Journal of Nonlinear Modeling and Analysis
Volume 2, Number 4, December 2020, 601–619
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.601

Global Attractor of Hindmarsh-Rose Equations in Neurodynamics*

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Abstract Global dynamics for a new mathematical model in neurodynamics of the diffusive Hindmarsh-Rose equations on a bounded domain is investigated in this paper. The existence of a global attractor and its regularity are proved through uniform estimates showing the dissipative properties and the asymptotically compact and smoothing characteristics.

Keywords Diffusive Hindmarsh-Rose equations, Global attractor, Absorbing property, Asymptotic compactness, Attractor regularity.

MSC(2010) 35B41, 35K58, 35Q92, 37N25, 92C20.

1. Introduction

The Hindmarsh-Rose equations as a three-dimensional mathematical model for neuronal spiking-bursting of the intracellular membrane potential observed in experiments was originally proposed in [11]. This model composed of three coupled ordinary differential equations has been studied through numerical simulations and bifurcation analysis in recent years, cf. [13, 15, 26, 32] and the references therein. It exhibits rich and interesting bursting patterns, especially chaotic bursting and dynamics such as self-excitation and self-oscillations.

In this work we present and study the global dynamics of the diffusive Hindmarsh-Rose equations as a new PDE model in neurodynamics:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \varphi(u) + v - w + J, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \psi(u) - v, \tag{1.2}$$

$$\frac{\partial w}{\partial t} = d_3 \Delta w + q(u - c) - rw, \tag{1.3}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ $(n \leq 3)$, where Ω is a bounded domain with locally Lipschitz continuous boundary. The nonlinear terms

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{1.4}$$

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*The authors were supported by USDA Grant No. 2018-38422-28564, USA.

In this system, the variable $u(t,x)$ refers to the membrane electric potential of a neuron cell, the variable $v(t,x)$ represents the transport rate of the ions of sodium and potassium through the fast channels and is called the spiking variable, while the variables $w(t,x)$ represents the transport rate across the cell membrane through slow channels of calcium and other ions correlated to the bursting phenomena and is called the bursting variable. The inject current J is treated as a constant.

All the involved parameters are positive constants except $c \in \mathbb{R}$ in the w-equation, which is a reference value of the membrane potential of neuron cells. In the original model [32], a set of typical parameters are

$$J = 3.281, \quad r = 0.0021, \quad S = 4.0, \quad q = rS, \quad c = -1.6,$$
$$\varphi(s) = 3.0\, s^2 - s^3, \quad \psi(s) = 1.0 - 5.0\, s^2.$$

We impose the Neumann boundary conditions for the three components,

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \quad \frac{\partial v}{\partial \nu}(t,x) = 0, \quad \frac{\partial w}{\partial \nu}(t,x) = 0, \qquad t > 0,\ x \in \partial\Omega, \tag{1.5}$$

and the initial conditions to be specified later,

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), \qquad x \in \Omega. \tag{1.6}$$

1.1. The Hindmarsh-Rose model in ODE

The original Hindmarsh-Rose model was developed [11] in 1984,

$$\begin{aligned} \frac{du}{dt} &= au^2 - bu^3 + v - w + J, \\ \frac{dv}{dt} &= \alpha - \beta u^2 - v, \\ \frac{dw}{dt} &= q(u - u_R) - rw. \end{aligned} \tag{1.7}$$

and was motivated by the discovery of neuronal cells in the pond snail *Lymnaea* which generated a burst after being depolarized by a short current pulse. This model characterizes the phenomena of synaptic bursting and especially chaotic bursting in a three-dimensional (u, v, w) space, which incorporates a third variable representing a slow current that hyperpolarizes the neuronal cell. This neurodynamics model is different from the four-dimensional highly nonlinear Hodgkin-Huxley equations [12] (1952) and from the two-dimensional FitzHugh-Nagumo equations [10] (1961-1962) for neuron dynamics in self-excitation and oscillation. The 2D FitzHugh-Nagumo model admits exquisite phase plane analysis showing sustained periodic spiking with refractory period, but it excludes chaotic solutions so that no chaotic bursting can be generated.

Neuronal signals are electrical pulses called spikes or the action potential. Neuron bursting of alternating phases of rapid firing spikes and then quiescence constitutes a mechanism to modulate and pace-setting for brain functionalities and to communicate signals with the neighbor or remote neurons. Bursting patterns occur in a variety of bio-systems such as pituitary melanotropic gland, thalamic neurons, respiratory pacemaker neurons, and insulin-secreting pancreatic β-cells, cf. [2, 3, 5, 11].

The mathematical analysis of several ODE models on bursting behavior has been studied by many authors [1, 8, 9, 15, 19, 26, 28, 29, 32] mainly using bifurcations together with numerical simulations . Neurons coordinate actions through synapses or called gap junction in neuroscience. Synchronization and stability of neural networks is another interesting topic in neurodynamics, cf. [6, 7, 9, 14, 20, 21, 23–25, 33].

The chaotic bursting exhibited in the simulations of this Hindmarsh-Rose model in ordinary differential equations shows more rapid synchronization and more effective regularization of synaptically coupled neurons due to lower threshold, which was rigorously proved in [26, 28, 32] that solutions can be quickly synchronized and regularized when the coupling strength is large enough to topologically change the bifurcation diagram, but the dynamics of chaotic bursting is highly complex.

Moreover, this 3D Hindmarsh-Rose model allows varying interspike interval. It is a suitable choice for investigating both the regular and the chaotic bursting when the parameters vary. In [16–18], the authors studied the synchronization of coupled Hindmarsh-Rose neurons, the exponential attractor of the Hindmarsh-Rose equations, and the global dynamics of the nonautonomous Hindmarsh-Rose equations. This paper aims to prove the existence of global attractor and its regularity for the diffusive Hindmarsh-Rose equations.

1.2. Formulation and preliminaries

Neuron as a specialized biological cell in the brain and the central nervous system has four parts: the central cell body containing the nucleus and intracellular organelles, the dendrites, the axon, and the terminals. The dendrites are the short branches near the nucleus receiving incoming signals of voltage pulse. The axon is a long branch to propagate outgoing signals. The nerve terminals communicate these signals to other neurons or cells. Neurons are immersed in aqueous chemical solutions consisting of different diffusive ions electrically charged.

