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Abundant Exact Explicit Solutions to a Modified cKdV Equation*

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Abstract In this paper, we construct abundant exact explicit solutions to a modified cKdV equation by employing the three forms of (ω/g)-expansion method, i.e., (g'/g^2)-expansion method, (g'/g)-expansion method and (g')-expansion method. The solutions obtained are under different constraint conditions and are in the form of hyperbolic, trigonometric and rational functions, respectively, including kink (antikink) wave solutions, singular wave solutions and periodic singular wave solutions which have potential applications in physical science and engineering.

Keywords Modified cKdV equation, Exact explicit solutions, (ω/g)-expansion method, (g'/g^2)-expansion method, (g'/g)-expansion method, (g')-expansion method.

MSC(2010) 34A05, 35C07, 36C09, 35A24.

1. Introduction

It is well known that the investigation of nonlinear wave equations and their solutions has been the field under discussion in different branches of engineering, physics and mathematics [1,2]. Many famous models, such as the Korteweg-de Vries (KdV) equation [3] and the Camassa-Holm (CH) eqation [4], have been proposed in different fields such as physics, chemistry, biology, mechanics, optics, etc. Among the study of these nonlinear models, their traveling wave solutions have gained considerate attention and a number of powerful methods have been developed to find their exact traveling wave solutions, such as the inverse scattering method [5], the Hirota's bilinear method [6], the Bäcklund transformation method [7], the bifurcation method of dynamical systems [8–18], the (g'/g)-expansion method [19] and so on.

In this paper, we aim to consider the following modified coupled Korteweg-de Vries (cKdV) equation [20],

$$\begin{cases} u_t = v_x - \frac{3}{2}uu_x + \alpha u_x, \\ v_t = \frac{1}{4}u_{xxx} - vu_x - \frac{1}{2}uv_x + \alpha v_x, \end{cases} \tag{1.1}$$

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where α is a constant.

When $\alpha = 0$, Eq.(1.1) is reduced to the cKdV equation, which is a general example of N-component systems, energy dependent schrödinger operators and bi-Hamiltonian structures for multi-component systems [20–23]. Many important equations, such as classical Boussinesq equation and the systems governing second harmonic generation, are connected to the cKdV equation through nonsingular transformations [20], which potentially enables solutions of cKdV equations to be interpreted in the context of these related equations. Therefore, because of its great importance, we will further study the exact explicit traveling wave solutions to Eq.(1.1). More preciously, we exploit the three forms of (ω/g)-expansion method [24], i.e., (g'/g^2)-expansion method, (g'/g)-expansion method and (g')-expansion method to obtain exact explicit expressions of traveling wave solutions to Eq.(1.1).

The rest of the paper is organized as follows. Section 2 is devoted to the description of the (ω/g)-expansion method. In Section 3, we apply the three forms of (ω/g)-expansion method to obtain exact explicit traveling wave solutions to Eq.(1.1). Finally, the paper ends with a brief conclusion.

2. Description of (ω/g)-expansion method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial of u and its various partial derivatives. Now we briefly show the main steps of the (ω/g)-expansion method.

Step 1 Suppose that $u(x,t) = u(\xi)$ with $\xi = x - ct$, where c is a parameter to be determined later, then from Eq.(2.1), one obtains

$$P(u, u', -cu', u'', -cu'', c^2u'', \cdots) = 0. \tag{2.2}$$

Step 2 Suppose the solutions of Eq.(2.2) can be expressed by a polynomial of (ω/g) as follows

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{\omega}{g}\right)^i,\ a_n \neq 0, \tag{2.3}$$

and $\omega,\ g$ satisfy the relation

$$\left(\frac{\omega}{g}\right)' = b_0 + b_1\left(\frac{\omega}{g}\right) + b_2\left(\frac{\omega}{g}\right)^2, \tag{2.4}$$

where $b_0,\ b_1$ and b_2 are arbitrary constants.

