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A Simple Proof of Regularity for $C^{1,\alpha}$ Interface Transmission Problems

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Abstract. We give an alternative proof of a recent result in [1] by Caffarelli, Soria-Carro, and Stinga about the $C^{1,\alpha}$ regularity of weak solutions to transmission problems with $C^{1,\alpha}$ interfaces. Our proof does not use the mean value property or the maximum principle, and also works for more general elliptic systems with variable coefficients. This answers a question raised in [1]. Some extensions to $C^{1,\mathrm{Dini}}$ interfaces and to domains with multiple sub-domains are also discussed.

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1 Introduction and main results

In a recent paper [1], Caffarelli, Soria-Carro, and Stinga studied the following transmission problem. Let $\Omega\in\mathbb{R}^d$ be a smooth bounded domain with $d\geq 2$, and Ω_1 be a sub-domain of Ω such that $\Omega_1\subset\subset\Omega$ and $\Omega_2=\Omega\setminus\overline{\Omega_1}$. Assume that the interfacial boundary $\Gamma(=\partial\Omega_1)$ between Ω_1 and Ω_2 is $C^{1,\alpha}$ for some $\alpha\in(0,1)$. Consider the elliptic problem with the transmission conditions

$$\begin{cases}\Delta u=0 & \text{in } \ \Omega_1\cup\Omega_2,\\ u=0 & \text{on } \ \partial\Omega,\\ u|^+_\Gamma=u|^-_\Gamma, \quad \partial_\nu u|^+_\Gamma-\partial_\nu u|^-_\Gamma=g, & \end{cases} \tag{1.1}$$

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where g is a given function on Γ, ν is the unit normal vector on Γ which is pointing inside Ω_1, and $u|_\Gamma^+$ and $u|_\Gamma^-$ (and $\partial_\nu u|_\Gamma^+$ and $\partial_\nu u|_\Gamma^-$) are the left and right limit of u (and its normal derivative, respectively) on Γ in Ω_1 and Ω_2. The main result of [1] can be formulated as the following theorem.

Theorem 1.1. *Under the assumptions above, for any $g\in C^\alpha(\Gamma)$, there is a unique weak solution $u\in H^1(\Omega)$ to* (1.1)*, which is piecewise $C^{1,\alpha}$ up to the boundary in Ω_1 and Ω_2 and satisfies*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega_1})}+\|u\|_{C^{1,\alpha}(\overline{\Omega_2})}\le N\|g\|_{C^\alpha(\Gamma)},$$

where $N=N(d,\alpha,\Omega,\Gamma)>0$ is a constant.

The proof in [1] uses the mean value property for harmonic functions and the maximum principle together with an approximation argument. We refer the reader to [1] for earlier results about the transmission problem with smooth interfacial boundaries. The main feature of Theorem 1.1 is that Γ is only assumed to be in $C^{1,\alpha}$, which is weaker than those in the literature. In Remark 4.5 of [1], the authors raised the question of transmission problems with variable coefficient operator and mentioned two main difficulties in carrying over their proof to the general case.

In this paper, we answer this question by giving a alternative proof of Theorem 1.1, which does not invoke the mean value property or the maximum principle, and also works for more general non-homogeneous elliptic systems in the form

$$\begin{cases} \mathcal{L}u:=D_k(A^{kl}D_lu)=\operatorname{div}F+f & \text{in } \ \Omega_1\cup\Omega_2,\\ u=0 & \text{on } \ \partial\Omega,\\ u|_\Gamma^+=u|_\Gamma^-, \quad A^{kl}D_lu\nu_k|_\Gamma^+-A^{kl}D_lu\nu_k|_\Gamma^-=g, \end{cases} \tag{1.2}$$

where the Einstein summation convention in repeated indices is used,

$$u=(u^1,\cdots,u^n)^\top,\quad F_k=(F_k^1,\cdots,F_k^n)^\top,\quad f=(f^1,\cdots,f^n)^\top,\quad g=(g^1,\cdots,g^n)^\top,$$

are (column) vector-valued functions, for k, l=$1,\cdots,d$, A^{kl}=$A^{kl}(x)$ are $n\times n$ matrices, which are bounded and satisfy the strong ellipticity with ellipticity constant $\kappa>0$:

$$\kappa|\xi|^2\le A^{kl}_{ij}\xi^i_k\xi^j_l,\qquad |A^{kl}|\le\kappa^{-1},$$

for any $\xi=(\xi^i_k)\in\mathbb{R}^{n\times d}$.