From biological and mathematical point of view, it is meaningful and useful to consider the diffusive Hindmarsh-Rose equations as a PDE model in neurodynamics, with the spatial variables x involved at least in $\mathbb{R}^1$. Here in the abstract extent, we present and study the diffusive Hindmarsh-Rose equations (1.1)-(1.3) in a bounded domain of space $\mathbb{R}^3$ and focus on the global dynamics of the solutions.

Define the Hilbert space $H = [L^2(\Omega)]^3 = L^2(\Omega, \mathbb{R}^3)$ and the Sobolev space $E = [H^1(\Omega)]^3 = H^1(\Omega, \mathbb{R}^3)$. The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. The norm of E will be denote by $\| \cdot \|_E$. We use $| \cdot |$ to denote a vector norm or a set measure in a Euclidean space. The solution problem of the Hindmarsh-Rose equations (1.1)–(1.6) is formulated to an initial value problem of the evolutionary equation:

$$\frac{\partial g}{\partial t} = Ag + f(g), \quad t > 0, \quad g(0) = g_0 \in H. \tag{1.8}$$

Here $g(t)$ is the column vector of $(u(t, \cdot), v(t, \cdot), w(t, \cdot))$ and g_0 is the column vector of (u_0, v_0, w_0). The nonpositive self-adjoint operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix} : D(A) \to H, \tag{1.9}$$

where $D(A) = \{g \in H^2(\Omega, \mathbb{R}^3) : \partial g/\partial\nu = 0\}$, is the generator of an analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on the Hilbert space H due to the Lumer-Philips theorem [22]. By the Sobolev embedding that the injection $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is continuous for space dimension $n \leq 3$ and by the Hölder inequality, there is a constant $C_0 > 0$ such that there is a constant $C_0 > 0$ such that

$$\|\varphi(u)\| \leq C_0(1 + \|u\|_{L^6}^3), \quad \|\psi(u)\| \leq C_0(1 + \|u\|_{L^4}^2) \quad \text{for } u \in H^1(\Omega).$$

Therefore, the nonlinear mapping

$$f(u, v, w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \\ \psi(u) - v, \\ \\ q(u - c) - rw \end{pmatrix} : E \to H \tag{1.10}$$

is a locally Lipschitz continuous.

Consider the weak solution of this initial value problem (1.8) defined in [4, Section XV.3]. The local existence and uniqueness of weak solutions in time stated in the following lemma can be proved by the Galerkin approximation method. Also see the corresponding propositions in [30, 31].

Lemma 1.1. *For any given initial state $g_0 \in H$, there exists a unique local weak solution $g(t, g_0) = (u(t), v(t), w(t))$, $t \in [0, T_{max})$, for some $T_{max} > 0$, of the initial value problem* (1.8)*, which satisfies*

$$g \in C([0, T_{max}); H) \cap C^1((0, T_{max}); H) \cap L^2_{loc}([0, T_{max}); E), \tag{1.11}$$

where $I_{max} = [0, T_{max})$ is the maximal interval of existence. Moreover, every weak solution $g(t, g_0)$ becomes a strong solution for $t > 0$ and has the regularity

$$g \in C([t_0, T_{max}); E) \cap C^1((t_0, T_{max}); H) \cap L^2_{loc}([t_0, T_{max}); H^2(\Omega, \mathbb{R}^3)) \tag{1.12}$$

for any $t_0 \in (0, T_{max})$. The weak solutions continuously depends on the initial data.

The goal of this paper is to prove the existence of a global attractor and its regularity properties, which will characterize qualitatively the longtime and global dynamics of the solution trajectories of the system (1.8). For clarity we list few concepts in the theory of infinite dimensional dynamical systems [4, 22, 27] and also in [18].

Definition 1.1. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A bounded set B^* of X is called an absorbing set for this semiflow, if for any given bounded subset $B \subset X$ there is a finite time $T_B \geq 0$ such that $S(t)B \subset B^*$ for all $t \geq T_B$.

Definition 1.2. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space X is called asymptotically compact, if for any bounded sequence $\{w_n\}$ in X and any monotone increasing sequences $0 < t_n \to \infty$, there exist subsequences $\{w_{n_k}\} \subset \{w_n\}$ and $\{t_{n_k}\} \subset \{t_n\}$ such that $\lim_{k\to\infty} S(t_{n_k})w_{n_k}$ exists in X.

Definition 1.3. A set $\mathscr{A}$ in a Banach space X is called a global attractor for a semiflow $\{S(t)\}_{t\geq 0}$ on X, if the following two conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the space X.

(ii) $\mathscr{A}$ attracts any given bounded set $B \subset X$ in the sense

$$\text{dist}_X(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_X \to 0, \text{ as } t \to \infty.$$

The following is the main existing result on the existence of a global attractor.

Proposition 1.1. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. If the following two conditions are satisfied*:

(i) *there exists a bounded absorbing set $B^* \subset X$ for $\{S(t)\}_{t\geq 0}$, and*

(ii) *the semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact on X,*

then there exists a unique global attractor $\mathscr{A}$ in X for the semiflow $\{S(t)\}_{t\geq 0}$ and

$$\mathscr{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} (S(t)B^*)}. \tag{1.13}$$

Definition 1.4. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X and let Y be a Banach space which is compactly embedded in X. Then a set $\mathscr{A} \subset Y$ is called an (X,Y)-global attractor for this semiflow if the following two conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in Y, and

(ii) $\mathscr{A}$ attracts any bounded set B of X with respect to the Y-norm.

The Young's inequality in the general form for any nonnegative x, y is

$$xy \leq \varepsilon x^p + C(\varepsilon, p) y^q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad C(\varepsilon, p) = \varepsilon^{-q/p}. \tag{1.14}$$

where $p, q > 1$ and constant $\varepsilon > 0$ can be arbitrarily small.

