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THE STABILITY OF A REACTION-DIFFUSION LOTKA-VOLTERRA COMPETITIVE SYSTEM WITH NONLOCAL DELAYS AND FEEDBACK CONTROLS*†

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Abstract

In this paper, we consider a Lotka-Volterra competitive system with nonlocal delays and feedback controls. Using the Lyapunov functional and iterative technique method, we investigate the global stability and extinction of the system. Also, we show the influence of feedback controls on dynamic behaviors of the system. Some examples are presented to verify our main results.

Keywords extinction; feedback control; reaction-diffusion; Lotka-Volterra; nonlocal delays

2000 Mathematics Subject Classification 35K57; 35B35; 35B40; 35Q92

1 Introduction and Main Results

In this paper, we consider the following reaction-diffusion Lotka-Volterra competitive system with nonlocal delays and feedback controls

$$
\begin{aligned}
&\frac{\partial u_1}{\partial t}-D_1\Delta u_1=u_1\left(b_1-a_{11}u_1-a_{12}\int_{-\infty}^{t}\int_0^{\pi}G_1(x,y,t-s)f_1(t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s-c_1v_1\right),\\
&\frac{\partial u_2}{\partial t}-D_2\Delta u_2=u_2\left(b_2-a_{21}\int_{-\infty}^{t}\int_0^{\pi}G_2(x,y,t-s)f_2(t-s)u_1(s,y)\mathrm{d}y\mathrm{d}s-a_{22}u_2-c_2v_2\right),\\
&\frac{\partial v_1}{\partial t}-D_3\Delta v_1=-e_1v_1+d_1u_1,\\
&\frac{\partial v_2}{\partial t}-D_4\Delta v_2=-e_2v_2+d_2u_2, \qquad (1.1)
\end{aligned}
$$

for $t>0,\ x\in(0,\pi)$, under the homogeneous Neumann boundary conditions

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$$\frac{\partial u_i}{\partial x} = \frac{\partial v_i}{\partial x} = 0, \quad t > 0,\ x = 0, \pi,\ i = 1, 2, \tag{1.2}$$

and initial conditions

$$\begin{aligned} &u_i(\theta, x) = \phi_i(\theta, x) \geq 0, \quad (\theta, x) \in (-\infty, 0] \times [0, \pi], \\ &v_i(0, x) = \psi_i(0, x) \geq 0, \quad x \in (0, \pi),\ i = 1, 2. \end{aligned} \tag{1.3}$$

In system (1.1), u_i denotes the population density of the i-th species; v_i denotes the feedback control variable; b_1 and b_2 are the intrinsic growth rates; a_{11} and a_{22} are the rates of the intra-specific competition of the first and second species respectively; a_{12} and a_{21} are the rates of the inter-specific competition of the first and second species respectively; c_i, e_i, d_i are coefficients of the feedback control variable; D_i is the diffusion rate. All the parameters in system (1.1) are positive constants. The boundary conditions (1.2) imply that the populations and feedback control variable do not move across the boundary $x = 0, \pi$. We assume that the kernel $G_i(x, y, t)f_i(t)$ depends on both the spatial and the temporal variables. The delay in this type of model formulation is called a spatio-temporal delay or nonlocal delay (as we shall show below how G_i are chosen).

The following two-species autonomous competitive system

$$\begin{aligned} x_1'(t) &= x_1(t)(b_1 - a_{11}x_1(t) - a_{12}x_2(t)), \\ x_2'(t) &= x_2(t)(b_2 - a_{21}x_1(t) - a_{22}x_2(t)), \end{aligned} \tag{1.4}$$

where $b_i, a_{ij},\ i, j = 1, 2$ are positive constants, has been discussed in many books on mathematical ecology (for example [1]). If the coefficients of system (1.4) satisfy $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, then system (1.4) has a unique positive equilibrium $(\overline{x}_1, \overline{x}_2)$ which is globally attractive, that is, all positive solutions of system (1.4) satisfy $\lim\limits_{t\to+\infty}(x_1(t), x_2(t)) = (\overline{x}_1, \overline{x}_2)$. If the coefficients of system (1.4) satisfy $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}},\ \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, then system (1.4) is extinct, that is, all positive solutions of system (1.4) satisfy $\lim\limits_{t\to+\infty}(x_1(t), x_2(t)) = (\frac{b_1}{a_{11}}, 0)$.

In [2], the authors argued that in some situation, the equilibrium is not the desirable one (or affordable) and a smaller value is required, which can be explained logically especially in a food limited environment since the circumstance can only withstand a certain amount of populations. Thus we must alter the system structurally by introducing a feedback control variable (Aizerman and Gantmacher [3] or Lefschetz [4]). On the other hand, ecosystem in the real world are continuously disturbed by unpredictable forces which can result in some changes of the biological parameters such as survival rates. We call the disturbance functions to be control variables. Gopalsamy and Weng [5] introduced a feedback control variable into a two species competitive system and discussed the existence of the globally attractive positive equilibrium of the system with feedback controls. For more details in this direction, please see [6-9].

In [10], Li, Han and Chen studied the following two-species autonomous Lotka-Volterra competitive system with infinite delays and feedback controls:

$$\begin{aligned}
x_1'(t) &= x_1(t)\biggl(b_1 - a_{11}x_1(t) - a_{12}\int_0^{+\infty} K_1(s)x_2(t-s)\mathrm{d}s - c_1u_1(t)\biggr),\\
x_2'(t) &= x_2(t)\biggl(b_2 - a_{21}\int_0^{+\infty} K_2(s)x_1(t-s)\mathrm{d}s - a_{22}x_2(t) - c_2u_2(t)\biggr),\\
u_1'(t) &= -e_1u_1(t) + d_1x_1(t),\\
u_2'(t) &= -e_2u_2(t) + d_2x_2(t),
\end{aligned}\tag{1.5}$$

where $b_i, a_{ij}, c_i, e_i, d_i,\ i,j = 1,2$, are positive constants; $x_i(t)$ denotes the density of the population x_i; $u_i(t)$ denotes the feedback control variable. By constructing suitable Lyapunov functional, the authors investigated the extinction and global stability of the equilibriums, and showed that the suitable feedback controls can retain or change the stability of system (1.5).

However, as argued in [11], in many ecological systems, the species under consideration may disperse spatially as well as evolve in time. This spatial dispersal or diffusion arises from the natural tendency of each species to diffuse to areas of lower population density. The role of diffusion in the ecological system has been extensively studied in [12-18].

In more realistic ecological models, any delays should be spatially inhomogeneous, that is, the delay affects both the temporal and spatial variables, due to the fact that any given individual may not necessarily have been at the same spatial location at the previous times. Such delays are called a spatio-temporal delay or nonlocal delay. In [19], Gourley and So considered the following food-limited reaction-diffusion population model with nonlocal delay

$$\frac{\partial u}{\partial t} - D\Delta u = u\left(\frac{1 - au - b(f*u)}{1 + acu + bc(f*u)}\right), \quad x \in (0,\pi),\ t \ge 0 \tag{1.6}$$

with homogeneous Neumann boundary conditions $\frac{\partial u}{\partial x} = 0,\ x = 0, \pi$, where the convolution $f*u$ is defined by

$$f*u = \int_{-\infty}^{t}\int_0^{\pi} G(x,y,t-s)f(t-s)u(y,s)\mathrm{d}y\mathrm{d}s,$$

here

$$G(x,y,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \mathrm{e}^{-Dn^2t}\cos nx \sin ny$$

is solution of

$$\frac{\partial G}{\partial t} = D\frac{\partial^2 G}{\partial y^2}$$

subject to

$$\frac{\partial G}{\partial y} = 0 \text{ at } y = 0, \pi, \text{ and } G(x, y, 0) = \delta(x - y);$$

the function $f(t)$ in (1.6) is called the delay kernel and satisfies $f(t) \geq 0$ for all $t \geq 0$ together with the normalization condition $\int_0^{+\infty} f(t)\mathrm{d}t = 1$. The authors [19] studied the the linear stability, boundedness, global convergence of solutions and bifurcations of system (1.6).

Gourley and Ruan [20] considered a two-species competition model described by a reaction-diffusion system with nonlocal delays. Using the energy function method, they studied the extinction and stability of the equilibria of the system. By employing linear chain techniques and geometric singular perturbation theory, they investigated the existence of traveling front solutions of the system.

Motivated by the works of Gourley and So [19] and Gourley and Ruan [20], in this paper, we discuss the extinction and global stability of a reaction-diffusion Lotka-Volterra competitive system (1.1), and show the effect of nonlocal delay and feedback control on system (1.1), that is, feedback control can retain or change the stability of system (1.1).

In system (1.1), we assume that

$$G_i(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \mathrm{e}^{-D_i n^2 t} \cos nx \sin ny \tag{1.7}$$

is the weight function describing the distribution at the past times of the individual of the species u_i at position x and time t, and satisfies $\frac{\partial G_i}{\partial t} = D_i \frac{\partial^2 G_i}{\partial y^2}$ subject to $\frac{\partial G_i}{\partial y} = 0$ at $y = 0, \pi$, and $G_i(x, y, 0) = \delta(x - y)$, $\int_0^{\pi} G_i(x, y, t)\mathrm{d}y = 1$, $i = 1, 2$; the delay kernel $f_i(t)$ satisfies

$$f_i(t) \geq 0, \quad t \geq 0, \quad \int_0^{+\infty} f_i(t)\mathrm{d}t = 1, \quad i = 1, 2. \tag{1.8}$$

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we study the extinction and global stability of system (1.1). To illustrate the feasibility of our main results, Section 4 is devoted to giving some numerical simulations. At last, we give a brief discussion of our result.

2 Preliminaries

In this section, we present some preliminary results required in the sequel.

Let $R = (-\infty, +\infty)$, $\Omega = (0, \pi)$. For $1 \leq p \leq \infty$, let $L^p(\Omega)$ denote the Banach space of Lesbegue measurable functions u on Ω satisfying

$$\|u\|_p = \begin{cases} \left(\displaystyle\int_\Omega |u(x)|^p \mathrm{d}x\right)^{1/p} < \infty, & 1 \le p < \infty; \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty, & p = \infty. \end{cases}$$

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with the usual inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2^2 = \langle\cdot,\cdot\rangle$. Let $|||\cdot|||_2$ denote the norm in $L^2((0,T);L^2(\Omega;R))$, that is,

$$|||u|||_2 = \left(\int_0^{\mathrm{T}} \|u(s)\|_2^2 \mathrm{d}s\right)^{1/2}.$$

Further, for $m \in N$, $1 \le p \le +\infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : \text{for any } |\alpha| \le m,\ \partial_x^\alpha f \in L^p(\Omega)\},$$

where $\alpha = (\alpha_1,\cdots,\alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the derivatives $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} f$ are taken in a weak sense. When endowed with the norm

$$\|f\|_{m,p,\Omega} = \sum_{|\alpha|\le m} \|\partial_x^\alpha f\|_p,$$

$W^{m,p}(\Omega)$ is a Banach space (see, for example [21]).

It can be easily seen that $(0,0,0,0)$ and (M_1,M_2,N_1,N_2) are a pair of coupled upper and lower solutions of problem (1.1)-(1.3), where

$$M_i = \max\left\{\frac{b_i}{a_{ii}}, \|\phi_i\|\right\}, \quad N_i = \max\left\{\frac{d_i}{e_i}M_i, \|\psi_i\|\right\}, \quad i = 1,2,$$

with

$$\|\phi_i\| = \sup_{(t,x)\in(-\infty,0]\times[0,\pi]} |\phi_i(\theta,x)|, \quad \|\psi_i\| = \sup_{x\in[0,\pi]} |\psi_i(0,x)|, \quad i = 1,2.$$

Hence, the global existence of solutions $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ of (1.1)-(1.3) can be derived based on the theory of upper-lower solution pairs (see, for example, Redlinger [22] or Pao [23]). It follows that $0 \le u_i(t,x) \le M_i$, $0 \le v_i(t,x) \le N_i$ $(i = 1,2)$ for $(t,x) \in R\times[0,\pi]$. In addition, if $\phi_i(0,x) \not\equiv 0$, $\psi_i(0,x) \not\equiv 0$ $(i = 1,2)$, then it follows from the strong maximum principle that $u_i(t,x) > 0$, $v_i(t,x) > 0$ $(i = 1,2)$ for all $t > 0$, $x \in [0,\pi]$.

