')\|^{\frac43}_{L^\infty_{\rm h}(L^2_{\rm v})}dt'$

Notice that $\dot{H}^{s-1,0}\hookrightarrow L^2_{\rm v}\big(B^{s-2}_{\infty,2}\big)_{\rm h}$, we deduce from Theorem 2.34 of [1] that

$$\begin{aligned}&\int_0^t \|\partial_3 u_L(t')\|^{\frac43}_{L^\infty_{\rm h}(L^2_{\rm v})}dt'\\ &\leq \Big(\int_0^t (t')^{2(s-1)}dt'\Big)^{\frac13}\Big(\int_{\mathbb{R}_z}\int_0^t (t')^{1-s}\|\partial_3 u_L(t',\cdot,z)\|^2_{L^\infty_{\rm h}}dt'dz\Big)^{\frac23}\\ &\leq C_s t^{\frac{2s-1}{3}}\Big(\int_{\mathbb{R}_z}\|\partial_3 u_0\|^2_{\left(B^{s-2}_{\infty,2}\right)_{\rm h}}dz\Big)^{\frac23}\\ &\leq C_s t^{\frac{2s-1}{3}}\|\partial_3 u_0\|^{\frac43}_{\dot{H}^{s-1,0}}.\end{aligned} \tag{3.11}$$

Estimate of $\int_0^t (t')^{1-s}\|u_L\cdot\nabla u_L(t')\|_{L^2}^2\,dt'$

We first get, by applying Hölder's inequality, that

$$\begin{aligned}&\int_0^t (t')^{1-s}\|u_L^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}u_L(t')\|_{L^2}^2\,dt'\\ \le&\Big(\int_0^t \|u_L^{\mathrm{h}}(t)\|_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})}^4\,dt'\Big)^{\frac12}\Big(\int_0^t (t')^{2(1-s)}\|\nabla_{\mathrm{h}}u_L(t')\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}^4\,dt'\Big)^{\frac12}.\end{aligned}$$

Observing that $\dot H^{s-1,0}\hookrightarrow L^2_{\mathrm{v}}\big(B^{s-\frac32}_{4,2}\big)_{\mathrm{h}}\hookrightarrow L^2_{\mathrm{v}}\big(B^{s-\frac32}_{4,4}\big)_{\mathrm{h}}$, and thanks to Theorem 2.34 of [1], we infer

$$\begin{aligned}&\Big(\int_0^t (t')^{2(1-s)}\|\nabla_{\mathrm{h}}u_L^{\mathrm{h}}(t')\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}^4\,dt'\Big)^{\frac14}\\ \le&\Big\|\Big(\int_0^t (t')^{2(1-s)}\|\nabla_{\mathrm{h}}u_L^{\mathrm{h}}(t',\cdot,z)\|_{L^4_{\mathrm{h}}}^4\,dt'\Big)^{\frac14}\Big\|_{L^2_{\mathrm{v}}}\\ \lesssim&\Big\|\|\nabla_{\mathrm{h}}u_0(\cdot,z)\|_{\big(B^{s-\frac32}_{4,2}\big)_{\mathrm{h}}}\Big\|_{L^2_{\mathrm{v}}}\lesssim\|\nabla_h u_0\|_{\dot H^{s-1,0}}.\end{aligned}$$

Whereas we deduce from (3.9) that

$$\begin{aligned}\int_0^t \|u_L^{\mathrm{h}}(t)\|_{L^4_{\mathrm{h}}(L^\infty_{\mathrm{v}})}^4\,dt'\lesssim&\int_0^t\Big(\sum_{k\in\mathbb{Z}}e^{-ct2^{2k}}2^{\frac k2}\sum_{\ell\in\mathbb{Z}}2^{\frac\ell2}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}u_0\|_{L^2}\Big)^4\,dt\\ \lesssim&\int_0^t (t')^{2(s-1)}\sup_{t>0}\Big(\sum_{k\in\mathbb{Z}}(t')^{\frac{1-s}{2}}2^{k(1-s)}\Big)^4\,dt'\|u_0\|_{H^s}^4\\ \le& C_s t^{2s-1}\|u_0\|_{H^s}^4.\end{aligned}\tag{3.12}$$