2. Global existence and absorbing properties

In this section, we shall prove the global existence of all the weak solutions of the problem (1.8) in time and the existence of an absorbing set in the space H for the solution semiflow.

Theorem 2.1. *For any given initial state $g_0 = (u_0, v_0, w_0) \in H$, there exists a unique global weak solution in time, $g(t) = (u(t), v(t), w(t))$, $t \in [0, \infty)$, of the initial value problem* (1.8) *for the diffusive Hindmarsh-Rose equations* (1.1)-(1.3). *The weak solution turns out to be a strong solution on the interval* $(0, \infty)$.

Proof. Taking the L^2 inner-product $\langle (1.1), C_1 u(t)\rangle$ with an adjustable constant $C_1 > 0$, we get

$$\frac{C_1}{2}\frac{d}{dt}\|u\|^2 + C_1 d_1 \|\nabla u\|^2 = \int_\Omega C_1 (au^3 - bu^4 + uv - uw + Ju)\, dx. \tag{2.1}$$

Taking the L^2 inner-products $\langle (1.2), v(t)\rangle$ and $\langle (1.3), w(t)\rangle$, by the Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 = \int_\Omega (\psi(u)v - v^2)\, dx = \int_\Omega (\alpha v - \beta u^2 v - v^2)\, dx \\
\leq &\int_\Omega \left(\alpha v + \frac{1}{2}(\beta^2 u^4 + v^2) - v^2\right) dx = \int_\Omega \left(\alpha v + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx \\
\leq &\int_\Omega \left(2\alpha^2 + \frac{1}{8}v^2 + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx = \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2 u^4 - \frac{3}{8}v^2\right) dx
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2 + d_3\|\nabla w\|^2 = \int_\Omega (q(u-c)w - rw^2)\,dx \\ &\le \int_\Omega \left(\frac{q^2}{2r}(u-c)^2 + \frac{1}{2}rw^2 - rw^2\right) dx \le \int_\Omega \left(\frac{q^2}{r}(u^2+c^2) - \frac{1}{2}rw^2\right) dx.\end{aligned} \tag{2.3}$$

Choose the scaling constant in (2.1) to be $C_1 = \frac{1}{b}(\beta^2+4)$ so that

$$\int_\Omega (-C_1 bu^4)\,dx + \int_\Omega (\beta^2 u^4)\,dx \le \int_\Omega (-4u^4)\,dx.$$

Then we estimate all the mixed product terms on the right-hand side of the above inequalities using the Young's inequality in appropriate ways. In (2.1),

$$\int_\Omega C_1 au^3\,dx \le \frac{3}{4}\int_\Omega u^4\,dx + \frac{1}{4}\int_\Omega (C_1 a)^4\,dx \le \int_\Omega u^4\,dx + (C_1 a)^4|\Omega|,$$

$$\int_\Omega C_1(uv - uw + Ju)\,dx \le \int_\Omega \left(2(C_1u)^2 + \frac{1}{8}v^2 + \frac{(C_1u)^2}{r} + \frac{1}{4}rw^2 + C_1u^2 + C_1J^2\right) dx,$$

where on the right-hand side of the second inequality we further treat the three terms involving u^2 as

$$\int_\Omega \left(2(C_1u)^2 + \frac{(C_1u)^2}{r} + C_1u^2\right) dx \le \int_\Omega u^4\,dx + \left[C_1^2\left(2+\frac{1}{r}\right) + C_1\right]^2 |\Omega|.$$

Then in (2.3),

$$\int_\Omega \frac{1}{r}q^2u^2\,dx \le \int_\Omega \left(\frac{u^4}{2} + \frac{q^4}{2r^2}\right) dx \le \int_\Omega u^4\,dx + \frac{q^4}{r^2}|\Omega|.$$

Substitute the above term estimates into (2.1) and (2.3) and then sum up the three inequalities (2.1)-(2.3) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) + (C_1d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2 + d_3\|\nabla w\|^2) \\ &\le \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\,dx \\ &+ \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2u^4 - \frac{3}{8}v^2\right) dx + \int_\Omega \left(\frac{q^2}{r}(u^2+c^2) - \frac{1}{2}rw^2\right) dx \\ &\le \int_\Omega (3-4)u^4\,dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8}\right) v^2\,dx + \int_\Omega \left(\frac{1}{4} - \frac{1}{2}\right) rw^2\,dx \\ &+ |\Omega| \left((C_1a)^4 + C_1J^2 + \left[C_1^2\left(2+\frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}\right) \\ &= -\int_\Omega \left(u^4(t,x) + \frac{1}{4}v^2(t,x) + \frac{1}{4}rw^2(t,x)\right) dx + C_2|\Omega|\end{aligned} \tag{2.4}$$

where $C_2 > 0$ is the constant given by

$$C_2 = (C_1a)^4 + C_1J^2 + \left[C_1^2\left(2+\frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}.$$

Set

$$d = 2\min\{d_1, d_2, d_3\}.$$

Then (2.4) yields the uniform estimate for all solutions in terms of the differential inequality

$$\begin{aligned}\frac{d}{dt}&(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2)\\ &+ \int_\Omega \left(2u^4(t,x) + \frac{1}{2}v^2(t,x) + \frac{1}{2}rw^2(t,x)\right) dx \le 2C_2|\Omega|,\end{aligned} \tag{2.5}$$

where $t \in I_{max} = [0, T_{max})$, which is the maximal time interval of solution existence. Since

$$2u^4 \ge \frac{1}{2}\left(C_1u^2 - \frac{C_1^2}{16}\right),$$

From (2.5) it follows that

$$\begin{aligned}\frac{d}{dt}&(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2)\\ &+ \int_\Omega \frac{1}{2}\left(C_1u^2(t,x) + v^2(t,x) + rw^2(t,x)\right) dx \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|.\end{aligned}$$

Set $r_1 = \frac{1}{2}\min\{1, r\}$. Then we have

$$\begin{aligned}\frac{d}{dt}&(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2)\\ &+ r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|, \quad t \in [0, T_{max}).\end{aligned} \tag{2.6}$$