By simple computation, system (1.1) has a trivial steady state solution $E_0(0,0,0,0)$, two semi-trivial steady state solutions

$$E_1(u_1^{**},0,v_1^{**},0) = \left(\frac{b_1e_1}{a_{11}e_1+c_1d_1}, 0, \frac{b_1d_1}{a_{11}e_1+c_1d_1}, 0\right),$$

$$E_2(0,u_2^{**},0,v_2^{**}) = \left(0, \frac{b_2e_2}{a_{22}e_2+c_2d_2}, 0, \frac{b_2d_2}{a_{22}e_2+c_2d_2}\right).$$

If
$$\frac{a_{11}e_1+c_1d_1}{a_{21}e_1} > \frac{b_1}{b_2} > \frac{a_{12}e_2}{a_{22}e_2+c_2d_2}, \tag{H$_1$}$$
then system (1.1) has a unique positive steady state $E^*(u_1^*, u_2^*, v_1^*, v_2^*)$, where
$$u_1^* = \frac{e_1(b_1(a_{22}e_2+c_2d_2)-b_2a_{12}e_2)}{(a_{11}e_1+c_1d_1)(a_{22}e_2+c_2d_2)-a_{12}a_{21}e_1e_2};$$
$$u_2^* = \frac{e_2(b_2(a_{11}e_1+c_1d_1)-b_1a_{21}e_1)}{(a_{11}e_1+c_1d_1)(a_{22}e_2+c_2d_2)-a_{12}a_{21}e_1e_2};$$
$$v_1^* = \frac{d_1u_1^*}{e_1}; \quad v_2^* = \frac{d_2u_2^*}{e_2}.$$

Lemma 2.1 *Let $(u_1(t,x), u_2(t,x), v_1(t,x), v_2(t,x))$ be a solution of system* (1.1) *with the boundary conditions* (1.2) *and initial conditions* (1.3) *satisfying $\phi_i(0,x) \not\equiv 0$ and $\psi_i(0,x) \not\equiv 0$, $i=1,2$. Then*
$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_i(t,x) \le \frac{b_i}{a_{ii}}, \quad \limsup_{t\to+\infty} \max_{x\in[0,\pi]} v_i(t,x) \le \frac{d_ib_i}{e_ia_{ii}}, \quad i=1,2.$$

Proof It follows from the first and second equations of system (1.1) that
$$\frac{\partial u_i}{\partial t} - D_i\Delta u_1 \le u_i(t,x)(b_i - a_{ii}u_i(t,x)), \quad i=1,2.$$
Let $z_i(t)$ be a solution of the following ordinary differential equation
$$z_i'(t) = z_i\left(b_i - a_{ii}z_i\right), \quad z_i(0) = \max_{x\in[0,\pi]} u_i(0,x), \quad i=1,2.$$
It is easy to see that $\lim\limits_{t\to+\infty} z_i(t) = \frac{b_i}{a_{ii}}$, $i=1,2$. From the comparison principle, we obtain $u_i(t,x) \le z_i(t)$, hence
$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_i(t,x) \le \frac{b_i}{a_{ii}}, \quad i=1,2.$$

From the above inequalities, for any $\varepsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that for any $x \in [0,\pi]$ and $t \ge T_1$, $u_i(t,x) \le \frac{b_i}{a_{ii}} + \varepsilon$. Therefore, if follow from the third and fourth equations of system (1.1) that
$$\frac{\partial v_i}{\partial t} - D_{i+2}\Delta v_i \le -e_iv_i(t,x) + d_i\left(\frac{b_i}{a_{ii}}+\varepsilon\right), \quad x\in[0,\pi],\ t\ge T_1,\ i=1,2.$$
Let $w_i(t)$ be a solution of the following ordinary differential equation
$$w_i'(t) = -e_iv_i + d_i\left(\frac{b_i}{a_{ii}}+\varepsilon\right), \quad w_i(T_1) = \max_{x\in[0,\pi]} u_i(T_1,x), \quad i=1,2.$$
It is easy to see that
$$\lim_{t\to+\infty} w_i(t) = \frac{d_i}{e_i}\left(\frac{b_i}{a_{ii}}+\varepsilon\right), \quad i=1,2.$$

It follows from the comparison principle that $v_i(t,x) \le w_i(t)$, then

$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} v_i(t,x) \le \frac{d_i}{e_i}\left(\frac{b_i}{a_{ii}}+\varepsilon\right), \quad i=1,2.$$

Setting $\varepsilon \to 0$, we obtain

$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} v_i(t,x) \le \frac{d_i b_i}{e_i a_{ii}}, \quad i=1,2.$$

This ends the proof of Lemma 2.1.

Similar to the proof of Lemma 2.5 in [14], we obtain the following lemma.

Lemma 2.2 *If $u(t,x)$ is a bounded nonnegative function, where $(t,x)\in(0,+\infty)\times(0,\pi)$, and $G_i(x,y,t), f_i(t)$, $i=1,2$ are defined by* (1.7) *and* (1.8) *respectively, then for $i=1,2$,*

$$\begin{aligned}\liminf_{t\to+\infty} \min_{x\in[0,\pi]} u(t,x) &\le \liminf_{t\to+\infty} \int_{-\infty}^{t}\int_0^{\pi} G_i(x,y,t-s) f_i(t-s) u(s,y)\mathrm{d}y\mathrm{d}s \\ &\le \limsup_{t\to+\infty} \int_{-\infty}^{t}\int_0^{\pi} G_i(x,y,t-s) f_i(t-s) u(s,y)\mathrm{d}y\mathrm{d}s \\ &\le \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u(t,x),\end{aligned}$$

uniformly for $x\in[0,\pi]$.

Lemma 2.3 *Let $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying $\phi_i(0,x)\not\equiv 0$ and $\psi_i(0,x)\not\equiv 0$, $i=1,2$. Assume that $\frac{b_1}{b_2}\big(1-\frac{c_1 d_1}{e_1 a_{11}}\big) > \frac{a_{12}}{a_{22}}$ holds, then there exists an $\alpha>0$ such that $\liminf\limits_{t\to+\infty} \min\limits_{x\in[0,\pi]} u_1(t,x) \ge \alpha$.*

Proof Note that $\frac{b_1}{b_2}\big(1-\frac{c_1 d_1}{e_1 a_{11}}\big) > \frac{a_{12}}{a_{22}}$, then we have $b_1 - a_{12}\frac{b_2}{a_{22}} - c_1\frac{d_1 b_1}{e_1 a_{11}} > 0$. Then for given $\varepsilon>0$ sufficiently small, we obtain

$$b_1 - a_{12}\left(\frac{b_2}{a_{22}}+\varepsilon\right) - c_1\left(\frac{d_1 b_1}{e_1 a_{11}}+\varepsilon\right) > 0. \tag{2.1}$$

It follows from Lemma 2.1 that

$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_2(t,x) \le \frac{b_2}{a_{22}}, \qquad \limsup_{t\to+\infty} \max_{x\in[0,\pi]} v_1(t,x) \le \frac{d_1 b_1}{e_1 a_{11}}.$$

According to Lemma 2.2, we obtain

$$\limsup_{t\to+\infty} \int_{-\infty}^{t}\int_0^{\pi} G_1(x,y,t-s) f_1(t-s) u_2(s,y)\mathrm{d}y\mathrm{d}s \le \limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_2(t,x) \le \frac{b_2}{a_{22}}$$

uniformly for $x\in[0,\pi]$.

Hence, for $\varepsilon>0$ sufficiently small satisfying (2.1), there is a $t_1>0$ such that

$$\int_{-\infty}^{t}\int_{0}^{\pi} G_1(x,y,t-s)f_1(t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s \le \frac{b_2}{a_{22}}+\varepsilon,$$
$$v_1(t,x) \le \frac{d_1b_1}{e_1a_{11}}+\varepsilon, \quad t>t_1,\ x\in[0,\pi].$$

From the above inequalities and the first equation of system (1.1), for $x\in[0,\pi],\ t\ge t_1$, we have

$$\frac{\partial u_1}{\partial t}-D_1\Delta u_1 \ge u_1(t,x)\left[b_1-a_{12}\left(\frac{b_2}{a_{22}}+\varepsilon\right)-c_1\left(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\right)-a_{11}u_1(t,x)\right].$$

From (2.1), a standard comparison argument shows that

$$\liminf_{t\to+\infty}\min_{x\in[0,\pi]} u_1(t,x) \ge \frac{b_1-a_{12}\big(\frac{b_2}{a_{22}}+\varepsilon\big)-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\big)}{a_{11}}>0.$$

Setting $\varepsilon\to 0$, it follows that

$$\liminf_{t\to+\infty}\min_{x\in[0,\pi]} u_1(t,x) \ge \frac{b_1-a_{12}\frac{b_2}{a_{22}}-c_1\frac{d_1b_1}{e_1a_{11}}}{a_{11}} \stackrel{\text{def}}{=} \alpha>0.$$

This ends the proof of Lemma 2.3.

Lemma 2.4 *Let* $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ *be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_i(0,x)\not\equiv 0$ *and* $\psi_i(0,x)\not\equiv 0,\ i=1,2$. *Assume that* $\frac{c_1d_1}{e_1a_{11}}<1$ *and* $\lim\limits_{t\to+\infty}u_2(t,x)=0$ *uniformly for* $x\in[0,\pi]$, *then*

$$\lim_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))=(u_1^{**},0,v_1^{**},0)$$

uniformly for $x\in[0,\pi]$.

Proof Note that $\lim\limits_{t\to+\infty}u_2(t,x)=0$ uniformly for $x\in[0,\pi]$, then it follows from Lemma 2.2 that there exist τ_n $(\tau_n<\tau_{n+1})$, $n=1,2,\cdots$, such that

$$0<\int_{-\infty}^{t}\int_{0}^{\pi} G_1(x,y,t-s)f_1(t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s \le \frac{\varepsilon}{n}, \quad \text{for any } t>\tau_n,\ x\in[0,\pi]. \tag{2.2}$$

From Lemma 2.1, for any $\varepsilon>0$ sufficiently small, there exists a $T_1>0$ such that for any $x\in[0,\pi]$ and $t\ge T_1$,

$$u_1(t,x)\le \frac{b_1}{a_{11}}+\varepsilon \stackrel{\text{def}}{=} \overline{u}_1^{(1)}, \quad v_1(t,x)\le \frac{d_1\overline{u}_1^{(1)}}{e_1}+\varepsilon \stackrel{\text{def}}{=} \overline{v}_1^{(1)}. \tag{2.3}$$

From the above inequalities, (2.2) and the first equation of system (1.1), for $x\in[0,\pi],\ t\ge t_1=\max\{T_1,\tau_1\}$, we have

$$\frac{\partial u_1}{\partial t}-D_1\Delta u_1 \ge u_1(t,x)(b_1-a_{12}\varepsilon-c_1\overline{v}_1^{(1)}-a_{11}u_1(t,x)).$$

Let $w_1(t)$ be a solution of the following ordinary differential equation

$$w_1'(t) = w_1(b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)} - a_{11}w_1), \quad w_1(t_1) = \min_{x\in[0,\pi]} u_1(t_1, x), \quad t \geq t_1.$$