Step 3 By substituting Eq.(2.3) into Eq.(2.2), making use of (2.4), and setting the coefficients of all powers of (ω/g) to be zeros, we will get a system of algebraic equations, from which c and $a_1, a_2, \cdots, a_n$ can be obtained explicitly.

Step 4 Substituting c and $a_1, a_2, \cdots, a_n$ obtained in Step 3 into Eq.(2.3), one get the possible solutions.

3. Exact explicit traveling wave solutions to the modified cKdV equation (1.1)

In this section, we will obtain exact explicit traveling wave solutions to Eq.(1.1) by employing the three forms of (ω/g)-expansion method, i.e., (g'/g^2)-expansion method, (g'/g)-expansion method and (g')-expansion method, which are resulted from respectively taking

$$\omega = g'/g, b_1 = 0,$$

$$\omega = g', b_0 = -\mu, b_1 = -\lambda, b_2 = -1,$$

and

$$\omega = gg',$$

in Eq.(2.4). Note that the (g'/g)-expansion method is exactly the (G'/G)-expansion method proposed by [19].

More preciously, substituting $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$ with $\xi = x - ct$ into Eq.(1.1), it follows,

$$\begin{cases} cu' + v' - \frac{3}{2}uu' + \alpha u' = 0, \\ cv' + \frac{1}{4}u''' - vu' - \frac{1}{2}uv' + \alpha v' = 0. \end{cases} \tag{3.1}$$

Integrating the first equation of (3.1) once leads to

$$v = \frac{3}{4}u^2 - (c+\alpha)u + h_1, \tag{3.2}$$

where h_1 is integral constant.

Substituting (3.2) into the second equation of (3.1) and integrating it once, it follows that

$$u'' - 2u^3 + 6(c+\alpha)u^2 - 4(c+\alpha)^2 u - 4h_1 u + 4h_2 = 0, \tag{3.3}$$

where h_2 is integral constant.

Considering the homogeneous balance between u'' and u^3 in Eq.(3.3) and exploiting (2.4), we derive that $n = 1$ in Eq.(2.3).

3.1. Exact explicit traveling wave solutions to Eq.(1.1) by (g'/g^2)-expansion method

As shown above, suppose

$$u(\xi) = a_0 + a_1 \left(\frac{g'}{g^2}\right), \tag{3.4}$$

with

$$\left(\frac{g'}{g^2}\right)' = b_0 + b_2 \left(\frac{g'}{g^2}\right)^2. \tag{3.5}$$

Substituting Eq.(3.4) with Eq.(3.5) into Eq.(3.3), collecting all terms with the same order of $\frac{g'}{g^2}$ and setting the coefficients of all powers of $\frac{g'}{g^2}$ to be zeros, one get a system of algebraic equations for a_0, a_1 and c as follows

$$\begin{aligned}
&2a_1\left(b_2^2-a_1^2\right)=0,\\
&6a_1^2(c+\alpha-a_0)=0,\\
&2a_1\left(b_0b_2-3a_0^2+6a_0(c+\alpha)-2(c+\alpha)^2-2h_1\right)=0,\\
&2\left(2h_2-2a_0\left((c+\alpha)^2+h_1\right)+3a_0^2(c+\alpha)-a_0^3\right)=0.
\end{aligned}$$

Solving the algebraic equations above, one has

$$a_1=\pm b_2, c=a_0-\alpha, h_1=\frac{a_0^2+b_0b_2}{2}, h_2=h_1a_0, \tag{3.6}$$

where a_0 is an arbitrary constant.

Substituting Eq.(3.6) and the general solution of Eq.(3.5) into Eq.(3.4), we therefore have the following theorem.