Apply the Gronwall inequality to the reduced differential inequality from (2.6),

$$\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|$$

and we obtain

$$C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le e^{-r_1 t}(C_1\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2) + M|\Omega| \tag{2.7}$$

for any $t \in [0, T_{max})$, where

$$M = \frac{1}{r_1}\left(2C_2 + \frac{C_1^2}{32}\right).$$

The estimate (2.7) shows that the weak solutions will never blow up at any finite time because it is uniformly bounded,

$$C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le C_1\|u_0\|^2 + \|v_0\|^2 + \|w_0|^2 + M|\Omega|.$$

Therefore the weak solution of the initial value problem (1.8) exists globally in time for any initial data. The time interval of maximal existence is $[0, \infty)$ for every weak solution. $\square$

The global existence and uniqueness of the weak solutions and their continuous dependence on the initial data enable us to define the solution semiflow of the diffusive Hindmarsh-Rose equations (1.1)-(1.3) on the space H to be

$$S(t) : g_0 \longmapsto g(t, g_0) = (u(t, \cdot), v(t, \cdot), w(t, \cdot)), \quad g_0 = (u_0, v_0, w_0) \in H, \quad t \geq 0,$$

where $g(t, g_0)$ is the weak solution with $g(0) = g_0$. We call this semiflow $\{S(t)\}_{t \geq 0}$ the Hindmarsh-Rose semiflow.

Theorem 2.2. *There exists an absorbing set in the space H for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$, which is the bounded ball*

$$B_H = \{g \in H : \|g\|^2 \leq K\} \tag{2.8}$$

where $K = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1$.

Proof. From the uniform estimate (2.7) in Theorem 2.1 we see that

$$\limsup_{t\to\infty} (\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) < K = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1 \tag{2.9}$$

for all weak solutions of (1.8) with any initial state $g_0 \in H$. Moreover, for any given bounded set $B = \{g \in H : \|g\|^2 \leq R\}$ in H, there exists a finite time

$$T_0(B) = \frac{1}{r_1} \log^+(R \max\{C_1, 1\}) \tag{2.10}$$

such that $\|u(t)\|^2+\|v(t)\|^2+\|w(t)\|^2 < K$ for all $t \geq T_0(B)$ and any $g_0 \in B$. According to Definition 1.1, the bounded ball B_H is an absorbing set for the Hindmarsh-Rose semiflow in the phase space H. □

3. Asymptotic compactness and global attractor

In this section, we show that the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact and then reach the main result on the existence of a global attractor for this dynamical system generated by the diffusive Hindmarsh-Rose equations.

Theorem 3.1. *For any given bounded set $B \in H$, there exists a finite time $T_1(B) > 0$ such that for any initial state $g_0 = (u_0, v_0, w_0) \in B$, the weak solution $g(t) = S(t)g_0 = (u(t), v(t), w(t))$ of the initial value problem* (1.8) *satisfies*

$$\|(u(t), v(t), w(t))\|_E^2 \leq Q, \quad \textit{for } \ t \geq T_1(B) \tag{3.1}$$

where $Q > 0$ is a constant independent of any initial data and $T_1(B) > 0$ depends only on the bounded set B.

Proof. Take the L^2 inner-product $\langle (1.1), -\Delta u(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 = \int_\Omega (-au^2\Delta u - 3bu^2|\nabla u|^2 - v\Delta u + w\Delta u - J\Delta u)\, dx \\
&\leq \int_\Omega \left(\frac{2v^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2w^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2J^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2a^2u^4}{d_1} + \frac{d_1}{8}|\Delta u|^2\right) dx \\
&\quad - \int_\Omega 3bu^2|\nabla u|^2\, dx, \quad \text{for } \ t > 0.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 + 6b\|u\nabla u\|^2 \le \frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{4J^2}{d_1}|\Omega| + \frac{4a^2}{d_1}\|u\|^4_{L^4}. \quad (3.2)$$

Next take the L^2 inner-product $\langle (1.2), -\Delta v(t)\rangle$ to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 = \int_\Omega (-\alpha \Delta v + \beta u^2 \Delta v - |\nabla v|^2)\, dx$$

$$\le \int_\Omega \left(\frac{\alpha^2}{d_2} + \frac{d_2}{4}|\Delta v|^2 + \frac{\beta^2 u^4}{d_2} + \frac{d_2}{4}|\Delta v|^2 \right) dx - \|\nabla v\|^2.$$

It follows that

$$\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + 2\|\nabla v\|^2 \le \frac{2\alpha^2}{d_2}|\Omega| + \frac{2\beta^2}{d_2}\|u\|^4_{L_4}, \quad t > 0. \quad (3.3)$$

Then taking the L_2 inner-product $\langle (1.3), -\Delta w(t)\rangle$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + d_3\|\Delta w\|^2 = \int_\Omega (qc\Delta w - qu\Delta w - r|\nabla w|^2)\, dx$$

$$\le \int_\Omega \left(\frac{q^2c^2}{d_3} + \frac{d_3}{4}|\Delta w|^2 + \frac{q^2u^2}{d_3} + \frac{d_3}{4}|\Delta w|^2 \right) dx - r\|\nabla w\|^2.$$

It follows that

$$\frac{d}{dt}\|\nabla w\|^2 + d_3\|\Delta w\|^2 + 2r\|\nabla w\|^2 \le \frac{2q^2c^2}{d_3}|\Omega| + \frac{2q^2}{d_3}\|u\|^2_{L_2}, \quad t > 0. \quad (3.4)$$

Sum up the above estimates (3.2), (3.3) and (3.4) to obtain

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + d_3\|\Delta w\|^2 \\ &+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2 \\ \le\ & \frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{2q^2}{d_3}\|u\|^2 + \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right)\|u\|^4_{L_4} + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|. \end{aligned} \quad (3.5)$$

Since $H^1(\Omega) \hookrightarrow L^4\Omega)$ and $H^1(\Omega) \hookrightarrow L^6(\Omega)$ are continuous embedding, there is a positive constant $\eta > 0$ such that

$$\|u\|_{L^4} \le \eta\|u\|_{H^1} = \eta\sqrt{\|u\|^2 + \|\nabla u\|^2}, \quad \|u\|_{L^6} \le \eta\|u\|_{H^1} = \eta\sqrt{\|u\|^2 + \|\nabla u\|^2}.$$