Theorem 1.2. *Assume that Ω_1, Ω_2, and Γ satisfy the conditions in Theorem* 1.1*, A^{kl} and F are piecewise C^α in Ω_1 and Ω_2, $g\in C^\alpha(\Gamma)$, and $f\in L_\infty(\Omega)$. Then there is a* *unique weak solution $u\in H^1(\Omega)$ to* (1.2)*, which is piecewise $C^{1,\alpha}$ up to the boundary in Ω_1 and Ω_2 and satisfies*

$$\sum_{j=1}^{2}\|u\|_{C^{1,\alpha}(\overline{\Omega_j})}\le N\|g\|_{C^{\alpha}(\Gamma)}+N\sum_{j=1}^{2}\|F\|_{C^{\alpha}(\Omega_j)}+N\|f\|_{L_\infty(\Omega)},$$

where $N=N(d,n,\kappa,\alpha,\Omega,\Gamma,[A]_{C^{\alpha}(\Omega_j)})>0$ is a constant.

We also consider the transmission problem with multiple disjoint sub-domains $\Omega_1,\cdots,\Omega_M$ with $C^{1,\alpha}$ interfacial boundaries in the setting of [5,6]. As in these papers, we assume that any point $x\in\Omega$ belongs to the boundaries of at most two of the $\Omega_j's$, so that if the boundaries of two Ω_j touch, then they touch on a whole component of such a boundary. Without loss of generality assume that $\Omega_j\subset\subset\Omega$, $j=1,\cdots,M-1$ and $\partial\Omega\subset\partial\Omega_M$. The transmission problem in this case is then given by

$$\begin{cases}\mathcal{L}u=\operatorname{div}F+f & \text{in } \bigcup_{j=1}^{M}\Omega_j,\\ u=0 & \text{on } \partial\Omega,\\ u|^{+}_{\partial\Omega_j}=u|^{-}_{\partial\Omega_j},\quad A^{kl}D_lu\nu_k|^{+}_{\partial\Omega_j}-A^{kl}D_lu\nu_k|^{-}_{\partial\Omega_j}=g_j, & j=1,\cdots,M-1.\end{cases}\tag{1.3}$$

In the following theorem, we obtain an estimate which is independent of the distance of interfacial boundaries, but may depend on the number of sub-domains M.

Theorem 1.3. *Assume that Ω_j satisfy the conditions above, A^{kl} and F are piecewise $C^{\alpha'}$ for some $\alpha'\in(0,\alpha/(1+\alpha)]$, $g_j\in C^{\alpha'}(\partial\Omega_j)$, $j=1,\cdots,M-1$, and $f\in L_\infty(\Omega)$. Then there is a unique weak solution $u\in H^1(\Omega)$ to* (1.3)*, which is piecewise $C^{1,\alpha'}$ up to the boundary in Ω_j, $j=1,\cdots,M$, and satisfies*

$$\sum_{j=1}^{M}\|u\|_{C^{1,\alpha'}(\overline{\Omega_j})}\le N\sum_{j=1}^{M-1}\|g\|_{C^{\alpha'}(\partial\Omega_j)}+N\sum_{j=1}^{M}\|F\|_{C^{\alpha'}(\Omega_j)}+N\|f\|_{L_\infty(\Omega)},$$

where $N=N(d,n,M,\kappa,\alpha,\alpha',\Omega_j,[A]_{C^{\alpha'}(\Omega_j)})>0$ is a constant.

It is worth noting that in the special case when $A^{\alpha\beta}$ and F are Hölder continuous in the whole domain, by the linearity the result of Theorem 1.3 still holds with $\alpha'=\alpha$.