As a result, it comes out

$$\int_0^t (t')^{1-s}\|u_L^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}u_L(t')\|_{L^2}^2\,dt'\lesssim t^{s-\frac12}\|u_0\|_{H^s}^4.\tag{3.13}$$

Exactly along the same line, we have

$$\int_0^t (t')^{1-s}\|u_L^3\partial_3u_L(t')\|_{L^2}^2\,dt'\lesssim t^{s-\frac12}\|\partial_3u_0\|_{\dot H^{s-1,0}}^2\|u_0\|_{H^s}^2.\tag{3.14}$$

By inserting the estimates (3.10), (3.11), (3.13) and (3.14) into (3.8), we achieve (3.1). This finishes the proof of Proposition 3.1. $\square$

4 The proof of Theorem 1.1

The goal of this section is to present the proof of Theorem 1.1. The key ingredient will be the following proposition:

Proposition 4.1. *Let* $s\in(1/2,1)$ *and* w *be a smooth enough solution of* (3.2) *on* $[0,T^*)$. *Then for any* $t<T^*$, *we have*

$$
\begin{aligned}
&\frac{d}{dt}\|w(t)\|_{\dot{H}^{0,s}}^2+\|\nabla_{\mathrm{h}}w\|_{\dot{H}^{0,s}}^2\\
\leq&\|\nabla u_L\|_{\dot{H}^{0,s}}^2+C\Big(\|\partial_3 u_L\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}^{\frac43}\|u_L\|_{\dot{H}^{0,s}}^{\frac23}\|w\|_{\dot{H}^{0,s}}^{\frac43}\\
&+\big(\|u_L\|_{L^\infty_{\mathrm{v}}(L^4)}^4+\|\nabla_{\mathrm{h}}u_L\|_{L^\infty_{\mathrm{v}}(L^2)}^2+\|\partial_3 u_L\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})}^{\frac43}\big)\big(\|w\|_{\dot{H}^{0,s}}^2+\|u_L\|_{\dot{H}^{0,s}}^2\big)\\
&+\|\nabla_{\mathrm{h}}w\|_{L^2}^2\big[\|w\|_{\dot{H}^{0,s}}^{\frac{4s}{2s-1}}+\big(\|u_L\|_{\dot{H}^{0,s}}\|w\|_{\dot{H}^{0,s}}\big)^{\frac{2s}{2s-1}}+\|w\|_{L^2}^2\|w\|_{\dot{H}^{0,s}}^{\frac{2(2s+1)}{2s-1}}\big]\Big). \qquad (4.1)
\end{aligned}
$$

Proof. By applying the operator Δ_ℓ^{v} to (3.2) and then taking L^2 inner product of the resulting equations with $\Delta_\ell^{\mathrm{v}}w$, we find

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\Delta_\ell^{\mathrm{v}}w(t)\|_{L^2}^2+\|\Delta_\ell^{\mathrm{v}}\nabla_h w\|_{L^2}^2\\
=&-\big(\Delta_\ell^{\mathrm{v}}((u_L+w)\cdot\nabla w)|\Delta_\ell^{\mathrm{v}}w\big)_{L^2}\\
&-\big(\Delta_\ell^{\mathrm{v}}(w\cdot\nabla u_L)|\Delta_\ell^{\mathrm{v}}w\big)_{L^2}-\big(\Delta_\ell^{\mathrm{v}}(u_L\cdot\nabla u_L)|\Delta_\ell^{\mathrm{v}}w\big)_{L^2}. \qquad (4.2)
\end{aligned}
$$

Let us now handle term by term in (4.2). We first observe that

$$
(\Delta_\ell^{\mathrm{v}}(w\cdot\nabla w)|\Delta_\ell^{\mathrm{v}}w)_{L^2}=\Big(\Delta_\ell^{\mathrm{v}}(w^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}w)|\Delta_\ell^{\mathrm{v}}w\Big)_{L^2}+\Big(\Delta_\ell^{\mathrm{v}}(w^3\partial_3 w)|\Delta_\ell^{\mathrm{v}}w\Big)_{L^2}.
$$