Then we have

$$\begin{aligned} \|u\|^4_{L_4} &\le \eta^4(\|u\|^2 + \|\nabla u\|^2)^2 \le 2\eta^4(\|u\|^4 + \|\nabla u\|^4), \\ \|u\|^3_{L_6} &\le \eta^3(\sqrt{\|u\|^2 + \|\nabla u\|^2})^3 \le 4\eta^3(\|u\|^3 + \|\nabla u\|^3). \end{aligned} \quad (3.6)$$

According to Theorem 2.2 and (2.10), there is a finite time $T_0(B) > 0$ such that the solution $g(t) = (u(t), v(t), w(t))$ with any initial state $g_0 \in B$ will permanently enter the absorbing ball B_0 shown in (2.8). It implies that the sum of the L^2-norms of all three components of the solution satisfies

$$\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le K, \quad \text{for any } t > T_0(B),\ g_0 \in B. \quad (3.7)$$

Then (3.5) yields the following differential inequality

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + d_3\|\Delta w\|^2 \\
&+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2 \\
\le\ & \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\eta^4\|\nabla u\|^4 \\
&+ \eta^4 K^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|, \quad t > T_0(B),\ g_0 \in B.
\end{aligned} \tag{3.8}
$$

The inequality (3.8) implies that for any initial data $g_0 \in B$ we have

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla u, \nabla v, \nabla w)\|^2 \\
\le\ & \eta^4\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|(\nabla u, \nabla v, \nabla w)\|^2\ \|(\nabla u, \nabla v, \nabla w)\|^2 \\
&+ \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K + \eta^4 K^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|
\end{aligned} \tag{3.9}
$$

for all $t > T_0(B)$.

We can apply the uniform Gronwall inequality [22, Lemma D.3] to the differential inequality (3.9), which is written as

$$
\frac{d}{dt}\sigma(t) \le \rho(t)\,\sigma(t) + h(t), \quad \text{for } t > T_0(B),\ g_0 \in B, \tag{3.10}
$$

where

$$
\sigma(t) = \|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2,
$$

$$
\rho(t) = \eta^4\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2,
$$

and $h(t)$ is a constant

$$
h(t) = \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K_1 + \eta^4 K_1^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|.
$$

For any $t > T_0(B)$, integrating (2.6) over the time interval $[t, t+1]$ implies that

$$
\begin{aligned}
&\int_t^{t+1} \min\{d_1, d_2, d_3\}(C_1\|\nabla u(s)\|^2 + \|\nabla v(s)\|^2 + \|\nabla w(s)\|^2)\, ds \\
\le\ & C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 + r_1 M|\Omega| \le \max\{1, C_1\}K + r_1 M|\Omega|, \ \ t > T_0(B).
\end{aligned}
$$

Here the constant $M > 0$ is shown in (2.7). Thus we get

$$
\int_t^{t+1} \sigma(s)\, ds \le \frac{r_1 M|\Omega| + \max\{1, C_1\}K}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}} \quad \text{for } t > T_0(B),\ g_0 \in B. \tag{3.11}
$$

Hence we also have

$$\int_t^{t+1} \rho(s)\, ds \le \eta^4 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\left(\frac{r_1 M|\Omega| + \max\{1, C_1\}K}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}}\right). \tag{3.12}$$

Denote by

$$N = \eta^4 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\left(\frac{r_1 M|\Omega| + \max\{1, C_1\}K}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}}\right).$$

The uniform Gronwall inequality applied to (3.10) together with (3.11) and (3.12) yields

$$\|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2 \le C_3\, e^N, \quad \text{for any } t \ge T_0(B) + 1,\ g_0 \in B, \tag{3.13}$$

where

$$\begin{aligned} C_3 \,&= \frac{r_1 M|\Omega| + \max\{1, C_1\}K}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}} + \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K \\ &\quad + \eta^4 K^2 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|. \end{aligned}$$

Finally, we complete the proof of (3.1):

$$\|(u(t), v(t), w(t))\|_E^2 = \|(u, v, w)\|^2 + \|\nabla(u, v, w)\|^2 \le Q = K + C_3\, e^N$$

for $t \ge T_1(B) = T_0(B) + 1$. The proof is completed. □

We now prove the main result on the existence of global attractor for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge0}$.

Theorem 3.2. *For any positive parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q, r, J$ and any $c \in \mathbb{R}$, there exists a global attractor $\mathscr{A}$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge0}$ generated by the weak solutions of the diffusive Hindmarsh-Rose equations* (1.8). *Moreover, the global attractor $\mathscr{A}$ is an (H, E)-global attractor.*

Proof. In Theorem 2.2 it has been shown that there is an absorbing set $B_H \in H$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge0}$. In Theorem 3.1, it is shown that for any given bounded set $B \subset H$,

$$\|S(t)g_0\|_E^2 \le Q, \quad \text{for } t \ge T_1(B) \text{ and all } g_0 \in B.$$

This implies that $\bigcup_{t\ge T_1(B)} S(t)B$ is a bounded set in E and consequently a precompact set in H due to the compact embedding $E \hookrightarrow H$. Therefore, the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge0}$ is asymptotically compact in H. Since the two conditions in Proposition (1.1) are satisfied, we conclude that there exists a global attractor $\mathscr{A}$ in the phase space H for this Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge0}$ and

$$\mathscr{A} = \bigcap_{\tau\ge0} \overline{\bigcup_{t\ge\tau}(S(t)B_H)}. \tag{3.14}$$