Our last result concerns the case when the interfaces are $C^{1,\mathrm{Dini}}$, and A^{kl} satisfy the piecewise Dini mean oscillation in Ω, i.e., the function

$$\omega_A(r):=\sup_{x_0\in\Omega}\inf_{\bar{A}\in\mathcal{A}}\Big(\fint_{\Omega_r(x_0)}|A(x)-\bar{A}|\,dx\Big)$$

satisfies the Dini condition, where $\Omega_r(x_0)=B_r(x_0)\cap\Omega$ and $\mathcal{A}$ is the set of piecewise constant functions in Ω_j, $j=1,\cdots,M$.

Theorem 1.4. *Assume that* Ω_j *satisfy the* $C^{1,Dini}$ *condition,* A^{kl} *and* F *are of piecewise Dini mean oscillation in* Ω, g_j *is Dini continuous on* $\partial\Omega_j$, $j=1,\cdots,M-1$, *and* $f\in L_\infty(\Omega)$. *Then there is a unique weak solution* $u\in H^1(\Omega)$ *to* (1.3), *which is piecewise* C^1 *up to the boundary in* Ω_j, $j=1,\cdots,M$.

We note that the piecewise Dini mean oscillation condition is weaker than the usual piecewise Dini continuity condition in the L_∞ sense.

2 Proofs

The idea of the proof is to reduce the transmission problem to an elliptic equation (system) with piecewise Hölder (or Dini) non-homogeneous terms, by solving a conormal boundary value problem. These equations arose from composite material and have been extensively studied in the literature. See, for instance, [5,6], and also recent papers [2,4]. We will apply the results in the latter two papers, the proofs of which in turn are based on Campanato's approach.

Proof of Theorem 1.2. Let $w\in H^1(\Omega)$ be the weak solution to the conormal boundary value problem

$$\begin{cases} \Delta w=c & \text{in } \ \Omega_1, \\ w_\nu=g & \text{on } \ \partial\Omega_1, \\ \displaystyle\int_{\Omega_1} w\,dx=0, \end{cases} \tag{2.1}$$

where $c=-|\Gamma|^{-1}\int_\Gamma g$ is a constant. The existence and uniqueness of such solution w follows from the trace theorem and the Lax–Milgram theorem, and

$$\|w\|_{H^1(\Omega_1)}\le N\|g\|_{L_2(\Gamma)}, \tag{2.2}$$

where $N=N(d,\Omega_1)$. Since $g\in C^\alpha(\Omega_1)$, by the classical elliptic theory (see, for instance, [7, Theorem 5.1]), we have

$$\|w\|_{C^{1,\alpha}(\Omega_1)}\le N\|g\|_{C^\alpha(\Gamma)}, \tag{2.3}$$

where $N=N(d,\alpha,\Omega_1)$. By using the weak formulation of solutions, from (2.1) it is easily seen that (1.2) is equivalent to

$$\begin{cases} \mathcal{L}u=\operatorname{div}\tilde{F}+\tilde{f} & \text{in } \ \Omega, \\ u=0 & \text{on } \ \partial\Omega, \end{cases} \tag{2.4}$$

where

$$\tilde{F}=1_{\Omega_1\cup\Omega_2}F-1_{\Omega_1}\nabla w,\qquad \tilde{f}=f+1_{\Omega_1}c.$$

By the Lax–Milgram theorem, there is a unique solution $u\in H^1(\Omega)$ to (2.4) and

$$\begin{aligned}\|u\|_{H^1(\Omega)}&\le N\|F\|_{L_2(\Omega)}+\|\nabla w\|_{L_2(\Omega_1)}+\|f\|_{L_2(\Omega)}+\|c\|_{L_2(\Omega_1)},\\ &\le N\|F\|_{L_2(\Omega)}+\|g\|_{L_2(\Gamma)}+\|f\|_{L_2(\Omega)},\end{aligned}\tag{2.5}$$

where we used (2.2) in the second inequality. Since $\tilde{F}$ and $A^{\alpha\beta}$ are piecewise C^α, it follows from [2, Corollary 2 and Remark 3(ii)], (2.3), and (2.5) that

$$\begin{aligned}&\sum_{j=1}^{2}\|u\|_{C^{1,\alpha}(\overline{\Omega_j})}\\ \le& N\|u\|_{L_2(\Omega)}+N\|F-\nabla w\|_{C^\alpha(\Omega_1)}+N\|F\|_{C^\alpha(\Omega_2)}+\|f+1_{\Omega_1}c\|_{L_\infty(\Omega)}\\ \le& N\|g\|_{C^\alpha(\Gamma)}+N\sum_{j=1}^{2}\|F\|_{C^\alpha(\Omega_j)}+N\|f\|_{L_\infty(\Omega)}.\end{aligned}$$