Applying Lemma 2.2 yields

$$
\begin{aligned}
\Big|\Big(\Delta_\ell^{\mathrm{v}}(w^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}w)|\Delta_\ell^{\mathrm{v}}w\Big)_{L^2}\Big|\leq&\|\Delta_\ell^{\mathrm{v}}(w^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}w)\|_{L^2_{\mathrm{v}}(L^{\frac43}_{\mathrm{h}})}\|\Delta_\ell^{\mathrm{v}}w\|_{L^2_{\mathrm{v}}(L^4_{\mathrm{h}})}\\
\lesssim&\|\Delta_\ell^{\mathrm{v}}(w^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}w)\|_{L^2_{\mathrm{v}}(L^{\frac43}_{\mathrm{h}})}\|\Delta_\ell^{\mathrm{v}}w\|_{L^2}^{\frac12}\|\Delta_\ell^{\mathrm{v}}\nabla_{\mathrm{h}}w\|_{L^2}^{\frac12}\\
\lesssim&c_\ell^2(t)2^{-2\ell s}\Big(\|w\|_{\dot{H}^{0,s}}\|\nabla_{\mathrm{h}}w\|_{\dot{H}^{0,s}}^{1+\frac{1}{2s}}\|\nabla_{\mathrm{h}}w\|_{L^2}^{1-\frac{1}{2s}}\\
&+\|w\|_{\dot{H}^{0,s}}^{\frac12+\frac{1}{4s}}\|\nabla_{\mathrm{h}}w\|_{\dot{H}^{0,s}}^{\frac32+\frac{1}{4s}}\big(\|w\|_{L^2}\|\nabla_{\mathrm{h}}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\Big).
\end{aligned}
$$

Applying Lemma 2.3 to $\big(\Delta_\ell^{\rm v}(w^3\partial_3 w)|\Delta_\ell^{\rm v}w\big)_{L^2}$ gives rise to the same estimate. Therefore, we achieve

$$\begin{aligned}\big|(\Delta_\ell^{\rm v}(w\cdot\nabla w)|\Delta_\ell^{\rm v}w)_{L^2}\big|\lesssim & c_\ell^2(t)2^{-2\ell s}\Big(\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{1+\frac{1}{2s}}\|\nabla_{\rm h}w\|_{L^2}^{1-\frac{1}{2s}}\\ &+\|w\|_{\dot H^{0,s}}^{\frac12+\frac{1}{4s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac32+\frac{1}{4s}}\big(\|w\|_{L^2}\|\nabla_{\rm h}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}}2^{2\ell s}\big|(\Delta_\ell^{\rm v}(w\cdot\nabla w)|\Delta_\ell^{\rm v}w)_{L^2}\big|\lesssim & \|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{L^2}^{1-\frac{1}{2s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{1+\frac{1}{2s}}\\ &+\|w\|_{\dot H^{0,s}}^{\frac12+\frac{1}{4s}}\big(\|w\|_{L^2}\|\nabla_{\rm h}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac32+\frac{1}{4s}}.\end{aligned}$$

Applying Young's inequality gives rise to

$$\begin{aligned}&\sum_{\ell\in\mathbb{Z}}2^{2\ell s}\big|(\Delta_\ell^{\rm v}(w\cdot\nabla w)|\Delta_\ell^{\rm v}w)_{L^2}\big|\\ \leq & C\big(\|w\|_{\dot H^{0,s}}^{\frac{4s}{2s-1}}+\|w\|_{L^2}^2\|w\|_{\dot H^{0,s}}^{\frac{2(1+2s)}{2s-1}}\big)\|\nabla_{\rm h}w\|_{L^2}^2+\frac18\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^2.\end{aligned}\tag{4.3}$$

Similarly, we write

$$(\Delta_\ell^{\rm v}(u_L\cdot\nabla w)|\Delta_\ell^{\rm v}w)_{L^2}=\Big(\Delta_\ell^{\rm v}(u_L^{\rm h}\cdot\nabla_{\rm h}w)|\Delta_\ell^{\rm v}w\Big)_{L^2}+\Big(\Delta_\ell^{\rm v}(u_L^3\partial_3 w)|\Delta_\ell^{\rm v}w\Big)_{L^2}.$$