Next we prove that this global attractor $\mathscr{A}$ is a bi-space (H, E)-global attractor. Theorem 3.1 actually shows that there is a bounded absorbing set B_E in E, which absorbs any bounded subset of H with respect to the E-norm for this semiflow $\{S(t)\}_{t\geq 0}$. Indeed,

$$B_E = \{g \in E : \|g\|_E^2 \leq Q\}. \tag{3.15}$$

Moreover, we can show that the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact not only on H but also with respect to the E-norm. Let $T > 0$ be arbitrarily given. For any time sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$, and any bounded sequence $\{g_n\} \in E$, there is an integer $n_0 \geq 1$ such that $t_n > T$ for all $n > n_0$. By Theorem 3.1 and the bounded sequence $\{g_n\}$ in E, we have

$$\{S(t_n - T)g_n\}_{n>n_0} \text{ is bounded set in } E.$$

Since E is a Hilbert space, there exists an increasing subsequence of integers $\{n_i\}_{i=1}^{\infty}$ where $n_i > n_0$ such that the following weak limit exists,

$$(w) \lim_{i\to\infty} S(t_{n_i} - T)g_{n_i} = g^* \in E.$$

Since E is compactly embedded in H, we can take subsequence of $\{n_i\}_{i=1}^{\infty}$ and relabel it as the same as $\{n_i\}_{i=1}^{\infty}$, such that the following strong convergence holds,

$$(s) \lim_{i\to\infty} S(t_{n_i} - T)g_{n_i} = g^* \in H.$$

Hence the following strong convergence in E holds,

$$\lim_{i\to\infty} S(t_{n_i})g_{n_i} = \lim_{i\to\infty} S(T)S(t_{n_i} - T)g_{n_i} = S(T)g^* \in E.$$

This proves that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E. Then by Proposition (1.1), there exists a global attractor $\mathscr{A}_E$ in E for the semiflow $\{S(t)\}_{t\geq 0}$.

Since Theorem 3.1 shows that $\mathscr{A}_E$ attracts the set B_H given in (2.8) with respect to the E-norm and, on the other hand, Theorem 2.2 shows that B_H absorbs any bounded subset B of H, then the global attractor $\mathscr{A}_E$ attracts any given bounded set $B \subset H$ in E-norm. Therefore, $\mathscr{A}_E$ is an (H, E) global attractor. Finally, since $\mathscr{A}$ is bounded and invariant in H and $\mathscr{A}_E$ is bounded and invariant in E, it holds that

$$\begin{aligned} &\mathscr{A}_E \text{ attracts } \mathscr{A} \text{ in } E, \text{ so that } \mathscr{A} \subset \mathscr{A}_E, \\ &\mathscr{A} \text{ attracts } \mathscr{A}_E \text{ in } H, \text{ so that } \mathscr{A}_E \subset \mathscr{A}. \end{aligned}$$

It concludes that $\mathscr{A} = \mathscr{A}_E$. Therefore, the global attractor $\mathscr{A}$ in H is also an (H, E) global attractor for the Hindmarsh-Rose semiflow. □

4. Regularity properties of the global attractor

In this section, we shall prove the regularity properties of the global attractor $\mathscr{A}$ in the spaces $L^\infty(\Omega, \mathbb{R}^3)$ and $H^2(\Omega, \mathbb{R}^3)$.

Theorem 4.1. *The global attractor $\mathscr{A}$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ in the space H is a bounded set in $L^\infty(\Omega, \mathbb{R}^3)$. There is a constant $C_\infty > 0$ such that*

$$\sup_{g\in\mathscr{A}} \|g\|_{L^\infty} \leq C_\infty. \tag{4.1}$$

Proof. The analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ has the regularity property [22, Theorem 38.10] that $e^{At} : L^p(\Omega) \to L^\infty(\Omega)$ for $p \geq 1, t > 0$, and there is a constant $c(p) > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(L^p, L^\infty)} \leq c(p)\, t^{-\frac{n}{2p}}, \quad \text{where } n = \dim \Omega. \tag{4.2}$$

Since every weak solution of (1.8) turns out to be a strong solution for time $t > 0$, which is a mild solution [22, Theorem 51.3] and the global attractor $\mathscr{A}$ is an invariant set, we have

$$\begin{aligned} &\|S(t)g\|_{L^\infty} \leq \|e^{At}\|_{\mathcal{L}(L^2,L^\infty)}\|g\| + \int_0^t \|e^{A(t-\sigma)}\|_{(L^2,L^\infty)} \|f(S(\sigma)g) - f(S(\sigma)0)\|\, d\sigma \\ &\leq c(2)t^{-\frac{3}{4}}\|g\| + \int_0^t c(2)(t-\sigma)^{-\frac{3}{4}} L(Q)(\|S(\sigma)g\|_E + \|S(\sigma)0\|_E)\, d\sigma, \quad t > 0,\ g \in \mathscr{A}, \end{aligned} \tag{4.3}$$

where $L(Q)$ is the Lipschitz constant of the nonlinear map f restricted on the closed, bounded ball B_E in E centered at the origin with radius $\sqrt{Q}$. The global attractor $\mathscr{A}$ is invariant so that

$$\{S(t)\mathscr{A} : t \geq 0\} \subset B_H\,(\subset H) \cap B_E\,(\subset E).$$

Then from (4.3) we obtain

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\leq c(2)Kt^{-\frac{3}{4}} + \int_0^t c(2)L(Q)\left(\sqrt{Q} + \sqrt{Q^*}\right)(t-\tau)^{-\frac{3}{4}}\, d\tau \\ &= c(2)[Kt^{-\frac{3}{4}} + 4L(Q)\left(\sqrt{Q} + \sqrt{Q^*}\right) t^{\frac{1}{4}}], \quad \text{for } 0 < t \leq 1, \end{aligned} \tag{4.4}$$

where

$$Q^* = \sup_{0\leq\tau\leq 1} \|S(\tau)0\,\|_E^2\,.$$

Take $t = 1$ in (4.4) and get

$$\|S(1)g\|_{L^\infty} \leq C_\infty = c(2)\left[K + 4L(Q)\left(\sqrt{Q} + \sqrt{Q^*}\right)\right], \quad \text{for any } g \in \mathscr{A}. \tag{4.5}$$

The invariance of $\mathscr{A}$ implies that $S(1)\mathscr{A} = \mathscr{A}$. Therefore, the global attractor $\mathscr{A}$ is a bounded subset in $L^\infty(\Omega)$. $\square$

Theorem 4.2. *The global attractor $\mathscr{A}$ in the space H for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is a bounded set in $H^2(\Omega, \mathbb{R}^3)$.*

Proof. Consider the solution trajectories inside the global attractor $\mathscr{A}$.

