Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 3, Number 2, June 2021, 263–273 DOI:10.12150/jnma.2021.263

# Mixed Boundary Value Problems for a Class of Fractional Differential Equations with Impulses*

Jin You[1] and Shurong Sun[1,†]

**Abstract** We investigate a class of mixed boundary value problem of nonlinear impulsive fractional differential equations with a parameter. The uniqueness of this problem is proved by applying Banach fixed point theorem.

**Keywords** Fractional differential equations, Boundary value problem, Impulsive, Uniqueness.

**MSC(2010)** 34A08, 34A37, 34A12.

## 1. Introduction

Fractional differential equation can depict the phenomenon, dynamic procedure, electroanalytical chemistry as well as control theory, signal processing in our life more perfectly [6, 8, 10]. Boundary value problem of fractional differential equation has been studied by many scholars [1, 4, 5, 11–13, 15–18]. The research of impulsive differential equations began in 20th century, through researchers cooperation, the basic theory of existence of solution for impulsive differential equations had been set up [3, 9], many essential results have also been obtained [7, 14].

In [14], Guo studied the hybrid boundary value problem of fractional differential equations with impulse

$$\begin{cases} D^\alpha u(t) = f\big(t, u(t)\big), \quad 0 < t < T, 1 < \alpha \le 2, \\ \triangle u(t_k) = I_k\big(u(t_k)\big), \triangle u'(t_k) = I_k^*\big(u(t_k)\big), k = 1, 2, \cdots, p, \\ Tu'(0) = -au(0) - bu(T), Tu'(T) = cu(0) + du(T), a, b, c, d \in R. \end{cases}$$

where $D^\alpha$ is the Caputo derivative, $1 < \alpha \le 2$, $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J \backslash \{t_1, \cdots, t_p\}$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$. By using Leray-schauder fixed point theorem, the author obtained the existence and uniqueness of this problem.

Motivated by the above work, in this paper, we consider the uniqueness of the solution for a class of mixed boundary value problem of nonlinear impulsive

†Corresponding author.
Email address: sshrong@163.com(S. Sun), youjin6666@163.com(J. You)
[1]School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, China
*The authors were supported by National Natural Science Foundation of China (61703180, 61803176) and Shandong Provincial Natural Science Foundation (ZR2017MA043, ZR2017MA043).

differential equations of order $\alpha \in (2,3]$ given by

$$\begin{cases} D^{\alpha}u(t) = \lambda f\big(t, u(t)\big), \quad t \in J', \\ \triangle u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{1.1}$$

where $D^{\alpha}$ is the Caputo derivative, $2 < \alpha \leq 3$, $J = [0,1]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $Q_k, R_k, S_k \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $J' = J \backslash \{t_1, \cdots, t_m\}$, $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_m = (t_m, 1]$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $\triangle u''(t_k) = u''(t_k^+) - u''(t_k^-)$.

The paper is organized as follows. In Section 2, we introduce some necessary notions, basic definitions and Lemmas. In Section 3, by using Banach fixed point theorem, the uniqueness of solutions is proved. In Section 4, we give an example to demonstrate the applications of our results.

## 2. Preliminaries

**Definition 2.1.** ( [10]) The Caputo derivative of order $\alpha > 0$ of a continuous function $u : J \to \mathbb{R}$ is given by

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, n = [\alpha] + 1, \alpha > 0.$$

**Lemma 2.1.** *( [10]) Let $(X, d)$ be a complete metric space, and $\overline{\Omega}$ be a convex closed subset of $X$. $T : \overline{\Omega} \to \overline{\Omega}$ is a contraction mapping:*

$$d(Tx, Ty) \leq kd(x, y),$$

*where $0 < k < 1$, for each $x, y \in \overline{\Omega}$. Then, there exists a unique fixed point $x$ of $T$ in $\overline{\Omega}$, i.e. $Tx = x$.*

**Lemma 2.2.** *( [2]) Let $u(t) \in C[0,1]$, and $\alpha > 0$, the solution of the fractional differential equation*

$$D^{\alpha}u(t) = 0$$

*is $u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, C_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n, n = [\alpha] + 1$.*