For any $\varepsilon > 0$ sufficiently small, it follows from $\frac{c_1 d_1}{e_1 a_{11}} < 1$ that

$$b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)} > 0, \tag{2.4}$$

then $\lim\limits_{t\to+\infty} w_1(t) = \frac{b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)}}{a_{11}}$. From the comparison principle, we obtain $u_1(t,x) \geq w_1(t)$. Then

$$\liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_1(t,x) \geq \frac{b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)}}{a_{11}}.$$

Hence, for any $\varepsilon > 0$ sufficiently small, there exists a $T_2' \geq t_1$ such that for any $x \in [0,\pi]$ and $t \geq T_2'$,

$$u_1(t,x) \geq \frac{b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)}}{a_{11}} - \varepsilon \stackrel{\text{def}}{=} \underline{u}_1^{(1)} > 0. \tag{2.5}$$

It follows from (2.5) and the third equation of system (1.1) that

$$\frac{\partial v_1}{\partial t} - D_3\Delta v_1 \geq -e_1 v_1(t,x) + d_1\underline{u}_1^{(1)}, \quad t \geq T_2'.$$

Let $p_1(t)$ be a solution of the following ordinary differential equation

$$p_1'(t) = -e_1 p_1(t) + d_1\underline{u}_1^{(1)}, \quad p_1(T_2') = \min_{x\in[0,\pi]} v_1(T_2', x), \quad t \geq T_2'.$$

Then solutions of the above equality satisfy $\lim\limits_{t\to+\infty} p_1(t) = \frac{d_1\underline{u}_1^{(1)}}{e_1}$. By the comparison theorem, we have $u_1(t,x) \geq p_1(t)$. Then

$$\liminf_{t\to+\infty} \min_{x\in[0,\pi]} v_1(t,x) \geq \frac{d_1\underline{u}_1^{(1)}}{e_1}.$$

Hence, for any $\varepsilon > 0$ sufficiently small, there exists a $T_2 \geq T_2'$ such that for any $x \in [0,\pi]$ and $t \geq T_2$,

$$v_1(t,x) \geq i\frac{d_1\underline{u}_1^{(1)}}{e_1} - \varepsilon \stackrel{\text{def}}{=} \underline{v}_1^{(1)} > 0. \tag{2.6}$$

By (2.2), (2.6) and the first equation of system (1.1), we have

$$\frac{\partial u_1}{\partial t} - D_1\Delta u_1 \leq u_1(b_1 - c_1\underline{v}_1^{(1)} - a_{11}u_1), \quad t \geq T_2.$$

It follows from (2.3), (2.4) and (2.6) that

$$b_1 - c_1\underline{v}_1^{(1)} \geq b_1 - a_{12}\varepsilon - c_1\overline{v}_1^{(1)} > 0.$$

Therefore, by the similar arguments as above, we have

$$\limsup_{t\to+\infty}\max_{x\in[0,\pi]} u_1(t,x)\le\frac{b_1-c_1\underline{v}_1^{(1)}}{a_{11}}.$$

For any $\varepsilon>0$ sufficiently small, there exists a $T_3'>T_2$ such that for any $x\in[0,\pi]$ and $t\ge T_3'$,

$$u_1(t,x)\le\frac{b_1-c_1\underline{v}_1^{(1)}}{a_{11}}+\frac{\varepsilon}{2}\overset{\text{def}}{=}\overline{u}_1^{(2)}. \tag{2.7}$$

If follows from (2.7) and the third equation of system (1.1) that

$$\frac{\partial v_1}{\partial t}-D_3\Delta v_1\le -e_1v_1(t,x)+d_1\overline{u}_1^{(2)},\quad x\in[0,\pi],\ t\ge T_3'.$$

By the similar arguments as above, we have

$$\limsup_{t\to+\infty}\max_{x\in[0,\pi]} v_1(t,x)\le\frac{d_1\overline{u}_1^{(2)}}{e_1}.$$

Hence, for any $\varepsilon>0$ sufficiently small, there exists a $T_3>T_3'$ such that for any $x\in[0,\pi]$ and $t\ge T_3$,

$$v_1(t,x)\le\frac{d_1\overline{u}_1^{(2)}}{e_1}+\frac{\varepsilon}{2}\overset{\text{def}}{=}\overline{v}_1^{(2)}. \tag{2.8}$$

From (2.2), (2.8) and the first equation of system (1.1), there exists a $t_2=\max\{T_3,\tau_2\}$ such that for any $x\in[0,\pi]$

$$\frac{\partial u_1}{\partial t}-D_1\Delta u_1\ge u_1\Big(b_1-a_{12}\frac{\varepsilon}{2}-c_1\overline{v}_1^{(2)}-a_{11}u_1\Big),\quad t\ge t_2.$$

It follows from (2.3), (2.4), (2.7) and (2.8) that

$$b_1-a_{12}\frac{\varepsilon}{2}-c_1\overline{v}_1^{(2)}\ge b_1-a_{12}\varepsilon-c_1\overline{v}_1^{(1)}>0.$$

By the similar arguments as above, one has

$$\liminf_{t\to+\infty}\min_{x\in[0,\pi]} u_1(t,x)\ge\frac{b_1-a_{12}\frac{\varepsilon}{2}-c_1\overline{v}_1^{(2)}}{a_{11}}.$$

Therefore, for any $\varepsilon>0$ sufficiently small, there exists a $T_4'\ge t_2$ such that for any $x\in[0,\pi]$ and $t\ge T_4'$,

$$u_1(t,x)\ge\frac{b_1-a_{12}\frac{\varepsilon}{2}-c_1\overline{v}_1^{(1)}}{a_{11}}-\frac{\varepsilon}{2}\overset{\text{def}}{=}\underline{u}_1^{(2)}>0. \tag{2.9}$$

If follows from (2.9) and the third equation of system (1.1) that

$$\frac{\partial v_1}{\partial t}-D_3\Delta v_1\ge -e_1v_1(t,x)+d_1\underline{u}_1^{(2)},\quad x\in[0,\pi],\ t\ge T_4'.$$

By the similar arguments as above, we have

$$\liminf_{t\to+\infty}\min_{x\in[0,\pi]} v_1(t,x)\ge \frac{d_1\underline{u}_1^{(2)}}{e_1}.$$

Hence, for any $\varepsilon>0$ sufficiently small, there exists a $T_4\ge T_4'$ such that for any $x\in[0,\pi]$ and $t\ge T_4$,

$$v_1(t,x)\ge \frac{d_1\underline{u}_1^{(2)}}{e_1}-\frac{\varepsilon}{2}\overset{\text{def}}{=}\underline{v}_1^{(2)}>0. \tag{2.10}$$

Obviously, from (2.3), (2.4) and (2.5)-(2.10), for any $x\in[0,\pi]$ and $t\ge T_4$, we have

$$0<\underline{u}_1^{(1)}<\underline{u}_1^{(2)}<u_1(t,x)<\overline{u}_1^{(2)}<\overline{u}_1^{(1)},\quad 0<\underline{v}_1^{(1)}<\underline{v}_1^{(2)}<v_1(t,x)<\overline{v}_1^{(2)}<\overline{v}_1^{(1)}.$$

Repeating the above procedure, we get four sequences $\overline{u}_1^{(n)},\underline{u}_1^{(n)},\overline{v}_1^{(n)}$ and $\underline{v}_1^{(n)}$, $n=1,2,\cdots$, such that for $n\ge 2$

$$\begin{aligned}
\overline{u}_1^{(n)}&=\frac{b_1-c_1\underline{v}_1^{(n-1)}}{a_{11}}+\frac{\varepsilon}{n},\quad \overline{v}_1^{(n)}=\frac{d_1\overline{u}_1^{(n)}}{e_1}+\frac{\varepsilon}{n},\\
\underline{u}_1^{(n)}&=\frac{b_1-a_{12}\frac{\varepsilon}{n}-c_1\overline{v}_1^{(n)}}{a_{11}}-\frac{\varepsilon}{n},\quad \underline{v}_1^{(n)}=\frac{d_1\underline{u}_1^{(n)}}{e_1}-\frac{\varepsilon}{n}.
\end{aligned} \tag{2.11}$$

Clearly, we have

$$\underline{u}_1^{(n)}<u_1(t,x)<\overline{u}_1^{(n)},\quad \underline{v}_1^{(n)}<v_1(t,x)<\overline{v}_1^{(n)},\quad \text{for any } x\in[0,\pi],\ t\ge T_{2n}.$$

We claim that the sequences $\overline{u}_1^{(n)},\overline{v}_1^{(n)}$ are non-increasing, and the sequences $\underline{u}_1^{(n)},\underline{v}_1^{(n)}$ are non-decreasing. To prove this claim, we will carry out by induction. Firstly, we immediately get

$$\overline{u}_1^{(2)}<\overline{u}_1^{(1)},\ \ \overline{v}_1^{(2)}<\overline{v}_1^{(1)},\ \ \underline{u}_1^{(1)}<\underline{u}_1^{(2)},\ \ \underline{v}_1^{(1)}<\underline{v}_1^{(2)}.$$

Assume that our claim is true for n, that is,

$$\overline{u}_1^{(n)}<\overline{u}_1^{(n-1)},\ \ \overline{v}_1^{(n)}<\overline{v}_1^{(n-1)},\ \ \underline{u}_1^{(n-1)}<\underline{u}_1^{(n)},\ \ \underline{v}_1^{(n-1)}<\underline{v}_1^{(n)}.$$

After a tedious but straightforward computation, we obtain that

$$\begin{aligned}
\overline{u}_1^{(n+1)}&=\frac{b_1-c_1\underline{v}_1^{(n)}}{a_{11}}+\frac{\varepsilon}{n+1}<\frac{b_1-c_1\underline{v}_1^{(n-1)}}{a_{11}}+\frac{\varepsilon}{n}=\overline{u}_1^{(n)};\\
\overline{v}_1^{(n+1)}&=\frac{d_1\overline{u}_1^{(n+1)}}{e_1}+\frac{\varepsilon}{n+1}<\frac{d_1\overline{u}_1^{(n)}}{e_1}+\frac{\varepsilon}{n}=\overline{v}_1^{(n)};\\
\underline{u}_1^{(n+1)}&=\frac{b_1-a_{12}\frac{\varepsilon}{n+1}-c_1\overline{v}_1^{(n)}}{a_{11}}-\frac{\varepsilon}{n+1}>\frac{b_1-a_{12}\frac{\varepsilon}{n}-c_1\overline{v}_1^{(n)}}{a_{11}}-\frac{\varepsilon}{n}=\underline{u}_1^{(n)};\\
\underline{v}_1^{(n+1)}&=\frac{d_1\underline{u}_1^{(n+1)}}{e_1}-\frac{\varepsilon}{n+1}>\frac{d_1\underline{u}_1^{(n)}}{e_1}-\frac{\varepsilon}{n}=\underline{v}_1^{(n)}.
\end{aligned}$$

Hence, the limits of $\overline{u}_1^{(n)},\underline{u}_1^{(n)},\overline{v}_1^{(n)}$ and $\underline{v}_1^{(n)}$, $n=1,2,\cdots$, exist. Denote that

$$\lim_{n\to+\infty}\overline{u}_1^{(n)}=\overline{u}_1,\quad \lim_{n\to+\infty}\underline{u}_1^{(n)}=\underline{u}_1,\quad \lim_{n\to+\infty}\overline{v}_1^{(n)}=\overline{v}_1,\quad \lim_{n\to+\infty}\underline{v}_1^{(n)}=\underline{v}_1.$$

Then $\overline{u}_1\geq\underline{u}_1$, $\overline{v}_1\geq\underline{v}_1$. To complete the proof, we only need to show $\overline{u}_1=\underline{u}_1$, $\overline{v}_1=\underline{v}_1$. Letting $n\to+\infty$ in (2.11), we obtain

$$\begin{aligned}&b_1-a_{11}\overline{u}_1-c_1\underline{v}_1=0,\quad -e_1\overline{v}_1+d_1\overline{u}_1=0,\\&b_1-a_{11}\underline{u}_1-c_1\overline{v}_1=0,\quad -e_1\underline{v}_1+d_1\underline{u}_1=0.\end{aligned}\tag{2.12}$$