Step 1. For the first component $u(t,x)$ of all the solution trajectories in $\mathscr{A}$, take the L^2 inner-product $\langle (1.1), u_t \rangle$ to obtain

$$\begin{aligned} \|u_t\|^2 + \frac{d_1}{2}\frac{d}{dt}\|\nabla u\|^2 &= \int_\Omega (au^2 - bu^2 + v - w + J)u_t\, dx \\ &\leq \int_\Omega \left(a\,C_\infty^2 + b\,C_\infty^3 + 2C_\infty + J\right)|u_t|\, dx \\ &= \frac{1}{2}\left(a\,C_\infty^2 + b\,C_\infty^3 + 2C_\infty + J\right)^2 |\Omega| + \frac{1}{2}\|u_t\|^2, \end{aligned}$$

where C_∞ is from (4.1) and shown in (4.5). Also take the L^2 inner-product $\langle (1.2), v_t\rangle$ to obtain

$$\begin{aligned}
\|v_t\|^2 + \frac{d_2}{2}\frac{d}{dt}\|\nabla v\|^2 &= \int_\Omega (\alpha - \beta u^2 - v) v_t \, dx \\
\leq \int_\Omega (\alpha + \beta {C_\infty}^2 + C_\infty)|v_t| \, dx &= \frac{1}{2}(\alpha + \beta {C_\infty}^2 + C_\infty)^2 |\Omega| + \frac{1}{2}\|v_t\|^2
\end{aligned}$$

for the second component $v(t,x)$ of all the solution trajectories in $\mathscr{A}$. Then take the L^2 inner-product $\langle (1.3), w_t\rangle$ to get

$$\begin{aligned}
\|w_t\|^2 + \frac{d_3}{2}\frac{d}{dt}\|\nabla w\|^2 &= \int_\Omega (qu - qc - rw) w_t \, dx \\
\leq \int_\Omega (qC_\infty + q|c| + rC_\infty)|w_t| \, dx &= \frac{1}{2}(qC_\infty + q|c| + rC_\infty)^2 |\Omega| + \frac{1}{2}\|w_t\|^2
\end{aligned}$$

for the third component $w(t,x)$ of all the solution trajectories in $\mathscr{A}$. Summing up the above three estimates we get, for $t > 0$,

$$\begin{aligned}
&\|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \frac{d}{dt}\left\{d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2 + d_3\|\nabla w\|^2\right\} \\
&\leq \left[(a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2 + (\alpha + \beta {C_\infty}^2 + C_\infty)^2 + (qC_\infty + q|c| + rC_\infty)^2\right]|\Omega|.
\end{aligned} \tag{4.6}$$

Integrate the inequality (4.6) over the time interval $[0,1]$. Then we obtain

$$\begin{aligned}
&\int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2) \, ds \\
&\leq d_1\|\nabla u(0)\|^2 + d_2\|\nabla v(0)\|^2 + d_3\|\nabla w(0)\|^2 \\
&\quad + (a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2|\Omega| + (\alpha + \beta{C_\infty}^2 + C_\infty)^2|\Omega| \\
&\quad + (qC_\infty + q|c| + rC_\infty)^2|\Omega| \\
&\leq (d_1 + d_2 + d_3)Q + (a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2|\Omega| \\
&\quad + (\alpha + \beta{C_\infty}^2 + C_\infty)^2|\Omega| + (qC_\infty + q|c| + rC_\infty)^2|\Omega|.
\end{aligned} \tag{4.7}$$

Step 2. Inside the global attractor $\mathscr{A}$ as an invariant set, we can differentiate the diffusive Hindmarsh-Rose equations (1.1), (1.2), and (1.3) in time t to get

$$\begin{aligned}
u_{tt} &= d_1 \Delta u_t + 2auu_t - 3bu^2 u_t + v_t - w_t, \\
v_{tt} &= d_2 \Delta v_t - 2\beta u u_t - v_t, \\
w_{tt} &= d_3 \Delta w_t + qu_t - rw_t\,.
\end{aligned} \tag{4.8}$$

Sum up the inner-products $\langle (1.1), t^2 u_t\rangle$, $\langle (1.2), t^2 v_t\rangle$, $\langle (1.3), t^2 w_t\rangle$ for $t > 0$. We have

$$
\begin{aligned}
& -t\|u_t\|^2 - t\|v_t\|^2 - t\|w_t\|^2 + \frac{1}{2}\frac{d}{dt}(\|tu_t\|^2 + \|tv_t\|^2 + \|tw_t\|^2) \\
& \qquad + t^2(d_1\|\nabla u_t\|^2 + d_2\|\nabla v_t\|^2 + d_3\|\nabla w_t\|^2) \\
& = \int_\Omega t^2(2auu_t^2 - 3bu^2u_t^2 + v_tu_t - w_tu_t - 2\beta uu_tv_t - v_t^2 + qu_tw_t - rw_t^2)\,dx \\
& \le \int_\Omega t^2\left[2aC_\infty u_t^2 + \frac{1}{2}(v_t^2+u_t^2) + \frac{1}{2}(w_t^2+u_t^2) + \beta C_\infty(u_t^2+v_t^2) + \frac{q}{2}(u_t^2+w_t^2)\right]dx \\
& = t^2\left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right)\|u_t\|^2 + t^2\left(\frac{1}{2} + \beta C_\infty\right)\|v_t\|^2 + t^2\left(\frac{1}{2}+\frac{q}{2}\right)\|w_t\|^2
\end{aligned}
\tag{4.9}
$$

where the u-component portion is deduced with the reference of the first equation in (4.8) as follows,

$$
\begin{aligned}
& -t\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|tu_t\|^2 = -t\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\langle tu_t, tu_t\rangle \\
& = -t\|u_t\|^2 + \left\langle \frac{d}{dt}(tu_t), tu_t\right\rangle = -t\|u_t\|^2 + \langle u_t, tu_t\rangle + \langle tu_{tt}, tu_t\rangle \\
& = -t\|u_t\|^2 + t\|u_t\|^2 + \langle u_{tt}, t^2u_t\rangle = \langle u_{tt}, t^2u_t\rangle. \\
& = \langle d_1\Delta u_t + 2auu_t - 3bu^2u_t + v_t - w_t,\, t^2u_t\rangle \\
& = -t^2d_1\|\nabla u_t\|^2 + \int_\Omega t^2(2auu_t^2 - 3bu^2u_t^2 + v_tu_t - w_tu_t)\,dx.
\end{aligned}
\tag{4.10}
$$

Similar derivation goes to the v-component portion and the w-component portion in (4.9) as well.