**Lemma 2.3.** *For a given $f \in C(J \times \mathbb{R}, \mathbb{R})$, a function $u$ is a solution of the following impulsive boundary value problem*

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), 2 < \alpha \leq 3, \quad t \in J', \\ \triangle u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{2.1}$$

*if and only if $u$ is a solution of the impulsive fractional integral equation*

$$
\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f\big(s,u(s)\big)ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k}Q_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t_k-t_i)R_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{k}(t-t_k)R_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}S_i\big(u(t_i)\big)-a_1-a_2t-a_3t^2,\quad t\in J_k, k=0,1,2,\cdots,m,
\end{aligned}
\tag{2.2}
$$

*where*

$$
\begin{aligned}
a_1 =& -\bigg(\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(t_m^2-4t_m+3)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}Q_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}(t_m-t_i)R_i\big(u(t_i)\big)+\sum_{i=1}^{m}\frac{2t_m^2-8t_m+3}{4}S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m}(2-t_m)R_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}S_i\big(u(t_i)\big)
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&-\frac{3}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\bigg),
\end{aligned}
$$

$$a_2=-a_1, \tag{2.4}$$

$$a_3=\frac{1}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds+\frac{1}{4}\sum_{i=1}^{m}S_i\big(u(t_i)\big). \tag{2.5}$$

**Proof.** Let $u$ be a solution of (2.1). Then by Lemma 2.2, we have

$$
\begin{aligned}
u(t)&=I^\alpha f\big(t,u(t)\big)-a_1-a_2t-a_3t^2\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,u(s)\big)ds-a_1-a_2t-a_3t^2,t\in J_0.
\end{aligned}
$$

So we have

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f\big(s,u(s)\big)ds-a_2-2a_3t,t\in J_0.$$

And

$$u''(t)=\frac{1}{\Gamma(\alpha-2)}\int_0^t(t-s)^{\alpha-3}f\big(s,u(s)\big)ds-2a_3,t\in J_0.$$

If $t\in J_1$, then we get

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}f\big(s,u(s)\big)ds-b_1-b_2(t-t_1)-b_3(t-t_1)^2,$$

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t(t-s)^{\alpha-2}f\big(s,u(s)\big)ds-b_2-2b_3(t-t_1),$$

and

$$u''(t)=\frac{1}{\Gamma(\alpha-2)}\int_{t_1}^t(t-s)^{\alpha-3}f\big(s,u(s)\big)ds-2b_3.$$

Through calculation,

$$u(t_1^-)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f\big(s,u(s)\big)ds-a_1-a_2t_1-a_3t_1^2,u({t_1}^+)=-b_1,$$

$$u'(t_1^-)=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f\big(s,u(s)\big)ds-a_2-2a_3t_1,u'({t_1}^+)=-b_2,$$

and

$$u''(t_1^-)=\frac{1}{\Gamma(\alpha-2)}\int_0^{t_1}(t_1-s)^{\alpha-3}f\big(s,u(s)\big)ds-2a_3,u''({t_1}^+)=-2b_3.$$

On account of $\triangle u(t_k)=u(t_k^+)-u(t_k^-)$, $\triangle u'(t_k)=u'(t_k^+)-u'(t_k^-)$, $\triangle u''(t_k)=u''(t_k^+)-u''(t_k^-)$, we have the following formulas

$$
\begin{aligned}
\triangle u(t_1)&=u(t_1^+)-u(t_1^-)=Q_1\big(u(t_1)\big),\\
\triangle u'(t_1)&=u'(t_1^+)-u'(t_1^-)=R_1\big(u(t_1)\big),\\
\triangle u''(t_1)&=u'(t_1^+)-u'(t_1^-)=S_1\big(u(t_1)\big),
\end{aligned}
$$

apply the above formulas, we can get

$$-b_1 = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\big(s, u(s)\big) ds - a_1 - a_2 t_1 - a_3 t_1^2 + Q_1\big(u(t_1)\big),$$

$$-b_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} f\big(s, u(s)\big) ds - a_2 - 2a_3 t_1 + R_1\big(u(t_1)\big),$$

$$-2b_3 = \frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} (t_1 - s)^{\alpha-3} f\big(s, u(s)\big) ds - 2a_3 + S_1\big(u(t_1)\big).$$