It follows from (2.12) that

$$\begin{aligned}&b_1-a_{11}\overline{u}_1-\tfrac{c_1d_1}{e_1}\underline{u}_1=0,\\&b_1-a_{11}\underline{u}_1-\tfrac{c_1d_1}{e_1}\overline{u}_1=0.\end{aligned}\tag{2.13}$$

Subtracting the first equality of (2.13) from the second equality, we obtain

$$\Big(a_{11}-\frac{c_1d_1}{e_1}\Big)(\overline{u}_1-\underline{u}_1)=0.$$

Since $\frac{c_1d_1}{e_1a_{11}}<1$, $\overline{u}_1=\underline{u}_1$, consequently, $\overline{v}_1=\underline{v}_1$. Also, (u_1^{**},v_1^{**}) satisfies (2.12). Hence $\overline{u}_1=\underline{u}_1=u_1^{**}$ and $\overline{v}_1=\underline{v}_1=v_1^{**}$, that is

$$\lim_{t\to+\infty}(u_1(t,x),v_1(t,x))=(u_1^{**},v_1^{**}),\quad \text{uniformly for } x\in[0,\pi].$$

Note that $\lim\limits_{t\to+\infty}u_2(t,x)=0$ uniformly for $x\in[0,\pi]$, then for any $\varepsilon>0$ sufficiently small, there exists a $T_0\geq0$ such that $u_2(t,x)<\varepsilon$, for any $x\in[0,\pi]$ and $t\geq T_0$. If follows from the fourth equation of system (1.1) that for any $x\in[0,\pi]$

$$\frac{\partial v_2}{\partial t}-D_4\Delta v_2\leq -e_2v_2(t,x)+d_2\varepsilon,\quad t\geq T_0.$$

Therefore, we have

$$\limsup_{t\to+\infty}\max_{x\in[0,\pi]}v_2(t,x)=\frac{d_2\varepsilon}{e_2}.$$

Setting $\varepsilon\to0$, we obtain

$$\lim_{t\to+\infty}v_2(t,x)=0,\quad \text{uniformly for } x\in[0,\pi].$$

This completes the proof of Lemma 2.4.

Take $c_1=c_2=0$ in system (1.1), that is, consider system (1.1) without feedback controls. Then system (1.1) is reduced to the following system

$$\begin{aligned}&\frac{\partial u_1}{\partial t}-D_1\Delta u_1=u_1\left(b_1-a_{11}u_1-a_{12}\int_{-\infty}^t\int_0^\pi G_1(x,y,t-s)f_1(t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s\right),\\&\frac{\partial u_2}{\partial t}-D_2\Delta u_2=u_2\left(b_2-a_{21}\int_{-\infty}^t\int_0^\pi G_2(x,y,t-s)f_2(t-s)u_1(s,y)\mathrm{d}y\mathrm{d}s-a_{22}u_2\right).\end{aligned}\tag{2.14}$$

Without lose of generality, it follows from Theorem 2.3 in [20] that we have the following theorem.

Theorem 2.1 *Let* $(u_1(t,x),u_2(t,x))^{\mathrm{T}}$ *be a solution of system* (2.14) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_1(0,x)\not\equiv 0$ *and* $\phi_2(0,x)\not\equiv 0$.

(i) *If* $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = (\overline{x}_1,\overline{x}_2)$ *uniformly for* $x\in[0,\pi]$, *where* $(\overline{x}_1,\overline{x}_2)=\big(\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}},\frac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\big)$ *is the unique positive steady state of system* (2.14).

(ii) *If* $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = \big(\frac{b_1}{a_{11}},0\big)$ *uniformly for* $x\in[0,\pi]$.

(iii) *If* $\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}} > \frac{b_1}{b_2}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = \big(0,\frac{b_2}{a_{22}}\big)$ *uniformly for* $x\in[0,\pi]$.

3 Main Results

The trivial steady state solution E_0 is of no interest here. In this paper, we discuss the stability of the equilibria E_1, E_2 and E^*, and shows the influence of feedback controls on the global stability of system (1.1). More precisely, we present the main results of this paper.

Theorem 3.1 *Let* $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ *be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_i(0,x)\not\equiv 0$ *and* $\psi_i(0,x)\not\equiv 0,\ i=1,2$. *Assume further that* (H_1) *and*

$$\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}. \tag{H_2}$$

Then $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x)) = (u_1^*,u_2^*,v_1^*,v_2^*)$ uniformly for $x\in[0,\pi]$.

Proof Define

$$\begin{aligned}
&V_1(t)=\sum_{i=1}^{2}\eta_i\int_\Omega\Big(u_i-u_i^*-u_i^*\ln\frac{u_i}{u_i^*}\Big)\mathrm{d}x+\sum_{i=1}^{2}\beta_i\int_\Omega(v_i-v_i^*)^2\mathrm{d}x,\\
&K_i(x,y,t)=G_i(x,y,t)f_i(t),\quad \Omega=(0,\pi),\ i=1,2,
\end{aligned}$$

where $\eta_2=1;\ \beta_i=\frac{c_i\eta_i}{2d_i},\ i=1,2;\ \eta_1$ is a positive constant to be determined below.

It is easy to see that the equations of (1.1) can be rewritten as

$$\begin{aligned}
\frac{\partial u_1}{\partial t}-D_1\Delta u_1=u_1(t,x)\Big(&-a_{11}(u_1(t,x)-u_1^*)-c_1(v_1(t,x)-v_1^*)\\
&-a_{12}\int_{-\infty}^{t}\int_\Omega K_1(x,y,t-s)(u_2(s,y)-u_2^*)\mathrm{d}y\mathrm{d}s\Big),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2(t,x)\Big(- a_{21}\int_{-\infty}^{t}\int_\Omega K_2(x,y,t-s)(u_1(s,y)-u_1^*)\mathrm{d}y\mathrm{d}s \\
&\qquad\qquad -a_{22}(u_2(t,x)-u_2^*) - c_2(v_2(t,x)-v_2^*)\Big), \\
&\frac{\partial v_1}{\partial t} - D_3\Delta v_1 = -e_1(v_1(t,x)-v_1^*) + d_1(u_1(t,x)-u_1^*), \\
&\frac{\partial v_2}{\partial t} - D_4\Delta v_2 = -e_2(v_2(t,x)-v_2^*) + d_2(u_2(t,x)-u_2^*).
\end{aligned} \tag{3.1}$$

Calculating the derivative of V_1 along the solution of system (3.1), it follows that

$$\begin{aligned}
\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} &= \sum_{i=1}^{2}\eta_i\int_\Omega \frac{\partial u_i}{\partial t}\left(1-\frac{u_i^*}{u_i}\right)\mathrm{d}x + \sum_{i=1}^{2}2\beta_i\int_\Omega\frac{\partial v_i}{\partial t}(v_i-v_i^*)\mathrm{d}x \\
&= -\sum_{i=1}^{2}\eta_iD_iu_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x - \sum_{i=1}^{2}2\beta_iD_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x \\
&\quad -\eta_1a_{11}\int_\Omega(u_1(t,x)-u_1^*)^2\mathrm{d}x - \int_\Omega a_{22}(u_2(t,x)-u_2^*)^2\mathrm{d}x \\
&\quad -\eta_1a_{12}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_1(x,y,t-s)(u_1(t,x)-u_1^*)(u_2(s,x)-u_2^*)\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&\quad -a_{21}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_2(x,y,t-s)(u_1(s,x)-u_1^*)(u_2(t,x)-u_2^*)\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&\quad -2\beta_1e_1\int_\Omega(v_1(t,x)-v_1^*)^2\mathrm{d}x - 2\beta_2e_2\int_\Omega(v_2(t,x)-v_2^*)^2\mathrm{d}x.
\end{aligned} \tag{3.2}$$

Noting that $ab \le \frac{\theta}{2}a^2 + \frac{1}{2\theta}b^2,\ \theta > 0$, we derive from (3.2) that

$$\begin{aligned}
\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} \le &-\sum_{i=1}^{2}\eta_iD_iu_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x - \sum_{i=1}^{2}2\beta_iD_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x \\
&-\eta_1a_{11}\int_\Omega(u_1(t,x)-u_1^*)^2\mathrm{d}x - a_{22}\int_\Omega(u_2(t,x)-u_2^*)^2\mathrm{d}x \\
&+\eta_1a_{12}\Big(\frac{\theta_1}{2}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_1(x,y,t-s)(u_1(t,x)-u_1^*)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&+\frac{1}{2\theta_1}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_1(x,y,t-s)(u_2(s,x)-u_2^*)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x\Big) \\
&+a_{21}\Big(\frac{\theta_2}{2}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_2(x,y,t-s)(u_1(s,x)-u_1^*)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&+\frac{1}{2\theta_2}\int_\Omega\int_{-\infty}^{t}\int_\Omega K_2(x,y,t-s)(u_2(t,x)-u_2^*)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x\Big) \\
&-2\beta_1e_1\int_\Omega(v_1(t,x)-v_1^*)^2\mathrm{d}x - 2\beta_2e_2\int_\Omega(v_2(t,x)-v_2^*)^2\mathrm{d}x.
\end{aligned} \tag{3.3}$$

Using the property of $K_i(x,y,t),\ i=1,2$ as described in (1.7) and (1.8), we have

$$\begin{aligned}&\int_\Omega\int_{-\infty}^t\int_\Omega K_i(x,y,t-s)(u_i(t,x)-u_i^*)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x\\ =&\int_\Omega\int_0^{+\infty}\int_\Omega K_i(x,y,r)(u_i(t,x)-u_i^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\ =&\int_\Omega\int_0^{+\infty}\int_\Omega G_i(x,y,r)f_i(r)(u_i(t,x)-u_i^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\ =&\int_\Omega\int_0^{+\infty} f_i(r)(u_i(t,x)-u_i^*)^2\mathrm{d}r\mathrm{d}x\\ =&\int_\Omega(u_i(t,x)-u_i^*)^2\mathrm{d}x,\quad i=1,2.\end{aligned}$$

Substituting the above equalities into (3.3) leads to

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}\le&-\sum_{i=1}^2\eta_iD_iu_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x-\sum_{i=1}^2 2\beta_iD_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x\\ &-\left(\eta_1a_{11}-\eta_1a_{12}\frac{\theta_1}{2}\right)\int_\Omega(u_1(t,x)-u_1^*)^2\mathrm{d}x\\ &-\left(a_{22}-a_{21}\frac{1}{2\theta_2}\right)\int_\Omega(u_2(t,x)-u_2^*)^2\mathrm{d}x\\ &+a_{21}\frac{\theta_2}{2}\int_\Omega\int_0^{+\infty}\int_\Omega K_2(x,y,r)(u_1(t-r,x)-u_1^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\ &+\eta_1a_{12}\frac{1}{2\theta_1}\int_\Omega\int_0^{+\infty}\int_\Omega K_1(x,y,r)(u_2(t-r,x)-u_2^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\ &-2\beta_1e_1\int_\Omega(v_1(t,x)-v_1^*)^2\mathrm{d}x-2\beta_2e_2\int_\Omega(v_2(t,x)-v_2^*)^2\mathrm{d}x.\end{aligned}\tag{3.4}$$