By applying the law of product (2.8), we get

$$\begin{aligned}\big|\big(\Delta_\ell^{\rm v}(u_L^h\cdot\nabla_{\rm h}w)|\,\Delta_\ell^{\rm v}w\big)_{L^2}\big|\leq & \|\Delta_\ell^{\rm v}(u_L^{\rm h}\cdot\nabla_{\rm h}w)\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\|\Delta_\ell^{\rm v}w\|_{L^2_{\rm v}(L^4_{\rm h})}\\ \lesssim & c_\ell^2(t)2^{-2\ell s}\Big(\|u_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac32}\\ &+\|u_L\|_{\dot H^{0,s}}^{\frac12}\|\nabla_h u_L\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{L^2}^{1-\frac{1}{2s}}\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac12+\frac{1}{2s}}\Big).\end{aligned}$$

Whereas applying the law of product (2.13) gives

$$\begin{aligned}&\big|\big(\Delta_\ell^{\rm v}(u_L^3\partial_3 w)|\,\Delta_\ell^{\rm v}w\big)_{L^2}\big|\\ \lesssim & c_\ell^2(t)2^{-2\ell s}\Big(\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\\ &+\|\nabla_h u_L\|_{\dot H^{0,s}}\big(\|w\|_{L^2}\|\nabla_{\rm h}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\Big).\end{aligned}$$

As a result, it comes out

$$
\begin{aligned}
&\big|(\Delta_\ell^{\rm v}\,(u_L\cdot\nabla w)|\Delta_\ell^{\rm v} w)_{L^2}\big|\\
\lesssim& c_\ell^2(t)2^{-2\ell s}\Big(\|u_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac32}+\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\\
&+\|u_L\|_{\dot H^{0,s}}^{\frac12}\|\nabla_h u_L\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{L^2}^{1-\frac{1}{2s}}\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac12+\frac{1}{2s}}\\
&+\|\nabla_h u_L\|_{\dot H^{0,s}}\big(\|w\|_{L^2}\|\nabla_{\rm h}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\Big).
\end{aligned}
$$

Then we get, by a similar derivation of (4.3), that

$$
\begin{aligned}
&\sum_{\ell\in\mathbb{Z}}2^{2\ell s}\big|(\Delta_\ell^{\rm v}(u_L\cdot\nabla w)|\Delta_\ell^{\rm v}w)_{L^2}\big|\\
\leq& C\Big(\|\nabla_{\rm h}w\|_{L^2}^2\big[\big(\|u_L\|_{\dot H^{0,s}}\|w\|_{\dot H^{0,s}}\big)^{\frac{2s}{2s-1}}+\|w\|_{L^2}^2\|w\|_{\dot H^{0,s}}^{\frac{2(2s+1)}{2s-1}}\big]\\
&+\big(\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2)}^2+\|u_L\|_{L^\infty_{\rm v}(L^4)}^4\big)\|w\|_{\dot H^{0,s}}^2\Big)+\frac18\big(\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^2+\|\nabla_h u_L\|_{\dot H^{0,s}}^2\big).
\end{aligned}\tag{4.4}
$$

To handle the last terms in (4.2) involving horizontal derivatives, we apply Remark 2.1 to get

$$
\begin{aligned}
&\big|(\Delta_\ell^{\rm v}(w\cdot\nabla_{\rm h}u_L)|\Delta_\ell^{\rm v}w)_{L^2}\big|\\
\leq&\|\Delta_\ell^{\rm v}(w\cdot\nabla_{\rm h}u_L)\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\|\Delta_\ell^{\rm v}w\|_{L^2_{\rm v}(L^4_{\rm h})}\\
\lesssim& c_\ell^2(t)2^{-2\ell s}\Big(\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2)}\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\\
&+\big(\|w\|_{L^2}\|\nabla_{\rm h}w\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\big(\|w\|_{\dot H^{0,s}}\|\nabla_h w\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\|\nabla_{\rm h}u_L\|_{\dot H^{0,s}}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\big|(\Delta_\ell^{\rm v}(u_L\cdot\nabla_{\rm h}u_L)|\Delta_\ell^{\rm v}w)_{L^2}\big|\leq&\|\Delta_\ell^{\rm v}(u_L\cdot\nabla_{\rm h}u_L)\|_{L^2_{\rm v}(L^{\frac43}_{\rm h})}\|\Delta_\ell^{\rm v}w\|_{L^2_{\rm v}(L^4_{\rm h})}\\
\lesssim& c_\ell^2(t)2^{-2\ell s}\Big(\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2)}\|u_L\|_{\dot H^{0,s}}^{\frac12}\|\nabla_h u_L\|_{\dot H^{0,s}}^{\frac12}\\
&+\|u_L\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\nabla_{\rm h}u_L\|_{\dot H^{0,s}}\Big)\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac12}.
\end{aligned}
$$