Consequently, for $t \in J_1$,

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big) ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\big(s, u(s)\big) ds \\ & + \frac{(t-t_1)}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} f\big(s, u(s)\big) ds + Q_1\big(u(t_1)\big) + (t-t_1) R_1\big(u(t_1)\big) \\ & + \frac{(t-t_1)^2}{2\Gamma(\alpha-2)} \int_0^{t_1} (t_1 - s)^{\alpha-3} f\big(s, u(s)\big) ds + \frac{1}{2}(t-t_1)^2 S_1\big(u(t_1)\big) \\ & - a_1 - a_2 t - a_3 t^2. \end{aligned}$$

We can make a conclusion via the same discussion in $J_1$,

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k-1} \frac{(t_k - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k} \frac{(t - t_k)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k-1} \frac{(t_k - t_i)(t - t_k)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) ds \\ & + \sum_{i=1}^{k} Q_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t_k - t_i) R_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} S_i\big(u(t_i)\big) \\ & + \sum_{i=1}^{k} (t - t_k) R_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) S_i\big(u(t_i)\big) \\ & + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} S_i\big(u(t_i)\big) - a_1 - a_2 t - a_3 t^2, \quad t \in J_k, k = 0, 1, 2, \cdots, m, \end{aligned}$$

On the condition that $u(0)+u'(0)=0$, we have

$$a_1=-a_2. \tag{2.6}$$

By $u(1)+u'(1)=0$, we have

$$\begin{aligned}
a_2+3a_3 =&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(t_m^2-4t_m+3)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}Q_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}(t_m-t_i)R_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m}\frac{(t_m^2-4t_m+3)}{2}S_i\big(u(t_i)\big)+\sum_{i=1}^{m}(2-t_m)R_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds.
\end{aligned} \tag{2.7}$$

Combining the (2.6) (2.7) with the condition $u''(1)+u''(0)=0$, we conclude that

$$\begin{aligned}
a_1 =& -\bigg(\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m-1}\frac{(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}\frac{(t_m^2-4t_m+3)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{m}Q_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}(t_m-t_i)R_i\big(u(t_i)\big)+\sum_{i=1}^{m}\frac{2t_m^2-8t_m+3}{4}S_i\big(u(t_i)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}(2-t_m)R_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&-\frac{3}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\Big),
\end{aligned}
$$

$$
a_3=\frac{1}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds+\frac{1}{4}\sum_{i=1}^{m}S_i\big(u(t_i)\big).
$$

Conversely, if we assume that $u$ is the solution of (2.2), through directly calculation, it shows that the solution given by (2.1). This completes the proof.

In the later section, we introduce the following space for the sake of convenience:

$PC(J,\mathbb{R})=\{u:J\to\mathbb{R}\mid u\in C(J_k),u(t_k^+),u(t_k^-)$ exist, $u(t_k^-)=u(t_k),k=0,1,\cdots,m\}$, with the norm $\|u\|=\sup_{t\in J}|u(t)|$. Clearly, $PC(J,\mathbb{R})$ is a Banach space.

Assume the following hold:

$(H_1)$ There exist constants $L_1,L_2,L_3,L_4>0,u,v\in\mathbb{R}$ such that

$$
\begin{aligned}
&|f(t,u)-f(t,v)|\le L_1|u-v|,|Q_k(u)-Q_k(v)|\le L_2|u-v|,\\
&|R_k(u)-R_k(v)|\le L_3|u-v|,|S_k(u)-S_k(v)|\le L_4|u-v|.
\end{aligned}
$$

We define an open bounded set $\Omega=\{u\in PC(J,\mathbb{R})|\ \|u\|<r\}$, then $\overline{\Omega}=\{u\in PC(J,\mathbb{R})|\ \|u\|\le r\}$, where $r$ is a positive real number.

# 3. Main results

In this section, we will prove the uniqueness of (1.1) by Lemma 2.1.