Theorem 3.1. *For given constants b_0, b_2 and arbitrary constants a_0, c_1, one has*

1. *When $b_0b_2>0$, Eq.(1.1) has solutions*

$$u(x,t)=a_0\pm\sqrt{b_0b_2}\tan\left(\sqrt{b_0b_2}(x-(a_0-\alpha)t)+c_1\right). \tag{3.7}$$

2. *When $b_0b_2<0$, Eq.(1.1) has solutions*

$$u(x,t)=a_0\pm\sqrt{-b_0b_2}\tanh\left(\sqrt{-b_0b_2}(x-(a_0-\alpha)t)+c_1\right), \tag{3.8}$$

and

$$u(x,t)=a_0\pm\sqrt{-b_0b_2}\coth\left(\sqrt{-b_0b_2}(x-(a_0-\alpha)t)+c_1\right). \tag{3.9}$$

3. *When $b_0=0, b_2\neq 0$, Eq.(1.1) has solutions*

$$u(x,t)=a_0\pm\frac{b_2}{b_2(x-(a_0-\alpha)t)+c_1}. \tag{3.10}$$

Taking $a_0=3, \alpha=1, c_1=0$, we illustrate the profiles of the solutions (3.7) with $b_0=1, b_2=1$ in Figures 1(a) and 1(b), the profiles of the solutions (3.8) with $b_0=-1, b_2=1$ in Figures 1(c) and 1(d), and the profiles of the solutions (3.9) with $b_0=-1, b_2=1$ in Figures 1(e) and 1(f), and the profiles of the solutions (3.10) with $b_0=0, b_2=1$ in Figures 1(g) and 1(h).

3.2. Exact explicit traveling wave solutions to Eq.(1.1) by (g'/g)-expansion method

Suppose

$$u(\xi)=a_0+a_1\left(\frac{g'}{g}\right), \tag{3.11}$$

with

$$g''+\lambda g'+\mu g=0. \tag{3.12}$$

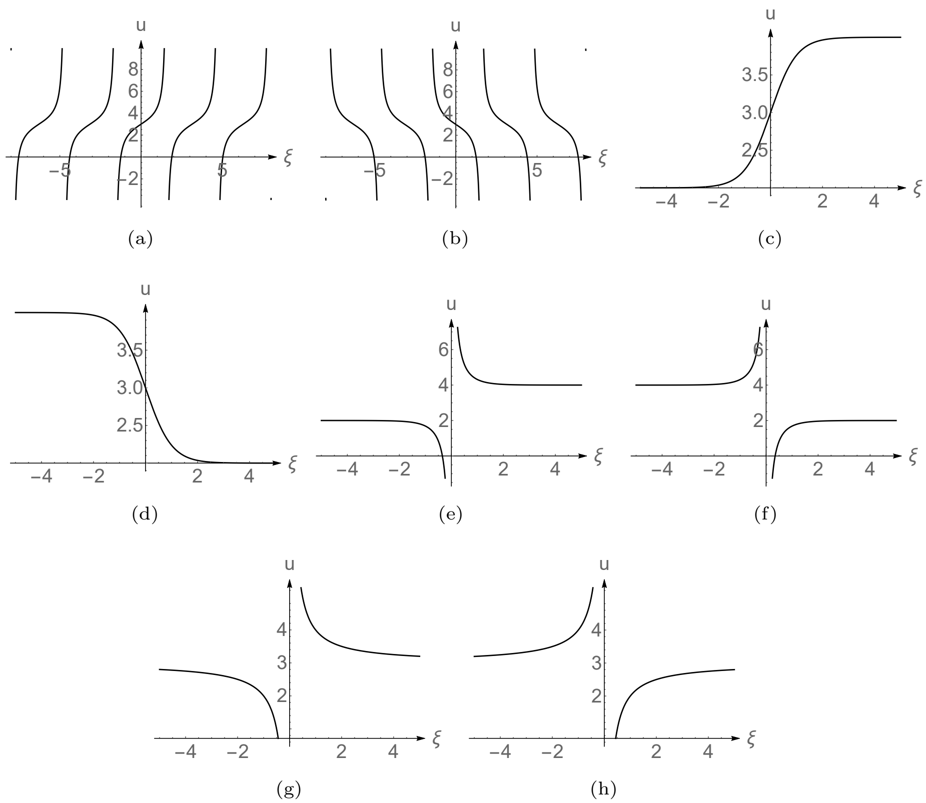

Figure 1. The illustrations of the profiles of the solutions (a) (3.7) with plus sign; (b) (3.7) with minus sign; (c) (3.8) with plus sign; (d) (3.8) with minus sign; (e) (3.9) with plus sign; (f) (3.9) with minus sign; (g) (3.10) with plus sign; (h) (3.10) with minus sign.