The theorem is proved. □

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2. In each Ω_j, $j=1,\cdots,M-1$, we find a weak solution to

$$\begin{cases}\Delta w_j=c_j & \text{in } \Omega_j,\\ \partial_\nu w_j|^{+}_{\partial\Omega_j}=g_j & \text{on } \partial\Omega_j,\\ \displaystyle\int_{\Omega_j} w_j\,dx=0,\end{cases}\tag{2.6}$$

where

$$c_j=-|\partial\Omega_j|^{-1}\int_{\partial\Omega_j}g_j$$

and w_j satisfies

$$\|w_j\|_{C^{1,\alpha'}(\Omega_j)}\le N\|g_j\|_{C^{\alpha'}(\partial\Omega_j)}.\tag{2.7}$$

By using the weak formulation of solutions, it is easily seen that (1.3) is equivalent to

$$\begin{cases}\mathcal{L}u=\operatorname{div}\tilde{F}+\tilde{f} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{2.8}$$

where

$$\tilde{F}=1_{\cup_{j=1}^{M}\Omega_j}F-\sum_{j=1}^{M-1}1_{\Omega_j}\nabla w_j,\qquad \tilde{f}=f+\sum_{j=1}^{M-1}1_{\Omega_j}c_j.$$

As before, by the Lax–Milgram theorem, there is a unique solution $u\in H^1(\Omega)$ to (2.8). Since $\tilde{F}$ and $A^{\alpha\beta}$ are piecewise $C^{\alpha'}$ and $\partial\Omega_j$ is piecewise $C^{1,\alpha}$, by using (2.7) and appealing to [4, Corollary 1.2 and Remark 1.4], we conclude the proof of the theorem. $\square$

Finally, we give

Proof of Theorem 1.4. We claim that under the conditions of the theorem, if w_j is the solution to (2.6), then Dw_j satisfies the L_2-Dini mean oscillation condition in Ω_j. Assuming this is true, then the conclusion of the theorem follows from the proof of Theorem 1.3 and [4, Theorem 1.1]. We remark that the C^1 continuity of w_j was proved in [7, Theorem 5.1] for more general quasilinear equations, but in general Dw_j may not be Dini continuous in the L_∞ sense.

To prove the claim, we follow the argument in the proof of Theorem 1.7 of [3]. We only give the boundary estimate since the corresponding interior estimate is simpler. By using the $C^{1,\mathrm{Dini}}$ regularity of Ω_j and locally flattening the boundary, it then suffices to verify Lemma 2.1 below. $\square$

In the sequel, we denote $x=(x',x_d)$, where $x'=(x_1,x_2,\cdots,x_{d-1})\in\mathbb{R}^{d-1}$, and $\Gamma_r(x):=B_r(x)\cap\{x_d=0\}$ for $x\in\mathbb{R}^d$ and $r>0$. We say that a function f satisfies the L_2-Dini mean oscillation in Ω if

$$\tilde{\omega}_f(r):=\sup_{x_0\in\Omega}\left(\fint_{\Omega_r(x_0)}|f(x)-(f)_{\Omega_r(x_0)}|^2dx\right)^{1/2}$$

satisfies the Dini condition.

Lemma 2.1. *Let $u\in H^1(B_4^+)$ be a weak solution to*

$$D_k(a^{kl}D_lu)=0 \quad in \quad B_4^+$$

with the conormal boundary condition $a^{dl}D_lu=g(x')$ on $\Gamma_4=B_4\cap\{x_d=0\}$, where $a^{kl}=a^{kl}(x)$ satisfy the uniform ellipticity condition and are of L_2-Dini mean oscillation in B_4^+, and g is a Dini continuous function on Γ_4. Then Du is of L_2-Dini mean oscillation in $\overline{B_1^+}$.