First of all, we define an operator $T:\overline{\Omega}\to\overline{\Omega}$ as

$$
\begin{aligned}
Tu(t)=&\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f\big(s,u(s)\big)ds+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{\lambda(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{\lambda(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k}\frac{\lambda(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k-1}\frac{\lambda(t_k-t_i)(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds\\
&+\sum_{i=1}^{k}\frac{\lambda(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)ds
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k} Q_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t_k-t_i)R_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{k}(t-t_k)R_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)S_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}S_i\big(u(t_i)\big)-a_1-a_2t-a_3t^2,\quad t\in J_k, k=0,1,2,\cdots,m,
\end{aligned}
$$

where $a_1, a_2$ and $a_3$ are defined as (2.3)-(2.5).

Problem (1.1) has a solution if and only if $u = T_\lambda u$ has a fixed point.

We denote

$$
p=\frac{1-4mL_2-(4m-3)L_3-\frac{(22m-13)}{4}L_4}{\frac{3(m+1)L_1}{\Gamma(\alpha+1)}+\frac{(8m-1)L_1}{\Gamma(\alpha)}+\frac{(43m-15)L_1}{4\Gamma(\alpha-1)}},
$$

**Theorem 3.1.** *We assume that $(H_1)$ holds, $|\lambda| < p$, the problem (1.1) has a unique solution.*

**Proof.** From $(H_1)$, for any $u, v \in \overline{\Omega}, t \in J_k, k = 0, 1, \cdots, m$, we can get

$$
\begin{aligned}
|Tu(t)-Tv(t)|\le&\frac{|\lambda|}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\frac{|\lambda|}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k-1}\frac{|\lambda|(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k-1}\frac{|\lambda|(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k}\frac{|\lambda|(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k-1}\frac{|\lambda|(t_k-t_i)(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k}\frac{|\lambda|(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,u(s))-f(s,v(s))\big)|ds\\
&+\sum_{i=1}^{k}|Q_i\big(u(t_i)\big)-Q_i\big(v(t_i)\big)|+\sum_{i=1}^{k-1}(t_k-t_i)|R_i\big(u(t_i)\big)-R_i\big(v(t_i)\big)|\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}|S_i\big(u(t_i)\big)-S_i\big(v(t_i)\big)|\\
&+\sum_{i=1}^{k}(t-t_k)|R_i\big(u(t_i)\big)-R_i\big(v(t_i)\big)|)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)\big|S_i\big(u(t_i)\big)-S_i\big(v(t_i)\big)\big| \\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}\big|S_i\big(u(t_i)\big)-S_i\big(v(t_i)\big)\big| \\
&+\frac{|\lambda|}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m-1}\frac{|\lambda|(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m}\frac{|\lambda|(1-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m-1}\frac{|\lambda|(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m}\frac{|\lambda|(t_m^2-4t_m+3)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m}\big|Q_i\big(u(t_i)\big)-Q_i\big(v(t_i)\big)\big|+\sum_{i=1}^{m-1}(t_m-t_i)\big|R_i\big(u(t_i)\big)-R_i\big(v(t_i)\big)\big| \\
&+\sum_{i=1}^{m-1}(2+t_m^2-4t_m)\big|S_i\big(u(t_i)\big)-S_i\big(v(t_i)\big)\big|+\sum_{i=1}^{m}(2-t_m)\big|R_i\big(u(t_i)\big)-R_i\big(v(t_i)\big)\big| \\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}\big|S_i\big(u(t_i)\big)-S_i\big(v(t_i)\big)\big| \\
&+\sum_{i=1}^{m+1}\frac{|\lambda|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
&+\sum_{i=1}^{m+1}\frac{|\lambda|}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\big|\big(f(s,u(s))-f(s,v(s)\big)\big|ds \\
\le&\bigg(\frac{(3m+1)|\lambda|L_1}{\Gamma(\alpha+1)}+\frac{(8m-1)|\lambda|L_1}{\Gamma(\alpha)}+\frac{(43m-15)|\lambda|L_1}{2\Gamma(\alpha-1)} \\
&+4mL_2+(4m-3)L_3+\frac{(22m-13)}{4}L_4\bigg)\|u-v\|.
\end{aligned}
$$

Therefore, $F$ is a contraction.