Then a similar derivation of (4.3) gives rise to

$$
\begin{aligned}
&\sum_{\ell\in\mathbb{Z}}2^{2\ell s}\big|(\Delta_\ell^{\rm v}(w\cdot\nabla_h u_L)|\Delta_\ell^{\rm v}w)_{L^2}\big|\\
\leq& C\Big(\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2)}^2\|w\|_{\dot H^{0,s}}^2+\|w\|_{L^2}^2\|\nabla_{\rm h}w\|_{L^2}^2\|w\|_{\dot H^{0,s}}^{\frac{2(2s+1)}{2s-1}}\Big)\\
&+\frac18\big(\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^2+\|\nabla_{\rm h}u_L\|_{\dot H^{0,s}}^2\big),
\end{aligned}\tag{4.5}
$$

and

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}} 2^{2\ell s}\big|(\Delta_\ell^{\rm v}(u_L\cdot\nabla_{\rm h}u_L)|\Delta_\ell^{\rm v}w)_{L^2}\big|\\
\leq& C\Big(\big(\|u_L\|_{L^\infty_{\rm v}(L^4)}^4+\|\nabla_{\rm h}u_L\|_{L^\infty_{\rm v}(L^2)}^2\big)\|w\|_{\dot H^{0,s}}^2+\|\nabla u_L\|_{L^\infty_{\rm v}(L^2)}^2\|u_L\|_{\dot H^{0,s}}^2\Big)\\
&+\frac{1}{8}\big(\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^2+\|\nabla_{\rm h}u_L\|_{\dot H^{0,s}}^2\big).
\end{aligned}\tag{4.6}$$

To handle the last two terms in (4.2) involving vertical derivative, we need the following lemma:

Lemma 4.1. *Let $a=(a^{\rm h},a^3)$ be a solenoidal vector field. Then we have*

$$\begin{aligned}
\Big|\Big(\Delta_\ell^{\rm v}(a^3\partial_3u_L)\mid\Delta_\ell^{\rm v}b\Big)_{L^2}\Big|\lesssim c_\ell^2 2^{-2\ell s}\Big(&\|a^3\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\partial_3u_L\|_{\dot H^{0,s}}\|b\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}b\|_{\dot H^{0,s}}^{\frac12}\\
&+\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|a^3\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}a^{\rm h}\|_{\dot H^{0,s}}^{\frac12}\|b\|_{\dot H^{0,s}}\Big).
\end{aligned}\tag{4.7}$$

Let us postpone the proof of this lemma till we finish the proof of this proposition.

Applying (2.4) and Lemma 4.1 yields

$$\begin{aligned}
&\Big|\Big(\Delta_\ell^{\rm v}(w^3\cdot\partial_3u_L)|\Delta_\ell^{\rm v}w\Big)_{L^2}\Big|\\
\lesssim& c_\ell^2(t)2^{-2\ell s}\Big(\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|w\|_{\dot H^{0,s}}^{\frac32}\|\nabla_{\rm h}w^{\rm h}\|_{\dot H^{0,s}}^{\frac12}\\
&+\big(\|w^3\|_{L^2}\|\nabla_{\rm h}w^3\|_{L^2}\big)^{\frac12-\frac{1}{4s}}\|\partial_3u_L\|_{\dot H^{0,s}}\big(\|w\|_{\dot H^{0,s}}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}\big)^{\frac12+\frac{1}{4s}}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Big|\Big(\Delta_\ell^{\rm v}(u_L^3\partial_3u_L)|\Delta_\ell^{\rm v}w\Big)_{L^2}\Big|\lesssim c_\ell^2(t)2^{-2\ell s}\Big(&\|u_L^3\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\partial_3u_L\|_{\dot H^{0,s}}\|w\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^{\frac12}\\
&+\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|u_L^3\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h}u_L^{\rm h}\|_{\dot H^{0,s}}^{\frac12}\|w\|_{\dot H^{0,s}}\Big).
\end{aligned}$$