Proof. We set

$$g^d(x)=g^d(x',x_d):=g(x'),$$

which satisfies $D_dg^d=0$. Therefore, the above problem is reduced to the standard conormal boundary problem

$$\begin{cases} D_k(a^{kl}D_lw)=D_dg^d & \text{in} \quad B_4^+, \\ a^{dl}D_lw=g^d & \text{on} \quad \Gamma_4. \end{cases}$$

Similar to [3, Section 3], for $x\in\overline{B_3^+}$ and $r\in(0,1)$, we define

$$\phi(x,r):=\left(\fint_{B_r(x)\cap B_4^+}|Du-(Du)_{B_r(x)\cap B_4^+}|^2\right)^{\frac{1}{2}},$$

where

$$(Du)_{B_r(x)\cap B_4^+}=\fint_{B_r(x)\cap B_4^+}Du.$$

Fix a smooth domain $\mathcal{D}$ satisfying

$$B_{1/2}^+\subset\mathcal{D}\subset B_1^+$$

and for $\bar{x}\in\partial\mathbb{R}^d_+$, we set $\mathcal{D}_r(\bar{x})=r\mathcal{D}+\bar{x}$. We decompose $u=w+v$, where $w\in H^1(\mathcal{D}_r(\bar{x}))$ is a weak solution of the problem

$$\begin{cases} D_k(\bar{a}^{kl}D_lw)=-D_k((a^{kl}-\bar{a}^{kl})D_lu)+D_d(g^d-\bar{g}^d) & \text{in } \ \mathcal{D}_r(\bar{x}),\\ \bar{a}^{kl}D_lw\nu_k=-(a^{kl}-\bar{a}^{kl})D_lu\nu_k+(g^d-\bar{g}^d)\nu_d & \text{on } \ \partial\mathcal{D}_r(\bar{x}),\end{cases}$$

where $\bar{a}^{kl}$ and $\bar{g}^d$ are the average of a^{kl} and g^d in $\mathcal{D}_r(\bar{x})$, respectively. By the H^1-estimate, we have

$$\left(\fint_{B_r^+(\bar{x})}|Dw|^2\right)^{1/2}\le N\tilde{\omega}_A(2r)\|Du\|_{L^\infty(B_{2r}^+(\bar{x}))}+N\breve{\omega}_g(2r). \tag{2.9}$$

Here $\breve{\omega}_g$ is the modulus of continuity of g in the L_∞ sense. Note that $v:=u-w$ satisfies

$$\begin{aligned} &D_k(\bar{a}^{kl}D_lv)=D_d\bar{g}^d && \text{in } \ B_r^+(\bar{x}),\\ &\bar{a}^{dl}D_lv=\bar{g}^d && \text{on } \ \Gamma_r(\bar{x}).\end{aligned}$$

Then for any $c\in\mathbb{R}$ and $k=1,2,\cdots,d-1$, $\tilde{v}:=D_kv-c$ satisfies

$$\begin{aligned} &D_k(\bar{a}^{kl}D_l\tilde{v})=0 && \text{in } \ B_r^+(\bar{x}),\\ &\bar{a}^{dl}D_l\tilde{v}=0 && \text{on } \ \Gamma_r(\bar{x}).\end{aligned}$$

By the standard elliptic estimates for equations with constant coefficients and zero conormal boundary data, we have for any $c\in\mathbb{R}$,

$$\|DD_kv\|_{L^\infty(B_{r/2}^+(\bar{x}))}\le Nr^{-1}\left(\fint_{B_r^+(\bar{x})}|D_kv-c|^2\right)^{1/2},\qquad k=1,\cdots,d-1.$$