Substituting Eq.(3.11) with Eq.(3.12) into Eq.(3.3), one get a system of algebraic equations for a_0, a_1 and c as follows

$$\begin{aligned}
&2a_1\left(1-a_1^2\right)=0,\\
&3a_1\left(\lambda-2a_0a_1+2(c+\alpha)a_1\right)=0,\\
&a_1\left(\lambda^2+2\mu-6a_0^2+12a_0(c+\alpha)-4\left((c+\alpha)^2+h_1\right)\right)=0,\\
&\lambda\mu a_1+4h_2-4a_0\left((c+\alpha)^2+h_1\right)+6a_0^2(c+\alpha)-2a_0^3=0.
\end{aligned}$$

Solving the algebraic equations above, one has

$$\begin{aligned}
&a_1=1, c=a_0-\alpha-\frac{\lambda}{2}, h_1=\frac{\lambda^2+2\mu-6a_0^2+12a_0(c+\alpha)-4(c+\alpha)^2}{4},\\
&h_2=a_0\left((c+\alpha)^2+h_1\right)-\frac{3}{2}a_0^2(c+\alpha)+\frac{1}{2}a_0^3-\frac{\lambda\mu}{4},
\end{aligned}\tag{3.13}$$

or

$$\begin{aligned}
&a_1=-1, c=a_0-\alpha+\frac{\lambda}{2}, h_1=\frac{\lambda^2+2\mu-6a_0^2+12a_0(c+\alpha)-4(c+\alpha)^2}{4},\\
&h_2=a_0\left((c+\alpha)^2+h_1\right)-\frac{3}{2}a_0^2(c+\alpha)+\frac{1}{2}a_0^3+\frac{\lambda\mu}{4},
\end{aligned}\tag{3.14}$$

where a_0 is an arbitrary constant.

Substituting Eq.(3.13) or Eq.(3.14) and the general solution of Eq.(3.12) into Eq.(3.11), we therefore have the following theorem.

Theorem 3.2. *For given constants λ, μ and arbitrary constants a_0, c_1, c_2, one has*

1. *When $\lambda^2-4\mu>0$, Eq.(1.1) has solutions*

$$\begin{aligned}
&u(x,t)=a_0-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\times\\
&\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)},
\end{aligned}\tag{3.15}$$

or

$$\begin{aligned}
&u(x,t)=a_0+\frac{\lambda}{2}-\frac{\sqrt{\lambda^2-4\mu}}{2}\times\\
&\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)}.
\end{aligned}\tag{3.16}$$

2. When $\lambda^2 - 4\mu < 0$, Eq.(1.1) has solutions

$$u(x,t) = a_0 - \frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \times \frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)\right)}, \tag{3.17}$$

or

$$u(x,t) = a_0 + \frac{\lambda}{2} - \frac{\sqrt{4\mu-\lambda^2}}{2} \times \frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)\right)}. \tag{3.18}$$

3. When $\lambda^2 - 4\mu = 0$, Eq.(1.1) has solutions

$$u(x,t) = a_0 - \frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\left(x-(a_0-\alpha-\frac{\lambda}{2})t\right)}, \tag{3.19}$$

or

$$u(x,t) = a_0 + \frac{\lambda}{2} - \frac{c_2}{c_1 + c_2\left(x-(a_0-\alpha+\frac{\lambda}{2})t\right)}. \tag{3.20}$$

Taking $a_0 = 3, \alpha = 1, c_1 = 1, c_2 = -2$, we illustrate the profiles of the solutions (3.15) and (3.16) with $\lambda = 3, \mu = 1$ in Figures 2(a) and 2(b), the profiles of the solutions (3.17) and (3.18) with $\lambda = 1, \mu = 1$ in Figures 2(c) and 2(d), and the profiles of the solutions (3.19) and (3.20) with $\lambda = 2, \mu = 1$ in Figures 2(e) and 2(f).