Now, define a new Lyapunov functional

$$\begin{aligned}V(t)=V_1(t)&+a_{21}\frac{\theta_2}{2}\int_\Omega\int_0^{+\infty}\int_\Omega\int_{t-r}^t K_2(x,y,r)(u_1(s,x)-u_1^*)^2\mathrm{d}s\mathrm{d}y\mathrm{d}r\mathrm{d}x\\ &+\eta_1a_{12}\frac{1}{2\theta_1}\int_\Omega\int_0^{+\infty}\int_\Omega\int_{t-r}^t K_1(x,y,r)(u_2(s,x)-u_2^*)^2\mathrm{d}s\mathrm{d}y\mathrm{d}r\mathrm{d}x.\end{aligned}\tag{3.5}$$

It is derived from (3.4) and (3.5) that

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le&-\sum_{i=1}^2\eta_iD_iu_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x-\sum_{i=1}^2 2\beta_iD_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x\\ &-\left(\eta_1a_{11}-\eta_1a_{12}\frac{\theta_1}{2}\right)\int_\Omega(u_1(t,x)-u_1^*)^2\mathrm{d}x\\ &-\left(a_{22}-a_{21}\frac{1}{2\theta_2}\right)\int_\Omega(u_2(t,x)-u_2^*)^2\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+a_{21}\frac{\theta_2}{2}\int_\Omega\int_0^{+\infty}\int_\Omega K_2(x,y,r)(u_1(t,x)-u_1^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\
&+\eta_1 a_{12}\frac{1}{2\theta_1}\int_\Omega\int_0^{+\infty}\int_\Omega K_1(x,y,r)(u_2(t,x)-u_2^*)^2\mathrm{d}y\mathrm{d}r\mathrm{d}x\\
&-2\beta_1 e_1\int_\Omega(v_1(t,x)-v_1^*)^2\mathrm{d}x-2\beta_2 e_2\int_\Omega(v_2(t,x)-v_2^*)^2\mathrm{d}x\\
=&-\sum_{i=1}^2\eta_i D_i u_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x-\sum_{i=1}^2 2\beta_i D_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x\\
&-\left(\eta_1 a_{11}-\eta_1 a_{12}\frac{\theta_1}{2}-a_{21}\frac{\theta_2}{2}\right)\int_\Omega(u_1(t,x)-u_1^*)^2\mathrm{d}x\\
&-\left(a_{22}-a_{21}\frac{1}{2\theta_2}-\eta_1 a_{12}\frac{1}{2\theta_1}\right)\int_\Omega(u_2(t,x)-u_2^*)^2\mathrm{d}x\\
&-2\beta_1 e_1\int_\Omega(v_1(t,x)-v_1^*)^2\mathrm{d}x-2\beta_2 e_2\int_\Omega(v_2(t,x)-v_2^*)^2\mathrm{d}x. \qquad (3.6)
\end{aligned}
$$

Denote $\delta_1=\eta_1 a_{11}-\eta_1 a_{12}\frac{\theta_1}{2}-a_{21}\frac{\theta_2}{2}$ and $\delta_2=a_{22}-a_{21}\frac{1}{2\theta_2}-\eta_1 a_{12}\frac{1}{2\theta_1}$. Then taking

$$\eta_1=\frac{a_{21}}{a_{12}},\quad \theta_1=\theta_2=\frac{2a_{21}a_{11}}{a_{11}a_{22}+a_{12}a_{21}}$$

can lead to

$$\delta_1=\frac{a_{11}a_{21}(a_{11}a_{22}-a_{21}a_{12})}{a_{12}(a_{11}a_{22}+a_{12}a_{21})},\quad \delta_2=\frac{a_{11}a_{22}-a_{21}a_{12}}{2a_{11}}.$$

From (H_2), we have $\delta_i>0,\ i=1,2$. It is easy to see that

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le&-\sum_{i=1}^2\eta_i D_i u_i^*\int_\Omega\frac{|\nabla u_i|^2}{u_i^2}\mathrm{d}x-\sum_{i=1}^2 2\beta_i D_{i+2}\int_\Omega|\nabla v_i|^2\mathrm{d}x\\
&-\sum_{i=1}^2\delta_i\int_\Omega(u_i(t,x)-u_i^*)^2\mathrm{d}x-\sum_{i=1}^2 2\beta_i e_i\int_\Omega(v_i(t,x)-v_i^*)^2\mathrm{d}x. \qquad (3.7)
\end{aligned}
$$

For any $T>0$, integrating (3.7) over $[0,T]$, we derive that

$$
\begin{aligned}
&\sum_{i=1}^2\eta_i D_i u_i^*\left|\left|\left|\frac{\nabla u_i}{u_i}\right|\right|\right|_2^2+\sum_{i=1}^2 2\beta_i D_{i+2}|||\nabla v_i|||_2^2\\
&+\sum_{i=1}^2\delta_i|||u_i-u_i^*|||_2^2+\sum_{i=1}^2 2\beta_i e_i|||v_i-v_i^*|||_2^2\le V(0). \qquad (3.8)
\end{aligned}
$$

From (3.8) we can conclude that

$$\left|\left|\left|\frac{\nabla u_i}{u_i}\right|\right|\right|_2\le C_i,\quad |||\nabla v_i|||_2\le D_i \qquad (3.9)$$

and

$$|||u_i - u_i^*|||_2 \le E_i, \quad |||v_i - v_i^*|||_2 \le F_i, \tag{3.10}$$

for some constants $C_i, D_i, E_i, F_i,\ i = 1, 2$ independent of T.

Noting that $u_i(t,x), v_i(t,x),\ i = 1,2$ are bounded, it follows from (3.9) that

$$|||\nabla u_i|||_2 \le Q_i, \tag{3.11}$$

for some constants $Q_i,\ i = 1, 2$ independent of T. We derive from (3.9)-(3.11) that $u_i(t,x) - u_i^*, v_i(t,x) - v_i^* \in L^2((0,\infty); W^{1,2}(\Omega; R)),\ i = 1,2$ thus

$$\lim_{t\to+\infty} \|u_i(t) - u_i^*\|_{W^{1,2}} = 0, \quad \lim_{t\to+\infty} \|v_i(t) - v_i^*\|_{W^{1,2}} = 0, \quad i = 1, 2.$$

We obtain from the Sobolev compact embedding theorem (see, for example [21]) that

$$\lim_{t\to+\infty} \|u_i(t) - u_i^*\|_{C(\overline{\Omega};R)} = 0, \quad \lim_{t\to+\infty} \|v_i(t) - v_i^*\|_{C(\overline{\Omega};R)} = 0, \quad i = 1, 2.$$

This completes the proof of Theorem 3.1.

Remark 3.1 If $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, from (i) of Theorem 2.1, the unique positive steady state $(\overline{x}_1, \overline{x}_2)$ of system (2.14) is globally stable. Note that $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ implies (H_1) and (H_2) hold. Thus it follows from Theorem 3.1 that the unique positive steady state $E^*(u_1^*, u_2^*, v_1^*, v_2^*)$ of system (1.1) is still globally stable, that is, in this case, feedback controls only change the position of the unique positive equilibrium and retain the stable property.

Remark 3.2 If $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$, from (ii) of Theorem 2.1, the species u_2 in system (2.14) is extinct. Choosing suitable values of c_1, e_1, d_1, by Theorem 3.1, the species u_2 in system (1.1) is globally stable, that is, in this case, the suitable feedback control variables can make extinct species u_2 become globally stable in system (1.1),

Remark 3.3 If $\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}} > \frac{b_1}{b_2}$, from (iii) of Theorem 2.1, the species u_1 in system (2.14) is extinct. Choosing suitable values of c_2, e_2, d_2, by Theorem 3.1, the species u_1 in system (1.1) is globally stable, that is, in this case, suitable feedback control variables can make extinct species u_1 become globally stable in system (1.1).

Now, we study the stability of semi-trivial steady state solution of system (1.1).

Theorem 3.2 *Let* $(u_1(t,x), u_2(t,x), v_1(t,x), v_2(t,x))$ *be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_i(0,x) \not\equiv 0$ *and* $\psi_i(0,x) \not\equiv 0,\ i = 1,2$. *Assume further that*

$$\frac{b_1}{b_2}\left(1 - \frac{c_1 d_1}{e_1 a_{11}}\right) \ge \frac{a_{11}}{a_{21}}, \quad \frac{b_1}{b_2}\left(1 - \frac{c_1 d_1}{e_1 a_{11}}\right) > \frac{a_{12}}{a_{22}}, \tag{H_3}$$

then $\lim\limits_{t\to+\infty}(u_1(t,x), u_2(t,x), v_1(t,x), v_2(t,x)) = (x_1^{**}, 0, u_1^{**}, 0)$ *uniformly for* $x \in [0, \pi]$.

Proof From Lemma 2.1 we have

$$\limsup_{t\to+\infty}\max_{x\in[0,\pi]} u_i(t,x)\le \frac{b_i}{a_{ii}},\quad \text{and}\quad \limsup_{t\to+\infty}\max_{x\in[0,\pi]} v_i(t,x)\le \frac{d_ib_i}{e_ia_{ii}},\quad i=1,2.$$

Hence, for any $\varepsilon>0$ sufficiently small, from Lemma 2.2, there exists a $T_1>0$ such that for any $x\in[0,\pi]$ and $t\ge T_1$,

$$u_2(t,x)\le \overline{u}_2^{(1)},\quad \int_{-\infty}^{t}\int_0^{\pi} K_1(x,y,t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s\le \overline{u}_2^{(1)}, \tag{3.12}$$

where

$$\overline{u}_2^{(1)}=\frac{b_2}{a_{22}}+\varepsilon,\quad K_1(x,y,t)=G_1(x,y,t)f_1(t).$$

For any $\varepsilon>0$ sufficiently small, it follows from Lemma 2.2 that there exist positive constants τ_n $(\tau_n<\tau_{n+1})$, $n=1,2,\cdots$ such that

$$v_1(t,x)\le \frac{d_1b_1}{e_1a_{11}}+\frac{\varepsilon}{n},\quad t>\tau_n,\ x\in[0,\pi]. \tag{3.13}$$

From (3.12), (3.13) and the first equation of system (1.1), for $x\in[0,\pi]$, $t\ge t_1=\max\{T_1,\tau_1\}$, we have

$$\frac{\partial u_1}{\partial t}-D_1\Delta u_1\ge u_1\Big(b_1-a_{12}\overline{u}_2^{(1)}-c_1\Big(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\Big)-a_{11}u_1\Big).$$

For any $\varepsilon>0$ sufficiently small, it follows from (H_3) that

$$b_1-a_{12}\overline{u}_2^{(1)}-c_1\Big(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\Big)>0, \tag{3.14}$$

then from the comparison principle, we obtain

$$\liminf_{t\to+\infty}\min_{x\in[0,\pi]} u_1(t,x)\ge \frac{b_1-a_{12}\overline{u}_2^{(1)}-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\big)}{a_{11}}.$$

Hence, for any $\varepsilon>0$ sufficiently small, from Lemma 2.2, there exists a $T_2\ge t_1$ such that for any $x\in[0,\pi]$ and $t\ge T_2$,

$$u_1(t,x)\ge \underline{u}_1^{(1)},\quad \int_{-\infty}^{t}\int_0^{\pi} K_2(x,y,t-s)u_1(s,y)\mathrm{d}y\mathrm{d}s\ge \underline{u}_1^{(1)}, \tag{3.15}$$

where

$$\begin{aligned}\underline{u}_1^{(1)}&=\frac{b_1-a_{12}\overline{u}_2^{(1)}-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\varepsilon\big)}{a_{11}}-\varepsilon>0,\\ K_2(x,y,t)&=G_2(x,y,t)f_2(t).\end{aligned}$$

It follows from (3.15) and the second equation of system (1.1) that

$$\frac{\partial u_2}{\partial t}-D_2\Delta u_2\le u_2(b_2-a_{21}\underline{u}_1^{(1)}-a_{22}u_2),\quad t\ge T_2.$$