This completes the proof. When $|\lambda|<p$ , by Lemma 2.1, problem (1.1) has a unique solution in $\overline{\Omega}$. $\square$

## 4. Examples

**Example 4.1.** Consider the following boundary value problem of impulsive fractional differential equation:

$$\begin{cases} D^\alpha u(t) = \lambda(\sin\frac{u}{2} + e^{-2t}),\ 2 < \alpha \leq 3, t \neq \frac{1}{4}, \\ \triangle u(\frac{1}{4}) = \frac{u^3}{3+u^2}, \\ \triangle u'(\frac{1}{4}) = \frac{u^3}{4+u^2}, \\ \triangle u''(\frac{1}{4}) = \frac{u^3}{5+u^2}, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0. \end{cases} \tag{4.1}$$

Let

$$p = \frac{347}{\left(\frac{3}{\Gamma(\alpha+1)} + \frac{7}{2\Gamma(\alpha)} + \frac{4}{\Gamma(\alpha-1)}\right)},$$

which satisfies

$$|f(t,u) - f(t,v)| \leq \frac{1}{20}|u-v|, |Q_k(u) - Q_k(v)| \leq \frac{1}{40}|u-v|,$$
$$|R_k(u) - R_k(v)| \leq \frac{1}{100}|u-v|, |S_k(u) - S_k(v)| \leq \frac{1}{60}|u-v|,$$

All the assumptions of Theorem 3.1 are satisfied. Thus, the problem (4.1) has a unique solution.

# References


[1] Z. Bai, Y. Chen, H. Lian and S. Sun, *On the existence of blow up solutions for a class of fractional differential equations*, Fractional Calculus and Applied Analysis, 2014, 17(4), 1175–1187.

[2] Z. Bai and H. Lu, *Positive solutions of boundary value problems of nonlinear fractional differential equations*, Mathematical Analysis and Applications, 2005, 311, 495–505.

[3] D. Bainov and P. Simenov, *Impulsive differential equations: Periodic solution and application*, Longman Scientific and Technical Harlow, New York, 1993.

[4] Y. Cui, W. Ma, Q. Sun and X. Su, *New uniqueness results for boundary value problem of fractional differential equation*, Nonlinear Analysis: Modelling and Control, 2018, 23(1), 31–39.

[5] Y. Cui, Q. Sun and X. Su, *Monotone iterative technique for nonlinear boundary value problems of fractional order $p \in (2, 3]$*, Advances in Difference Equations, 2017, 248.

[6] K. Diethelm, *The analysis of fractional differential Equations*, Springer, New York, 2004.

[7] M. Feckan, Y. Zhou and J. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Communication in Nonlinear Science and Numerical Simulation, 2012, 17, 3050–3060.

[8] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[9] V. Lakshmikantham, D. Bainov and P. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[11] K. Sheng, W. Zhang and Z. Bai, *Positive solutions to fractional boundary value problems with p-Laplacian on time scales*, Boundary Value Problems, 2018, 70.

[12] Q. Song and Z. Bai, *Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition*, Advances in Difference Equations, 2018, 183.

[13] Y. Tian, Y. Wei and S. Sun, *Multiplicity for fractional differential equations with p-Laplacian*, Boundary Value Problems, 2018, 127.

[14] G. Wang and B Ahmad, *Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions*, Computers and Mathematics with Application, 2011, 62, 1389–1397.

[15] J. You and S. Sun, *On impulsive coupled hybrid fractional differential systems in Banach algebras*, Journal of Applied Mathematics and Computing, 2020, 62, 189–205.

[16] X. Zhang, Z, Shao and Q. Zhong, *Positive solutions for semipositone $(k, n-k)$ conjugate boundary value roblems with singularities on space variables*, Applied Mathematics Letters, 2017, 72, 50–57.

[17] X. Zhang and Q. Zhong, *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Applied Mathematics Letters, 2018, 80, 12–19.

[18] Y. Zhao and S. Sun, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Applied Mathematics and Computation, 2011, 217, 6950–6958.