3.3. Exact explicit traveling wave solutions to Eq.(1.1) by (g')-expansion method

Suppose

$$u(\xi) = a_0 + a_1 g', \tag{3.21}$$

with

$$g'' = b_0 + b_1 g' + b_2 (g')^2. \tag{3.22}$$

Substituting Eq.(3.21) with Eq.(3.22) into Eq.(3.3), one get a system of algebraic equations for a_0, a_1 and c as follows

$$\begin{aligned}
&2a_1\left(b_2^2 - a_1^2\right) = 0,\\
&3a_1(b_1 b_2 - 2a_0 a_1 + 2(c+\alpha)a_1) = 0,\\
&a_1\left(b_1^2 + 2b_0 b_2 - 6a_0^2 + 12a_0(c+\alpha) - 4\left((c+\alpha)^2 + h_1\right)\right) = 0,\\
&a_1 b_0 b_1 + 4h_2 - 4a_0\left((c+\alpha)^2 + h_1\right) + 6a_0^2(c+\alpha) - 2a_0^3 = 0.
\end{aligned}$$

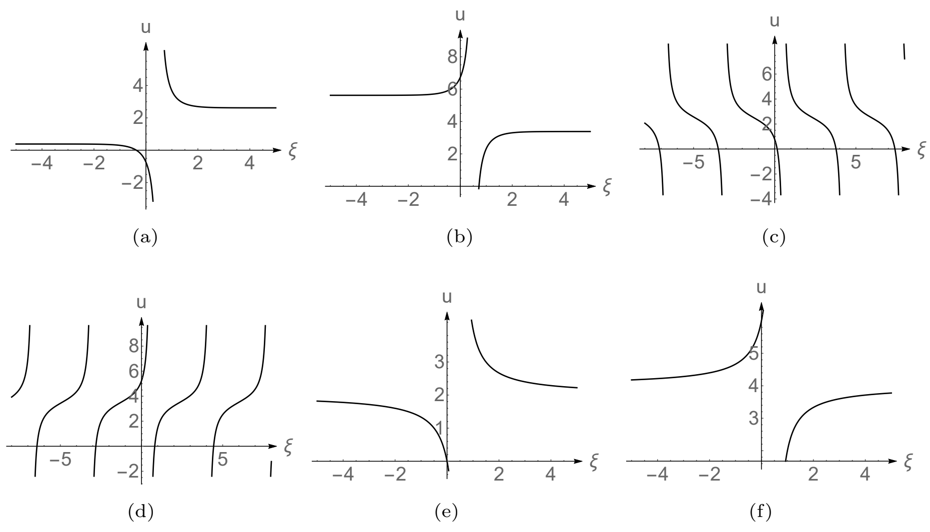

Figure 2. The illustrations of the profiles of the solutions (a) (3.15); (b) (3.16); (c) (3.17); (d) (3.18); (e) (3.19); (f) (3.20).