If $b_2 - a_{21}\underline{u}_1^{(1)} \le 0$, then from the comparison principle, we obtain $\lim\limits_{t\to+\infty} u_2(t,x) = 0$ uniformly for $x \in [0,\pi]$. Note that (H$_3$) implies $\frac{c_1 d_1}{e_1 a_{11}} < 1$, by Lemma 2.4, then $\lim\limits_{t\to+\infty}(u_1(t,x), u_2(t,x), v_1(t,x), v_2(t,x)) = (x_1^{**}, 0, u_1^{**}, 0)$ uniformly for $x \in [0,\pi]$, that is, the proof is completed. Next, we consider $b_2 - a_{21}\underline{u}_1^{(1)} > 0$. By the comparison principle, we have

$$\limsup_{t\to+\infty} \max_{x\in[0,\pi]} u_2(t,x) \le \frac{b_2 - a_{21}\underline{u}_1^{(1)}}{a_{22}}.$$

Hence, for any $\varepsilon > 0$ sufficiently small, from Lemma 2.2, there exists a $T_3 > T_2$ such that for any $x \in [0,\pi]$ and $t \ge T_3$,

$$u_2(t,x) \le \overline{u}_2^{(2)}, \quad \int_{-\infty}^{t}\int_{\Omega} K_1(x,y,t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s \le \overline{u}_2^{(2)}, \tag{3.16}$$

where

$$\overline{u}_2^{(2)} = \frac{b_2 - a_{21}\underline{u}_1^{(1)}}{a_{22}} + \frac{\varepsilon}{2}.$$

By (3.13), (3.16) and the first equation of system (1.1), for $x \in [0,\pi]$, $t \ge t_2 = \max\{T_3, \tau_2\}$, we have

$$\frac{\partial u_1}{\partial t} - D_1\Delta u_1 \ge u_1\Big(b_1 - a_{12}\overline{u}_2^{(2)} - c_1\Big(\frac{d_1 b_1}{e_1 a_{11}} + \frac{\varepsilon}{2}\Big) - a_{11}u_1\Big).$$

For any $\varepsilon > 0$ sufficiently small, it follows from (3.12), (3.14) and (3.16) that

$$b_1 - a_{12}\overline{u}_2^{(2)} - c_1\Big(\frac{d_1 b_1}{e_1 a_{11}} + \frac{\varepsilon}{2}\Big) > 0.$$

By the comparison principle, we have

$$\liminf_{t\to+\infty} \min_{x\in[0,\pi]} u_1(t,x) \ge \frac{b_1 - a_{12}\overline{u}_2^{(2)} - c_1\big(\frac{d_1 b_1}{e_1 a_{11}} + \frac{\varepsilon}{2}\big)}{a_{11}}.$$

Hence, for any $\varepsilon > 0$ sufficiently small, from Lemma 2.2, there exists a $T_4 \ge t_2$ such that for any $x \in [0,\pi]$ and $t \ge T_4$,

$$u_1(t,x) \ge \underline{u}_1^{(2)}, \quad \int_{-\infty}^{t}\int_{\Omega} K_2(x,y,t-s)u_1(s,y)\mathrm{d}y\mathrm{d}s \ge \underline{u}_1^{(2)}, \tag{3.17}$$

where

$$\underline{u}_1^{(2)} = \frac{b_1 - a_{12}\overline{u}_2^{(2)} - c_1\big(\frac{d_1 b_1}{e_1 a_{11}} + \frac{\varepsilon}{2}\big)}{a_{11}} - \frac{\varepsilon}{2} > 0.$$

From (3.12) and (3.15)-(3.17), for any $x \in [0,\pi]$ and $t \ge T_4$, we have

$$0 < u_2(t,x) < \overline{u}_2^{(2)} < \overline{u}_2^{(1)}, \quad 0 < \underline{u}_1^{(1)} < \underline{u}_1^{(2)} < u_1(t,x).$$

Repeating the above procedure, we get two sequences $\overline{u}_2^{(n)}$ and $\underline{u}_1^{(n)}$, $n=1,2,\cdots$, such that for $n\geq 2$

$$\begin{aligned} \overline{u}_2^{(n)} &= \frac{b_2-a_{21}\underline{u}_1^{(n-1)}}{a_{22}}+\frac{\varepsilon}{n},\\ \underline{u}_1^{(n)} &= \frac{b_1-a_{12}\overline{u}_2^{(n)}-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\frac{\varepsilon}{n}\big)}{a_{11}}-\frac{\varepsilon}{n}>0. \end{aligned} \tag{3.18}$$

Without loss of generality, we assume $b_2-a_{21}\underline{u}_1^{(n)}>0,\ n=1,2\cdots$. Then

$$0<u_2(t,x)<\overline{u}_2^{(n)},\quad 0<\underline{u}_1^{(n)}<u_1(t,x),\quad \text{for any } x\in[0,\pi],\ t\geq T_{2n}.$$

We claim that the sequences $\overline{u}_2^{(n)}$ are non-increasing, and the sequences $\underline{u}_1^{(n)}$ are non-decreasing. To prove this claim, we will carry out by induction. Firstly, we immediately get

$$\overline{u}_2^{(2)}<\overline{u}_2^{(1)},\quad \underline{u}_1^{(1)}<\underline{u}_1^{(2)}.$$

Assume that our claim is true for n, that is,

$$\overline{u}_2^{(n)}<\overline{u}_2^{(n-1)},\quad \underline{u}_1^{(n-1)}<\underline{u}_1^{(n)}.$$

By computation, we have

$$\begin{aligned} \overline{u}_2^{(n+1)} &= \frac{b_2-a_{21}\underline{u}_1^{(n)}}{a_{22}}+\frac{\varepsilon}{n+1}<\frac{b_2-a_{21}\underline{u}_1^{(n-1)}}{a_{22}}+\frac{\varepsilon}{n}=\overline{u}_2^{(n)},\\ \underline{u}_1^{(n+1)} &= \frac{b_1-a_{12}\overline{u}_2^{(n+1)}-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\frac{\varepsilon}{n+1}\big)}{a_{11}}-\frac{\varepsilon}{n+1}\\ &> \frac{b_1-a_{12}\overline{u}_2^{(n)}-c_1\big(\frac{d_1b_1}{e_1a_{11}}+\frac{\varepsilon}{n}\big)}{a_{11}}-\frac{\varepsilon}{n}=\underline{u}_1^{(n)}. \end{aligned}$$

Therefore, the limits of $\overline{u}_2^{(n)}$ and $\underline{u}_1^{(n)}$ exist. Denote that

$$\lim_{n\to+\infty}\overline{u}_2^{(n)}=\overline{u}_2,\quad \lim_{n\to+\infty}\underline{u}_1^{(n)}=\underline{u}_1.$$

Note that (H_3) holds, then if follows from Lemma 2.3 that $\liminf\limits_{t\to+\infty}\min\limits_{x\in\overline{\Omega}}u_1(t,x)>\alpha$, that is $\underline{u}_1\geq\alpha>0$. Obviously $\overline{u}_2\geq 0$. To prove $\lim\limits_{t\to+\infty}u_2(t,x)=0$ uniformly for $x\in[0,\pi]$, it suffices to show that $\overline{u}_2=0$. Otherwise, we suppose that $\overline{u}_2>0$. Letting $n\to+\infty$ in (3.18), we obtain

$$\begin{aligned} &b_1\Big(1-\frac{c_1d_1}{e_1a_{11}}\Big)-a_{11}\underline{u}_1-a_{12}\overline{u}_2=0,\\ &b_2-a_{21}\underline{u}_1-a_{22}\overline{u}_2=0. \end{aligned} \tag{3.19}$$

Multiplying the second equation of (3.19) by $-\frac{b_1}{b_2}\big(1-\frac{c_1d_1}{e_1a_{11}}\big)$ and adding it to the first equation of (3.19), we obtain

$$\Big[a_{22}\frac{b_1}{b_2}\Big(1-\frac{c_1d_1}{e_1a_{11}}\Big)-a_{12}\Big]\overline{u}_2=\Big[a_{11}-a_{21}\frac{b_1}{b_2}\Big(1-\frac{c_1d_1}{e_1a_{11}}\Big)\Big]\underline{u}_1. \tag{3.20}$$

From the first inequality in condition (H$_3$), $\underline{u}_1>0$ and (3.20), we have

$$\Big[a_{22}\frac{b_1}{b_2}\Big(1-\frac{c_1d_1}{e_1a_{11}}\Big)-a_{12}\Big]\overline{u}_2\leq 0. \tag{3.21}$$

It follows from the second inequality in condition (H$_3$) and (3.21) that $\overline{u}_2\leq 0$, which is a contradiction, then we obtain $\lim\limits_{t\to+\infty}u_2(t,x)=0$ uniformly for $x\in[0,\pi]$. Hence, by Lemma 2.4, we have $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))=(x_1^{**},0,u_1^{**},0)$ uniformly for $x\in[0,\pi]$. This ends the proof of Theorem 3.3.

Using Lyapunov functional method, another sufficient conditions which guarantee the stability of semi-trivial steady state solution of system (1.1) are obtained.

Theorem 3.3 *Let* $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ *be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_i(0,x)\not\equiv 0$ *and* $\psi_i(0,x)\not\equiv 0$, $i=1,2$. *Assume further that*

$$\frac{b_1}{b_2}>\frac{a_{11}e_1+c_1d_1}{a_{21}e_1},\qquad \frac{a_{11}}{a_{21}}>\frac{a_{12}}{a_{22}}, \tag{H$_4$}$$

then $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))=(u_1^{**},0,v_1^{**},0)$ *uniformly for* $x\in[0,\pi]$.

Proof Define

$$V_1(t)=\eta_1\int_\Omega\Big(u_1-u_1^{**}-u_1^{**}\ln\frac{u_1}{u_1^{**}}\Big)\mathrm{d}x+\eta_2\int_\Omega u_2\mathrm{d}x+\beta_1\int_\Omega(v_1-v_1^{**})^2\mathrm{d}x+\beta_2\int_\Omega v_2^2\mathrm{d}x,$$

$$K_i(x,y,t)=G_i(x,y,t)f_i(t),\quad \Omega=(0,\pi),\ i=1,2,$$

where $\eta_2=1$; $\beta_i=\frac{c_i\eta_i}{2d_i}$, $i=1,2$; η_1 is a positive constant to be determined below.

System (1.1) can be rewritten as

$$\begin{aligned}
\frac{\partial u_1}{\partial t}-D_1\Delta u_1&=u_1(t,x)\Big(-a_{11}(u_1(t,x)-u_1^{**})-c_1(v_1(t,x)-v_1^{**})\\
&\qquad -a_{12}\int_{-\infty}^t\int_\Omega K_1(x,y,t-s)u_2(s,y)\mathrm{d}y\mathrm{d}s\Big)\\
\frac{\partial u_2}{\partial t}-D_2\Delta u_2&=u_2(t,x)\Big(b_2-a_{21}u_1^{**}-a_{22}u_2(t,x)-c_2v_2(t,x)\\
&\qquad -a_{21}\int_{-\infty}^t\int_\Omega K_2(x,y,t-s)(u_1(s,y)-u_1^{**})\mathrm{d}y\mathrm{d}s\Big),\\
\frac{\partial v_1}{\partial t}-D_3\Delta v_1&=-e_1(v_1(t,x)-v_1^{*})+d_1(u_1(t,x)-u_1^{*}),\\
\frac{\partial v_2}{\partial t}-D_4\Delta v_2&=-e_2v_2(t,x)+d_2u_2(t,x).
\end{aligned}\tag{3.22}$$

If follows from the first inequality in condition (H_4) that $b_2 - a_{21}u_1^{**} < 0$. Using similar arguments to those in the proof of Theorem 3.1, we have

$$\lim_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x)) = (u_1^{**},0,v_1^{**},0) \quad \text{uniformly for } x\in[0,\pi].$$

This ends the proof of Theorem 3.3.