Then a similar derivation of (4.3) gives rise to

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}}2^{2\ell s}\Big|\Big(\Delta_\ell^{\rm v}(w^3\partial_3u_L)|\Delta_\ell^{\rm v}w\Big)_{L^2}\Big|\\
\leq& C\Big(\|\partial_3u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}^{\frac43}\|w\|_{\dot H^{0,s}}^2+\|w^3\|_{L^2}^2\|\nabla_{\rm h}w\|_{L^2}^2\|w\|_{\dot H^{0,s}}^{\frac{2(2s+1)}{2s-1}}\Big)\\
&+\frac18\big(\|\partial_3u_L\|_{\dot H^{0,s}}^2+\|\nabla_{\rm h}w\|_{\dot H^{0,s}}^2\big),
\end{aligned}\tag{4.8}$$

and

$$
\begin{aligned}
&\sum_{\ell\in\mathbb{Z}} 2^{2\ell s}\Big|\Big(\Delta_\ell^{\rm v}(u_L^3\partial_3 u_L)|\Delta_\ell^{\rm v} w\Big)_{L^2}\Big|\\
\leq & C\Big(\|u_L\|_{L^\infty_{\rm v}(L^4)}^4\|w\|_{\dot H^{0,s}}^2+\|\partial_3 u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}^{\frac43}\|u_L\|_{\dot H^{0,s}}^{\frac23}\|w\|_{\dot H^{0,s}}^{\frac43}\Big)\\
&+\frac18\big(\|\nabla_h u_L\|_{\dot H^{0,s}}^2+\|\nabla_{\rm h} w\|_{\dot H^{0,s}}^2+\|\partial_3 u_L\|_{\dot H^{0,s}}^2\big).
\end{aligned}\tag{4.9}
$$

Multiplying (4.2) by $2^{2\ell s}$ and summing up the resulting inequalities over $\ell\in Z$, and then inserting the estimates (4.3), (4.4), (4.5), (4.6), (4.8) and (4.9) into the resulting inequality, we obtain (4.1). This completes the proof. □

Proposition 4.1 is proved provided that we present the proof of Lemma 4.1.

Proof of Lemma 4.1. We first get, by applying Bony's decomposition (2.2) in the vertical variable $a^3\partial_3 u_L$, that

$$
a^3\partial_3 u_L=T^{\rm v}_{a^3}\partial_3 u_L+R^{\rm v}(a^3,\partial_3 u_L).
$$

Considering the support properties to the Fourier transform of the terms in $T^{\rm v}_{a^3}\partial_3 u_L$, we infer

$$
\begin{aligned}
\big|\big(\Delta_\ell^{\rm v}(T^{\rm v}_{a^3}\partial_3 u_L)\mid\Delta_\ell^{\rm v} b\big)_{L^2}\big|
&\lesssim\sum_{|j-\ell|\leq 5}\|S^{\rm v}_{j-1}a^3\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_j^{\rm v}\partial_3 u_L\|_{L^2}\|\Delta_\ell^{\rm v} b\|_{L^2_{\rm v}(L^4_{\rm h})}\\
&\lesssim\sum_{|j-\ell|\leq 5}\|a^3\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\Delta_j^{\rm v}\partial_3 u_L\|_{L^2}\|\Delta_\ell^{\rm v} b\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h} b\|_{L^2}^{\frac12}\\
&\lesssim c_\ell^2 2^{-2\ell s}\|a^3\|_{L^\infty_{\rm v}(L^4_{\rm h})}\|\partial_3 u_L\|_{\dot H^{0,s}}\|b\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h} b\|_{\dot H^{0,s}}^{\frac12}.
\end{aligned}
$$