Solving the algebraic equations above, one has

$$\begin{aligned} &a_1 = b_2, c = a_0 - \alpha - \frac{b_1}{2}, h_1 = \frac{b_1^2 + 2b_0b_2 - 6a_0^2 + 12a_0(c+\alpha) - 4(c+\alpha)^2}{4}, \\ &h_2 = a_0\left((c+\alpha)^2 + h_1\right) - \frac{3}{2}a_0^2(c+\alpha) + \frac{1}{2}a_0^3 - \frac{b_0b_1b_2}{4}, \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} &a_1 = -b_2, c = a_0 - \alpha + \frac{b_1}{2}, h_1 = \frac{b_1^2 + 2b_0b_2 - 6a_0^2 + 12a_0(c+\alpha) - 4(c+\alpha)^2}{4}, \\ &h_2 = a_0\left((c+\alpha)^2 + h_1\right) - \frac{3}{2}a_0^2(c+\alpha) + \frac{1}{2}a_0^3 + \frac{b_0b_1b_2}{4}, \end{aligned} \tag{3.24}$$

where a_0 is an arbitrary constant.

Substituting Eq.(3.23) or Eq.(3.24) and the general solution of Eq.(3.22) into Eq.(3.21), we therefore have the following theorem.

Theorem 3.3. *For given constants b_0, b_1, b_2 and arbitrary constants a_0, c_1, one has*

1. When $b_1^2 - 4b_0b_2 > 0$, Eq.(1.1) has solutions

$$u(x,t) = a_0 - \frac{b_1}{2} + \frac{1}{2}\sqrt{b_1^2 - 4b_0b_2}\tanh\left(-\frac{\sqrt{b_1^2 - 4b_0b_2}}{2}\left(x - (a_0 - \alpha - \frac{b_1}{2})t\right) + c_1\right), \tag{3.25}$$

$$u(x,t) = a_0 - \frac{b_1}{2} + \frac{1}{2}\sqrt{b_1^2 - 4b_0b_2}\coth\left(-\frac{\sqrt{b_1^2 - 4b_0b_2}}{2}\left(x - (a_0 - \alpha - \frac{b_1}{2})t\right) + c_1\right), \tag{3.26}$$

or

$$u(x,t) = a_0 + \frac{b_1}{2} - \frac{1}{2}\sqrt{b_1^2 - 4b_0b_2}\tanh\left(-\frac{\sqrt{b_1^2 - 4b_0b_2}}{2}\left(x - (a_0 - \alpha + \frac{b_1}{2})t\right) + c_1\right), \tag{3.27}$$

$$u(x,t) = a_0 + \frac{b_1}{2} - \frac{1}{2}\sqrt{b_1^2 - 4b_0b_2}\coth\left(-\frac{\sqrt{b_1^2 - 4b_0b_2}}{2}\left(x - (a_0 - \alpha + \frac{b_1}{2})t\right) + c_1\right). \tag{3.28}$$

2. When $b_1^2 - 4b_0b_2 < 0$, Eq.(1.1) has solutions

$$u(x,t) = a_0 - \frac{b_1}{2} + \frac{1}{2}\sqrt{4b_0b_2 - b_1^2}\tan\left(-\frac{\sqrt{4b_0b_2 - b_1^2}}{2}\left(x - (a_0 - \alpha - \frac{b_1}{2})t\right) + c_1\right), \tag{3.29}$$

or

$$u(x,t) = a_0 + \frac{b_1}{2} - \frac{1}{2}\sqrt{4b_0b_2 - b_1^2}\tan\left(-\frac{\sqrt{4b_0b_2 - b_1^2}}{2}\left(x - (a_0 - \alpha + \frac{b_1}{2})t\right) + c_1\right). \tag{3.30}$$

3. When $b_1^2 - 4b_0b_2 = 0$, Eq.(1.1) has solutions

$$u(x,t) = a_0 - \frac{b_1}{2} - \frac{1}{x - (a_0 - \alpha - \frac{b_1}{2})t + c_1}, \tag{3.31}$$

or

$$u(x,t) = a_0 + \frac{b_1}{2} + \frac{1}{x - (a_0 - \alpha - \frac{b_1}{2})t + c_1}. \tag{3.32}$$

Taking $a_0 = 3, \alpha = 1, c_1 = 0$, we illustrate the profiles of the solutions (3.25) and (3.26) with $b_0 = 1, b_1 = 3, b_2 = 1$ in Figures 3(a) and 3(b), the profiles of the solutions (3.27) and (3.28) with $b_0 = 1, b_1 = 3, b_2 = 1$ in Figures 3(c) and 3(d), and the profiles of the solutions (3.29) and (3.30) with $b_0 = 1, b_1 = 1, b_2 = 1$ in Figures 3(e) and 3(f), and the profiles of the solutions (3.31) and (3.32) with $b_0 = 1, b_1 = 2, b_2 = 1$ in Figures 3(g) and 3(h).