Then by using

$$D_{dd}v=-\frac{1}{\bar{a}^{dd}}\sum_{(i,j)\neq(d,d)}\bar{a}^{ij}D_{ij}v,$$

we obtain

$$\|D^2v\|_{L^\infty(B^+_{r/2}(\bar{x}))}\le N\|DD_{x'}v\|_{L^\infty(B^+_{r/2}(\bar{x}))}\le Nr^{-1}\left(\fint_{B^+_r(\bar{x})}|D_{x'}v-c|^2\right)^{1/2},$$

where we used the notation $D_{x'}v=(D_1v,\cdots,D_{d-1}v)$. Therefore, we have

$$\|D^2v\|_{L^\infty(B^+_{r/2}(\bar{x}))}\le Nr^{-1}\left(\fint_{B^+_r(\bar{x})}|Dv-q|^2\right)^{1/2},\qquad \forall q\in\mathbb{R}^d.$$

Let $\mu\in(0,1/2)$ be a small number. Since

$$\left(\fint_{B^+_{\mu r}(\bar{x})}\left|Dv-(Dv)_{B^+_{\mu r}(\bar{x})}\right|^2\right)^{1/2}\le 2\mu r\|D^2v\|_{L^\infty(B^+_{\mu r}(\bar{x}))},$$

we see that there is a constant $N_0=N_0(d,\kappa)>0$ such that

$$\left(\fint_{B^+_{\mu r}(\bar{x})}\left|Dv-(Dv)_{B^+_{\mu r}(\bar{x})}\right|^2\right)^{1/2}\le N_0\mu\left(\fint_{B^+_r(\bar{x})}|Dv-q|^2\right)^{1/2},\qquad \forall q\in\mathbb{R}^d.$$

By using the decomposition $u=v+w$, we obtain from the above and the triangle inequality that

$$\begin{aligned}&\left(\fint_{B^+_{\mu r}(\bar{x})}\left|Du-(Dv)_{B^+_{\mu r}(\bar{x})}\right|^2\right)^{1/2}\\ \le&\left(\fint_{B^+_{\mu r}(\bar{x})}\left|Dv-(Dv)_{B^+_{\mu r}(\bar{x})}\right|^2\right)^{1/2}+\left(\fint_{B^+_{\mu r}(\bar{x})}|Dw|^2\right)^{1/2}\\ \le& N_0\mu\left(\fint_{B^+_r(\bar{x})}|Du-q|^2\right)^{1/2}+N\mu^{-d/2}\left(\fint_{B^+_r(\bar{x})}|Dw|^2\right)^{1/2}.\end{aligned}$$

Setting $q=(Du)_{B^+_r(\bar{x})}$ and using (2.9), we obtain

$$\phi(\bar{x},\mu r)\le N_0\mu\phi(\bar{x},r)+N\mu^{-d/2}\Big(\tilde{\omega}_A(2r)\|Du\|_{L^\infty(B^+_{2r}(\bar{x}))}+\breve{\omega}_g(2r)\Big).\tag{2.10}$$

By using an iteration argument as in the proof of [3, Theorem 1.7], from (2.10) and the corresponding interior estimate, it is easily seen that Du is of L_2-Dini mean oscillation in $\overline{B^+_1}$ with a modulus of continuity depending on d, κ, $\|Du\|_{L_2(B^+_4)}$, $\breve{\omega}_g$, and $\hat{\omega}_A$. The lemma is proved. □

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References

[1] L. A. Caffarelli, M. Soria-Carro, and P. R. Stinga, Regularity for $C^{1,\alpha}$ interface transmission problems, arXiv:2004.07322.

[2] Hongjie Dong, Gradient estimates for parabolic and elliptic systems from linear laminates, Arch. Rational Mech. Anal., 205 (2012), pp. 119–149.

[3] Hongjie Dong, Jihoon Lee, and Seick Kim, On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients, Indiana Univ. Math. J., 69(6) (2020), pp. 1815–1853.

[4] Hongjie Dong, and Longjuan Xu, Gradient estimates for divergence form elliptic systems arising from composite material, SIAM J. Math. Anal., 59(3) (2019), pp. 2444–2478.

[5] Yanyan Li, and Louis Nirenberg, Estimates for elliptic systems from composite material, Commun. Pure Appl. Math., 56 (2003), pp. 892–925.

[6] Y. Y. Li, and M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Rational Mech. Anal., 153 (2000), pp. 91–151.

[7] G. M. Lieberman, Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions, Ann. Mat. Pura Appl., 148(4) (1987), pp. 77–99.