Along the same line, due to $\partial_3 a^3=-{\rm div}_{\rm h} a^{\rm h}$, we deduce that

$$
\begin{aligned}
&\Big|\Big(\Delta_\ell^{\rm v}(R^{\rm v}(a^3,\partial_3 u_L))\mid\Delta_\ell^{\rm v} b\Big)_{L^2}\Big|\\
\lesssim&\sum_{j\geq\ell-N_0}\|\Delta_j^{\rm v} a^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|S^{\rm v}_{j+2}\partial_3 u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta_\ell^{\rm v} b\|_{L^2}\\
\lesssim&\sum_{j\geq\ell-N_0}\|\Delta_j^{\rm v} a^3\|_{L^2}^{\frac12}\|\Delta_j^{\rm v}\partial_3 a^3\|_{L^2}^{\frac12}\|\partial_3 u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta_\ell^{\rm v} b\|_{L^2}^{\frac12}\|\Delta_\ell^{\rm v}\nabla_{\rm h} b\|_{L^2}^{\frac12}\\
\lesssim& c_\ell^2 2^{-2\ell s}\|\partial_3 u_L\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|a^3\|_{\dot H^{0,s}}^{\frac12}\|\nabla_{\rm h} a^{\rm h}\|_{\dot H^{0,s}}^{\frac12}\|b\|_{\dot H^{0,s}}.
\end{aligned}
$$

This leads to (4.7). □

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The local well-posedness part of Theorem 1.1 has been proved in [4, 11]. Let us denote $T^*(u_0)$ be the maximal existence of such a solution. It remains to prove (1.4). Indeed, we first get, by a similar derivation of (3.10) that

$$\begin{aligned}\int_0^t \|\nabla_{\mathrm{h}} u_L(t')\|^2_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})} dt' &\leq C\int_0^t \Big(\sum_{k\in\mathbb{Z}} e^{-ct'2^{2k}} 2^k \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} u_0\|_{L^2}\Big)^2 dt' \\ &\leq C\int_0^t (t')^{s-\frac{3}{2}} dt' \|u_0\|^2_{\dot{H}^s} \sup_{t'\in[0,t]} \Big(\sum_{k\in\mathbb{Z}} e^{-ct'2^{2k}} (t')^{\frac{3}{4}-\frac{s}{2}} 2^{k\left(\frac{3}{2}-s\right)}\Big)^2 \\ &\leq C_s t^{s-\frac{1}{2}} \|u_0\|^2_{\dot{H}^s}. \end{aligned} \tag{4.10}$$

Now let us define

$$\begin{aligned} &T_1^\star \stackrel{\text{def}}{=} \sup\Big\{ T\leq T^*(u_0);\ \sup_{0\leq t\leq T} \big(\|w(t)\|^2_{\dot{H}^{0,s}} + \|\nabla_{\mathrm{h}} w\|^2_{L^2_t(\dot{H}^{0,s})}\big) \leq 4[u_0]_s^2 \Big\} \\ &\text{with} \qquad [u_0]_s^2 \stackrel{\text{def}}{=} \|u_0\|^2_{\dot{H}^{0,s}} + \|\partial_3 u_0\|^2_{\dot{H}^{s-1,0}} + \|\partial_3 u_0\|^2_{\dot{H}^{-1,s}}. \end{aligned} \tag{4.11}$$

Then for $t\leq T_1^\star$, by virtue of (3.11) and (3.12), we have

$$\int_0^t \|\partial_3 u_L\|^{\frac{4}{3}}_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} \|u_L\|^{\frac{2}{3}}_{\dot{H}^{0,s}} \|w\|^{\frac{4}{3}}_{\dot{H}^{0,s}} dt' \leq C_s t^{\frac{2s-1}{3}} \|\partial_3 u_0\|^{\frac{4}{3}}_{\dot{H}^{s-1,0}} [u_0]_s^2, \tag{4.12}$$

and

$$\begin{aligned} &\int_0^t \big(\|u_L\|^4_{L^\infty_{\mathrm{v}}(L^4)} + \|\nabla_{\mathrm{h}} u_L\|^2_{L^\infty_{\mathrm{v}}(L^2)} + \|\partial_3 u_L\|^{\frac{4}{3}}_{L^\infty_{\mathrm{v}}(L^2)}\big)\big(\|w\|^2_{\dot{H}^{0,s}} + \|u_L\|^2_{\dot{H}^{0,s}}\big) dt' \\ &\leq C_s \big(t^{2s-1}\|u_0\|^4_{\dot{H}^s} + t^{\frac{2s-1}{3}} \|\partial_3 u_0\|^{\frac{4}{3}}_{\dot{H}^{s-1,0}} + t^{s-\frac{1}{2}} \|u_0\|^2_{\dot{H}^s}\big)[u_0]_s^2. \end{aligned} \tag{4.13}$$