Now we integrate the differential inequality (4.9) on $[0,t]$ to obtain

$$
\begin{aligned}
& \frac{1}{2}(\|tu_t\|^2 + \|tv_t\|^2 + \|tw_t\|^2) \\
& \le \int_0^t s^2\left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right)\|u_t(s)\|^2\,ds \\
& \quad + \int_0^t s^2\left(\frac{1}{2}+\beta C_\infty\right)\|v_t(s)\|^2\,ds + \int_0^t s^2\left(\frac{1}{2}+\frac{q}{2}\right)\|w_t(s)\|^2\,ds \\
& \quad + \int_0^t s(\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2)\,ds.
\end{aligned}
\tag{4.11}
$$

In the above inequality we can take $t = 1$ and get

$$
\begin{aligned}
&\|u_t(1)\|^2 + \|v_t(1)\|^2 + \|w_t(1)\|^2 \\
&\le 2\int_0^1 \left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right)\|u_t(s)\|^2\, ds \\
&\quad + 2\int_0^1 \left(\frac{1}{2} + \beta C_\infty\right)\|v_t(s)\|^2\, ds + 2\int_0^1 \left(\frac{1}{2} + \frac{q}{2}\right)\|w_t(s)\|^2\, ds \\
&\quad + 2\int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2)\, ds \\
&\le 2\left(2aC_\infty + 2 + 2\beta C_\infty + \frac{q}{2}\right)\int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2)\, ds \le D
\end{aligned} \tag{4.12}
$$

where, by the inequality in (4.7),

$$
\begin{aligned}
D &= (4aC_\infty + 4 + 4\beta C_\infty + q)\,\{(d_1 + d_2 + d_3)Q \\
&\quad + ({aC_\infty}^2 + {bC_\infty}^3 + 2C_\infty + J)^2|\Omega| \\
&\quad + (\alpha + {\beta C_\infty}^2 + C_\infty)^2|\Omega| + (qC_\infty + q|c| + rC_\infty)^2|\Omega|\}\,.
\end{aligned}
$$

where the constant Q is given in (3.1).

Step 3. Since the global attractor $\mathscr{A}$ is an invariant set, for any trajectory $g(t) = (u(t), v(t), w(t)) \in \mathscr{A}$, one has $\tilde{g}(t) = g(t-1) \in \mathscr{A}$ such that $g(t) = S(1)\tilde{g}(t)$. Then the inequality (4.12) together with the equations (1.1), (1.2) and (1.3) implies that

$$
\begin{aligned}
&d_1\|\Delta u(t)\| + d_2\|\Delta v(t)\| + d_3\|\Delta w(t)\| \\
&\le \|u_t(t)\| + \|v_t(t)\| + \|w_t(t)\| + a\|u^2(t)\| + b\|u^3(t)\| + \|v(t)\| + \|w(t)\| \\
&\quad + \beta\|u^2(t)| + \|v(t)\| + q\|u(t)\| + r\|w(t)\| + (J + \alpha + q|c|)|\Omega|^{\frac{1}{2}} \\
.\quad &= \|\tilde{u}_t(t+1)\| + \|\tilde{v}_t(t+1)\| + \|\tilde{w}_t(t+1)\| + q\|u(t)\| + 2\|v(t)\| \\
&\quad + (1+r)\|w(t)\| + (a+\beta)\|u(t)\|^2_{L^4} + b\|u(t)\|^3_{L^6} + (J + \alpha + q|c|)|\Omega|^{\frac{1}{2}} \\
&\le D^{\frac{1}{2}} + (q + 3 + r)K^{\frac{1}{2}} + 2(a+\beta)\eta^2(K + Q) \\
&\quad + 2b\eta^3(K^{\frac{3}{2}} + Q^{\frac{3}{2}}) + (J + \alpha + q|c|)|\Omega|^{\frac{1}{2}}
\end{aligned} \tag{4.13}
$$

where K and Q are given in Theorems 2.2 and 3.1, and (3.6) is used in the last step.

The Laplacian operator $A_0 = \Delta$ with the Neumann boundary condition (1.5) is self-adjoint and negative definite modulo constant functions. Hence the Sobolev space norm of any $g \in H^2(\Omega, \mathbb{R}^3)$ is equivalent to $\|g\| + \|\nabla g\| + \|\Delta g\|$. The inequality (4.13) together with Theorem 2.2, Theorem 3.1, and Theorem 3.2 shows that

$$
\begin{aligned}
&\|g\|_{H^2(\Omega,\mathbb{R}^3)} \cong \|g\| + \|\nabla g\| + \|\Delta g\| \\
&\le K^{\frac{1}{2}} + Q^{\frac{1}{2}} + \frac{1}{d}\Big(D^{\frac{1}{2}} + (q + 3 + r)K^{\frac{1}{2}} + 2(a+\beta)\eta^2(K + Q) \\
&\quad + 2b\eta^3(K^{\frac{3}{2}} + Q^{\frac{3}{2}}) + (J + \alpha + q|c|)|\Omega|^{\frac{1}{2}}\Big), \quad \text{for any } g \in \mathscr{A},
\end{aligned} \tag{4.14}
$$

where $d = \min\{d_1, d_2, d_3\}$. Therefore, the global attractor $\mathscr{A}$ is a bounded set in $H^2(\Omega, \mathbb{R}^3)$. $\square$

The global attractor $\mathscr{A}$ shown in this paper has a finite fractal dimension in the space H, which can be proved via the existence of an exponential attractor in the space H shown by the authors in [17].

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