Remark 3.4 If $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$, from (ii) of Theorem 2.1, the species u_2 in system (2.14) is extinct. By Theorem 3.2 or 3.3, choosing suitable values of c_1, e_1, d_1, the species u_2 in system (1.1) is still extinct, that is, in this case, the suitable feedback control variables can make extinct species u_2 still keep the property of extinction in system (1.1).

Similar to the proofs of Theorems 3.2 and 3.3, we have the following theorem.

Theorem 3.4 *Let* $(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x))$ *be a solution of system* (1.1) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_i(0,x)\not\equiv 0$ *and* $\psi_i(0,x)\not\equiv 0,\ i=1,2$. *Assume further that*

$$\frac{b_1}{b_2} < \frac{a_{11}}{a_{21}}\left(1-\frac{c_2d_2}{e_2a_{22}}\right), \quad \frac{b_1}{b_2} \le \frac{a_{12}}{a_{22}}\left(1-\frac{c_2d_2}{e_2a_{22}}\right) \tag{H$_5$}$$

or

$$\frac{b_1}{b_2} < \frac{a_{12}e_2}{a_{22}e_2+c_2d_2}, \quad \frac{a_{12}}{a_{22}} < \frac{a_{11}}{a_{21}} \tag{H$_6$}$$

hold, then $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x),v_1(t,x),v_2(t,x)) = (0,x_2^{**},0,u_2^{**})$ *uniformly for* $x\in[0,\pi]$.

Note that when $c_i=0,\ i=1,2$, system (1.1) is reduced to system (2.14). Similar to the analysis of Theorems 3.1, 3.2 and 3.4, we have the following corollary.

Corollary 3.1 *Let* $(u_1(t,x),u_2(t,x))^{\mathrm{T}}$ *be a solution of system* (2.14) *with the boundary conditions* (1.2) *and the initial conditions* (1.3) *satisfying* $\phi_1(0,x)\not\equiv 0$ *and* $\phi_2(0,x)\not\equiv 0$.

(i) *If* $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = (\overline{x}_1,\overline{x}_2)$ *uniformly for* $x\in[0,\pi]$, *where* $(\overline{x}_1,\overline{x}_2)$ *is the unique positive steady state given by Theorem* 2.1.

(ii) *If* $\frac{b_1}{b_2} \ge \frac{a_{11}}{a_{21}}, \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = (\frac{b_1}{a_{11}},0)$ *uniformly for* $x\in[0,\pi]$.

(iii) *If* $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2}, \frac{a_{12}}{a_{22}} \ge \frac{b_1}{b_2}$, *then* $\lim\limits_{t\to+\infty}(u_1(t,x),u_2(t,x)) = (0,\frac{b_2}{a_{22}})$ *uniformly for* $x\in[0,\pi]$.

Remark 3.5 When $c_i=0,\ i=1,2$, conditions (H_3) and (H_4) are changed into $\frac{b_1}{b_2} \ge \frac{a_{11}}{a_{21}}, \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ and $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$, respectively. Obviously, $\frac{b_1}{b_2} \ge \frac{a_{11}}{a_{21}}$ and $\frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ are weaker than $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$. Then, the conditions of Corollary 3.1 are weaker than those of Theorem 2.1. Hence, Theorems 3.1-3.4 and Corollary 3.1 generalize and improve the results of [20].

Remark 3.6 If system (1.1) is reduced to system (1.5), Theorems 3.1-3.4 generalize the main results of [10]. Especially, it is hard to construct the extinction of Lyapunov functional to study the extinction of system (1.1) as in [10]. Hence, in Theorem 3.2, we use the iterative technique method to investigate the extinction of system (1.1).

4 Example

In this section, we give some examples to show the feasibility of our results.

In the following, we always take $f_i(t) = \frac{1}{\tau_i}\mathrm{e}^{-\frac{t}{\tau_i}}$ and

$$G_i(x,y,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\mathrm{e}^{-D_i n^2 t}\cos nx \sin ny.$$

However, it is difficult for us to carry out numerical simulations directly because of nonlocal term. Define

$$Q_i(t,x) = \int_{-\infty}^{t}\int_{0}^{\pi} G_i(x,y,t-s)\frac{1}{\tau_i}\mathrm{e}^{-\frac{t-s}{\tau_i}}u_j(s,y)\mathrm{d}y\mathrm{d}s, \quad i \neq j,\ i,j=1,2. \tag{4.1}$$

Similar to [19], the equations of (1.1) are rewritten as:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = u_1\big(b_1 - a_{11}u_1 - a_{12}Q_1 - c_1v_1\big),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2\big(b_2 - a_{21}Q_2 - a_{22}u_2 - c_2v_2\big),\\
&\frac{\partial Q_1}{\partial t} - D_1\Delta Q_1 = \frac{1}{\tau_1}(u_2 - Q_1), \quad \frac{\partial Q_2}{\partial t} - D_2\Delta Q_2 = \frac{1}{\tau_2}(u_1 - Q_2),\\
&\frac{\partial v_1}{\partial t} - D_3\Delta v_1 = -e_1v_1 + d_1u_1, \quad \frac{\partial v_2}{\partial t} - D_4\Delta v_2 = -e_2v_2 + d_2u_2.
\end{aligned} \tag{4.2}$$

Each component is considered with homogeneous Neumann boundary conditions; additionally, we need the following initial condition

$$Q_i(0,x) = \int_{-\infty}^{t}\int_{0}^{\pi} G_i(x,y,-s)\frac{1}{\tau_i}\mathrm{e}^{\frac{s}{\tau_i}}u_j(s,y)\mathrm{d}y\mathrm{d}s, \quad i \neq j,\ i,j=1,2. \tag{4.3}$$

Similar to (4.1)-(4.3), the equations of (2.14) are rewritten as:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = u_1\big(b_1 - a_{11}u_1 - a_{12}Q_1\big),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2\big(b_2 - a_{21}Q_2 - a_{22}u_2\big),\\
&\frac{\partial Q_1}{\partial t} - D_1\Delta Q_1 = \frac{1}{\tau_1}(u_2 - Q_1),\\
&\frac{\partial Q_2}{\partial t} - D_2\Delta Q_2 = \frac{1}{\tau_2}(u_1 - Q_2).
\end{aligned} \tag{4.4}$$

Consider the following system

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = u_1\left(4 - 2u_1 - 2\int_{-\infty}^{t}\int_0^{\pi} G_1(x,y,t-s)\mathrm{e}^{-(t-s)}u_2(s,y)\mathrm{d}y\mathrm{d}s\right),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2\left(1 - \int_{-\infty}^{t}\int_0^{\pi} G_2(x,y,t-s)\frac{1}{2}\mathrm{e}^{-\frac{t-s}{2}}u_1(s,y)\mathrm{d}y\mathrm{d}s - 2u_2\right),
\end{aligned} \tag{4.5}$$

where $b_1 = 4$; $a_{11} = 2$; $a_{12} = 2$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 2$; $\tau_1 = 1$; $\tau_2 = 2$. Obviously $\frac{b_1}{b_2} > \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}$ holds, it follows from (i) of Theorem 2.1 that system (4.5) has a semi-trivial steady state $(3, 0)$, which attracts all positive solutions of system (4.5).

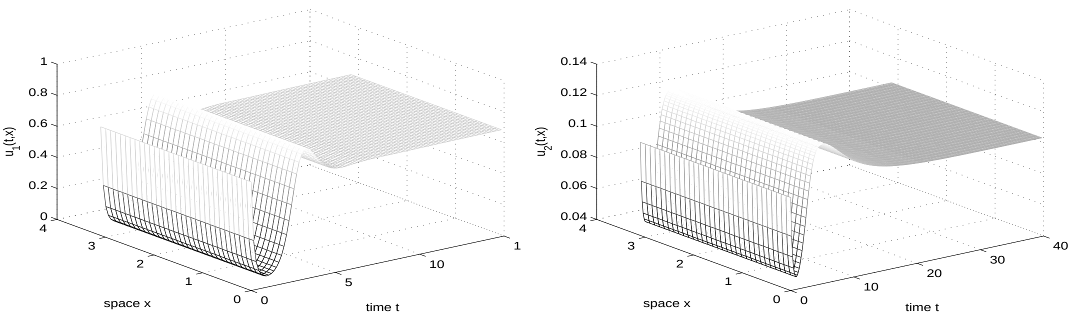
Now, we show the influence of feedback controls on dynamic behaviors of system (4.5), and consider the following feedback controls system (4.6)

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = u_1\left(4 - 2u_1 - 2\int_{-\infty}^{t}\int_0^{\pi} G_1(x,y,t-s)\mathrm{e}^{-(t-s)}u_2(s,y)\mathrm{d}y\mathrm{d}s - c_1v_1\right),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2\left(1 - \int_{-\infty}^{t}\int_0^{\pi} G_2(x,y,t-s)\frac{1}{2}\mathrm{e}^{-\frac{t-s}{2}}u_1(s,y)\mathrm{d}y\mathrm{d}s - 2u_2 - c_2v_2\right),\\
&\frac{\partial v_1}{\partial t} - D_3\Delta v_1 = -e_1v_1 + d_1u_1, \quad \frac{\partial v_2}{\partial t} - D_4\Delta v_2 = -e_2v_2 + d_2u_2.
\end{aligned} \tag{4.6}$$

Example 4.1 In system (4.6), set $c_1 = 7$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 2$; $e_2 = 2$; $d_2 = 1$. By computation, one has

$$\frac{a_{11}e_1 + c_1d_1}{a_{21}e_1} = 5.5 > \frac{b_1}{b_2} = 4 > \frac{a_{12}e_2}{a_{22}e_2 + c_2d_2} \approx 0.666\,7 \quad \text{and} \quad \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}},$$

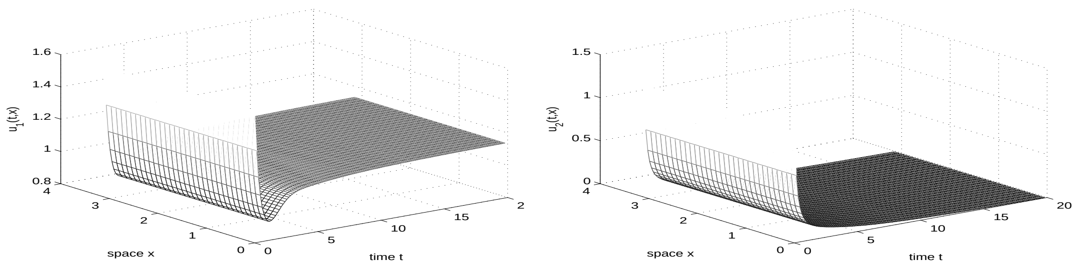
then conditions (H_1) and (H_2) hold. It follows from Theorem 3.1 that system (4.6) has a unique positive steady state $E^*(0.689\,7, 0.103\,4, 0.344\,8, 0.051\,7)$, which attracts all positive solutions of system (4.6). Note that species u_2 is extinct in system (4.5). However, species u_2 is globally stable in system (4.6), that is, feedback controls can make an extinct species in system (4.5) become globally stable. Figure 1 shows the dynamics behavior of system (4.6).



(a) u_1 (b) u_2

Figure 1: Dynamics behavior of system (4.6) with $c_1 = 7$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 2$; $e_2 = 2$; $d_2 = 1$.