In order to deal with the integral of the last term in (4.1), we define

$$T_2^\star \stackrel{\text{def}}{=} \min\Big(T_1^\star, \varepsilon [u_0]_s^{-\frac{1}{\gamma}} \Big). \tag{4.14}$$

Then for $t\leq T_2^\star$, we deduce from Proposition 3.1 that

$$\frac{\|w(t)\|^2_{L^2}}{t^s} + \int_0^t \frac{\|\nabla_{\mathrm{h}} w(t')\|^2_{L^2}}{(t')^s} dt' \leq C t^{s-\frac{1}{2}} [u_0]_s^4. \tag{4.15}$$

As a result, we find

$$\begin{aligned}
&\int_0^t \|\nabla_{\rm h} w\|_{L^2}^2 \big[\|w\|_{\dot H^{0,s}}^{\frac{4s}{2s-1}} + \big(\|u_L\|_{\dot H^{0,s}}\|w\|_{\dot H^{0,s}}\big)^{\frac{2s}{2s-1}} + \|w\|_{L^2}^2\|w\|_{\dot H^{0,s}}^{\frac{2(2s+1)}{2s-1}}\big]\,dt' \\
\leq & C\Big(t^s[u_0]_{\dot H^{0,s}}^{\frac{4s}{2s-1}} + Ct^{2s}[u_0]_{\dot H^{0,s}}^{\frac{2(2s+1)}{2s-1}} \sup_{t'\in[0,t]} \frac{\|w(t')\|_{L^2}^2}{(t')^s}\Big)\int_0^t \frac{\|\nabla_{\rm h} w(t')\|_{L^2}^2}{(t')^s}dt' \\
\leq & C\big(t^{2s-\frac12}[u_0]_s^{\frac{4s}{2s-1}} + t^{4s-1}[u_0]_s^{\frac{12s-2}{2s-1}}\big)[u_0]_s^4.
\end{aligned} \tag{4.16}$$

Let $s=\frac12+2\gamma$ for $\gamma\in(0,1/4)$, for $t\leq T_2^\star$, by integrating (4.1) over $[0,t]$ and then inserting the estimates, (4.12), (4.13) and (4.16), into the resulting inequality, we obtain

$$\begin{aligned}
&\|w(t)\|_{\dot H^{0,s}}^2 + \|\nabla_{\rm h} w\|_{L^2_t(\dot H^{0,s})}^2 \\
\leq & 2\|u_0\|_{\dot H^{0,s}}^2 + \|\partial_3 u_0\|_{\dot H^{-1,s}}^2 \\
&+ C_s\Big(t^{\frac{4\gamma}{3}}[u_0]_s^{\frac43} + t^{2\gamma}[u_0]_s^2 + t^{\frac{1+8\gamma}{2}}[u_0]_s^{\frac{1+8\gamma}{2\gamma}} + t^{1+8\gamma}[u_0]_s^{\frac{1+8\gamma}{\gamma}}\Big)[u_0]_s^2.
\end{aligned} \tag{4.17}$$

Therefore as long as ε is sufficiently small in (4.14), we deduce from (4.17) that

$$\|w(t)\|_{\dot H^{0,s}}^2 + \|\nabla_{\rm h} w\|_{L^2_t(\dot H^{0,s})}^2 \leq 3[u_0]_s^2 \quad \text{for} \quad t\leq T_2^\star. \tag{4.18}$$

Then a standard continuous argument shows that $T_1^\star \geq \varepsilon[u_0]_s^{-\frac{1}{\gamma}}$. Hence we conclude that

$$T^*(u_0) \geq \varepsilon[u_0]_s^{-\frac{1}{\gamma}}.$$

This completes the proof of Theorem 1.1. $\square$

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