Remark 3.1. We have employed the software Mathematica to check the correctness of the above traveling wave solutions. In addition, all the solutions are newly obtained. Furthermore, it is worth to mention that the solutions in the form of coth, such as (3.9), (3.26) and (3.28), are not given in the original (ω/g)-expansion method [24].

4. Conclusion

In this paper, three forms of (ω/g)-expansion method are successfully exploited to construct abundant exact explicit traveling wave solutions to a modified cKdV equation (1.1) under different constraint conditions. The solutions obtained include kink (antikink) wave solutions, singular wave solutions and periodic singular wave solutions, which are in the form of hyperbolic, trigonometric and rational functions, respectively.

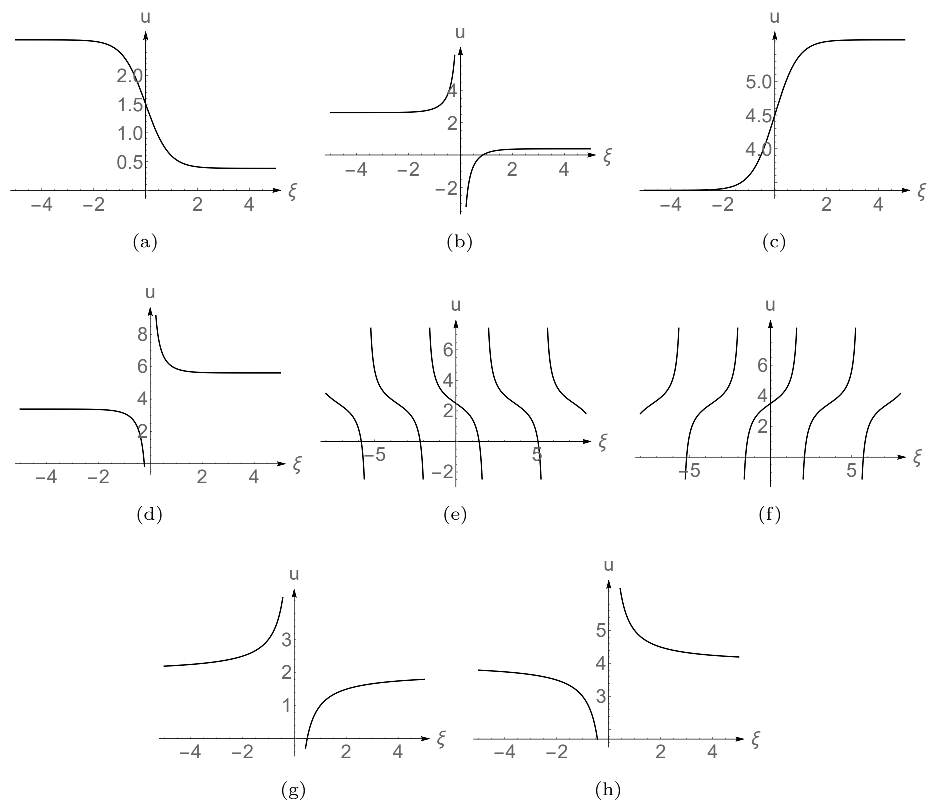

Figure 3. The illustrations of the profiles of the solutions (a) (3.25); (b) (3.26); (c) (3.27); (d) (3.28); (e) (3.29); (f) (3.30); (g) (3.31); (h) (3.32).

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