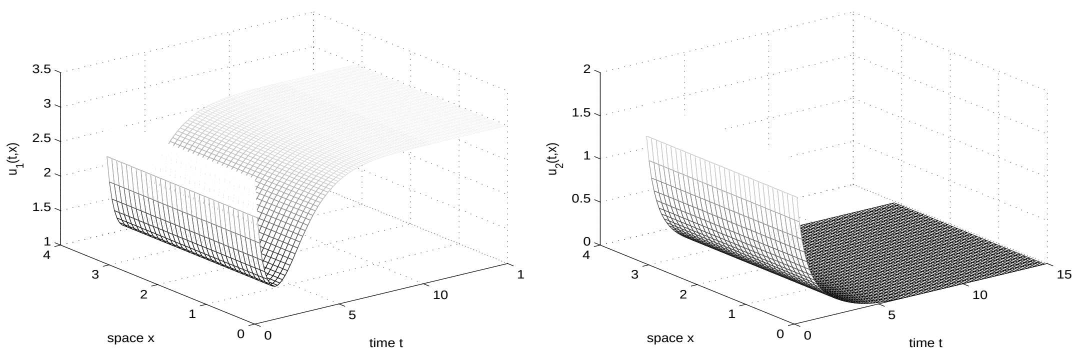
Example 4.2 In system (4.6), set $c_1 = 3$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 2$; $e_2 = 2$; $d_2 = 1$. By computation, (H_4) holds, but (H_3) is not satisfied, then it follows from Theorem 3.2 that system (4.6) has a semi-trivial steady state $E_1(1.142\,9, 0, 0.571\,4, 0)$, which attracts all positive solutions of system (4.6). Hence, by choosing suitable feedback controls variables, the extinct species u_2 in system (4.5) is still extinct. Figure 2 shows the dynamics behavior of system (4.6).



(a) u_1 (b) u_2

Figure 2: Dynamics behavior of system (4.6) with $c_1 = 3$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 2$; $e_2 = 2$; $d_2 = 1$.

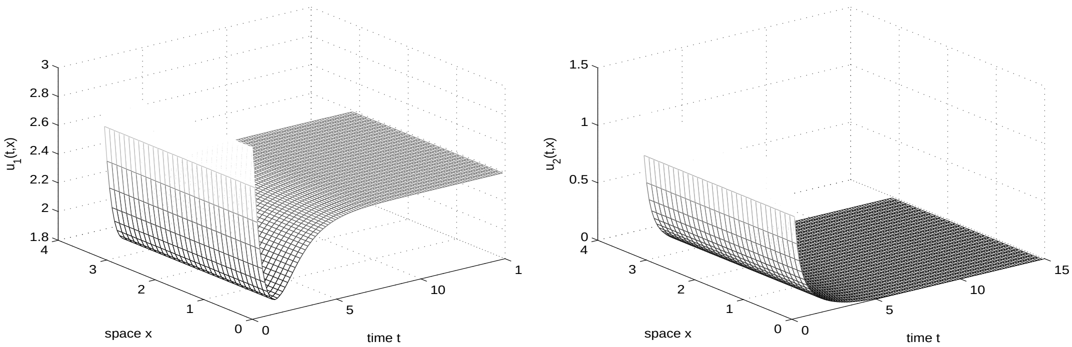
Example 4.3 In system (2.14), set $b_1 = 3$; $a_{11} = 1$; $a_{12} = 2$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 1$; $\tau_1 = 1$; $\tau_2 = 2$. Obviously $\frac{a_{11}}{a_{21}} < \frac{a_{12}}{a_{22}}$, which does not satisfy the condition of Theorem 2.1 (ii), thus (ii) of Theorem 2.1 fails to study system (2.14). But $\frac{b_1}{b_2} \geq \frac{a_{11}}{a_{21}}$, $\frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ hold, then it follows from (ii) of Corollary 3.1 that system (2.14) has a semi-trivial steady state $(3, 0)$, which attracts all positive solutions of system (2.14). Figure 3 shows the dynamics behavior of system (2.14).



(a) u_1 (b) u_2

Figure 3: Dynamics behavior of system (2.14) with $b_1 = 3$; $a_{11} = 1$; $a_{12} = 2$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 1$; $\tau_1 = 1$; $\tau_2 = 2$.

Example 4.4 In system (1.1), $b_i, a_{ij}, \tau_i,\ i,j = 1,2$ are the same as those in Example 4.3. Let $c_1 = 0.5;\ e_1 = 1;\ d_1 = 0.5;\ c_2 = 2;\ e_2 = 2;\ d_2 = 1$. Obviously ($H_4$) holds, but ($H_3$) is not satisfied, then it follows from Theorem 3.3 that system (1.1) has a semi-trivial steady state $E_1(2.4, 0, 1.2, 0)$, which attracts all positive solutions of system (1.1). Then, the extinct species u_2 in system (1.1) retains the property of extinction under suitable feedback controls variables. Figure 4 shows the dynamics behavior of system (1.1).



(a) u_1 (b) u_2

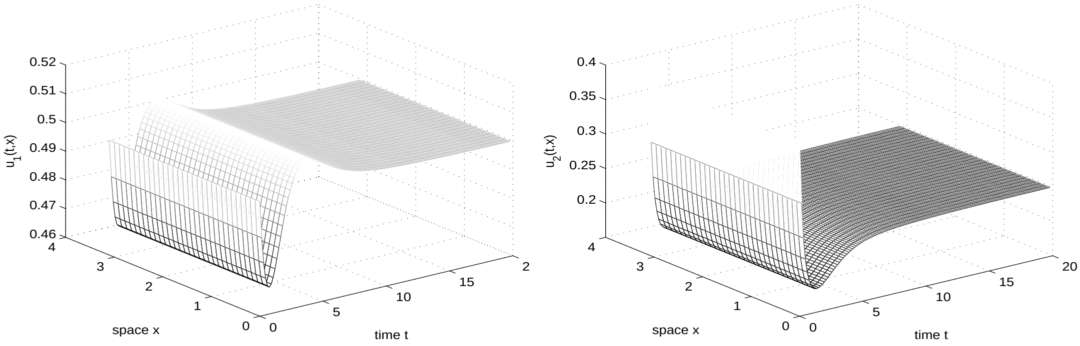
Figure 4: Dynamics behavior of system (1.1) with $b_1 = 3;\ a_{11} = 1;\ a_{12} = 2;\ b_2 = 1;\ a_{21} = 1;\ a_{22} = 1;\ c_1 = 0.5;\ e_1 = 1;\ d_1 = 0.5;\ c_2 = 2;\ e_2 = 2;\ d_2 = 1;\ \tau_1 = 1;\ \tau_2 = 2$.

Consider the following system

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - D_1\Delta u_1 = u_1\left(1 - 3u_1 - 2\int_{-\infty}^{t}\int_0^{\pi} G_1(x,y,t-s)\mathrm{e}^{-(t-s)}u_2(s,y)\mathrm{d}y\mathrm{d}s\right),\\
&\frac{\partial u_2}{\partial t} - D_2\Delta u_2 = u_2\left(1 - \int_{-\infty}^{t}\int_0^{\pi} G_2(x,y,t-s)\frac{1}{2}\mathrm{e}^{-\frac{t-s}{2}}u_1(s,y)\mathrm{d}y\mathrm{d}s - u_2\right),
\end{aligned} \tag{4.7}$$

where $b_1 = 2;\ a_{11} = 3;\ a_{12} = 1;\ b_2 = 1;\ a_{21} = 1;\ a_{22} = 1;\ \tau_1 = 1;\ \tau_2 = 2$. Then $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ holds. It follows from (i) of Theorem 2.1 that system (4.7) has a unique positive steady state $(0.5, 0.5)$, which attracts all positive solutions of system (4.7).

Example 4.5 In system (1.1), $b_i, a_{ij}, \tau_i,\ i,j = 1,2$ are chosen the same as those in system (4.7). Let $c_1 = 1;\ e_1 = 1;\ d_1 = 0.5;\ c_2 = 2;\ e_2 = 2;\ d_2 = 1$. Note that $\frac{a_{11}}{a_{21}} > \frac{b_1}{b_2} > \frac{a_{12}}{a_{22}}$ implies (H_1) and (H_2) hold, then it follows from Theorem 3.1 that system (1.1) has a semi-trivial steady state $E^*(0.5, 0.25, 0.25, 0.125)$, which attracts all positive solutions of system (1.1). Hence, in this case, feedback controls have no influence on the stability of system (1.1), that is, feedback controls only change the value of the positive steady state and keep the property of stability. Figure 5 shows the dynamics behavior of system (1.1).



(a) u_1 (b) u_2

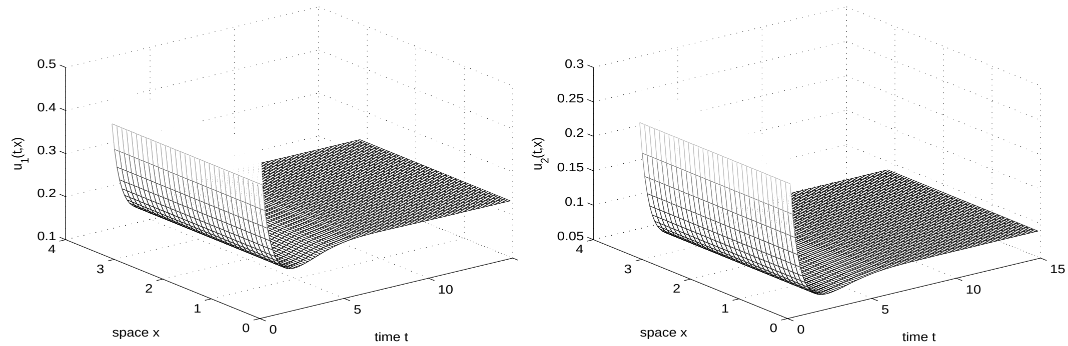
Figure 5: Dynamics behavior of system (1.1) with $b_1 = 2$; $a_{11} = 3$; $a_{12} = 1$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 1$; $c_1 = 1$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 2$; $e_2 = 2$; $d_2 = 1$; $\tau_1 = 1$; $\tau_2 = 2$.

Consider the following system

$$\begin{aligned}&\frac{\partial u_1}{\partial t}-D_1\Delta u_1=u_1\left(1-3u_1-2\int_{-\infty}^{t}\int_0^{\pi}G_1(x,y,t-s)\mathrm{e}^{-(t-s)}u_2(s,y)\mathrm{d}y\mathrm{d}s\right),\\&\frac{\partial u_2}{\partial t}-D_2\Delta u_2=u_2\left(1-\int_{-\infty}^{t}\int_0^{\pi}G_2(x,y,t-s)\frac{1}{2}\mathrm{e}^{-\frac{t-s}{2}}u_1(s,y)\mathrm{d}y\mathrm{d}s-u_2\right),\end{aligned}\tag{4.8}$$

where $b_1 = 1$; $a_{11} = 3$; $a_{12} = 2$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 1$; $\tau_1 = 1$; $\tau_2 = 2$. Then $\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}} > \frac{b_1}{b_2}$ holds. It follows from (iii) of Theorem 2.1 that system (4.8) has a semi-trivial steady state $(0, 1)$, which attracts all positive solutions of system (4.8).

Example 4.6 In system (1.1), $b_i, a_{ij}, \tau_i,\ i, j = 1, 2$ are chosen the same as those in system (4.8). Let $c_1 = 1$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 3$; $e_2 = 2$; $d_2 = 5$. Obviously, (H_1) and (H_2) hold, then it follows from Theorem 3.1 that system (1.1) has a semi-trivial steady state $E^*(0.23, 0.09, 0.11, 0.23)$, which attracts all positive solutions of system (1.1). Hence, suitable feedback controls can make an extinct species u_1 in system (4.8) become globally stable. Figure 6 shows the dynamics behavior of system (1.1).



(a) u_1 (b) u_2

Figure 6: Dynamics behavior of system (1.1) with $b_1 = 1$; $a_{11} = 3$; $a_{12} = 2$; $b_2 = 1$; $a_{21} = 1$; $a_{22} = 1$; $c_1 = 1$; $e_1 = 1$; $d_1 = 0.5$; $c_2 = 3$; $e_2 = 2$; $d_2 = 5$; $\tau_1 = 1$; $\tau_2 = 2$.

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