

Bifurcation of a Modified Leslie-Gower System with Discrete and Distributed Delays*

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Abstract A modified Leslie-Gower predator-prey system with discrete and distributed delays is introduced. By analyzing the associated characteristic equation, stability and local Hopf bifurcation of the model are studied. It is found that the positive equilibrium is asymptotically stable when τ is less than a critical value and unstable when τ is greater than this critical value and the system can also undergo Hopf bifurcation at the positive equilibrium when τ crosses this critical value. Furthermore, using the normal form theory and center manifold theorem, the formulae for determining the direction of periodic solutions bifurcating from positive equilibrium are derived. Some numerical simulations are also carried out to illustrate our results.

Keywords Modified Leslie-Gower system, discrete and distributed delays, stability, Hopf bifurcation.

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1. Introduction

Recently, the dynamics (including stability, persistence, periodic oscillation, bifurcation and chaos, etc.) of predator-prey system has long been one of the dominant themes in mathematical ecology due to its universal existence and importance (for example, see [1, 7, 9, 11, 19, 26, 29]). [14] first proposed and discussed the following predator-prey system

$$\begin{cases} \dot{x}(t) = x(a - bx) - p(x)y, \\ \dot{y}(t) = y[s(1 - h\frac{y}{x})], \end{cases} \quad (1.1)$$

where $p(x)$ is the predator functional response to prey, a, s are the intrinsic growth rate of prey $x(t)$ and predator $y(t)$, respectively, b measures the strength of competition among individuals of species $x(t)$, $\frac{a}{b}$ is the environmental carrying capacity for the prey, the environmental carrying capacity of predator K_y is a function of the

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available prey quantity, where $K_y = \frac{x}{h}$. The predator subsistence exclusively dependent on prey population in system (1.1). However, in the case of severe scarcity of the prey x , the predator y can switch to other population, but its growth will be limited. By adding a positive constant to the K_y , that is, $K_y = \frac{x+c}{h}$, then in this case, a modified Leslie-Gower predator-prey system with functional response can be described by the following system

$$\begin{cases} \dot{x}(t) = x(a - bx) - p(x)y, \\ \dot{y}(t) = y[s(1 - h\frac{y}{x+c})]. \end{cases} \quad (1.2)$$

The system (1.2) have been studied by many authors (for example, see [3, 4, 25, 30]). When $p(x)$ is Holling type II functional response, then system (1.2) can be written as

$$\begin{cases} \dot{x}(t) = x(t)(a_1 - bx(t) - \frac{c_1y(t)}{x(t) + k_1}), \\ \dot{y}(t) = y(t)(a_2 - \frac{c_2y(t)}{x(t) + k_2}). \end{cases} \quad (1.3)$$

For system (1.3), [2] investigated the boundedness of solution, existence of an attracting set and global stability of the coexisting interior equilibrium.

Time delays in mathematical models of population dynamics are usually due to gestation time, maturation time, capturing time or some other reasons. Therefore, a more realistic predator-prey system should be described by delay differential equations. In fact, delay differential equations are capable of generating rich effective and accurate dynamics compared to ordinary differential equations (for example, see [5, 6, 8, 10, 15–17, 20, 22–24, 27, 31]). [18] investigated the following delayed Leslie-Gower predator-prey system

$$\begin{cases} \dot{x}(t) = x(t)(a_1 - bx(t) - \frac{c_1y(t)}{x(t) + k_1}), \\ \dot{y}(t) = y(t)(a_2 - \frac{c_2y(t-\tau)}{x(t-\tau) + k_2}). \end{cases} \quad (1.4)$$

For system (1.4), Nindjin et al. investigated the permanence and global stability of positive equilibrium without considering the effects of time delays on the prey. [4] studied the following Leslie-Gower predator-prey system with delay and investigated Hopf bifurcations at the positive equilibrium

$$\begin{cases} \dot{x}(t) = x(t)(a_1 - bx(t) - \frac{c_1y(t-\tau)}{x(t-\tau) + k_1}), \\ \dot{y}(t) = y(t)(a_2 - \frac{c_2y(t-\tau)}{x(t-\tau) + k_2}). \end{cases} \quad (1.5)$$

Based on the above, in this paper, we will investigate the following modified Leslie-Gower predator-prey model with discrete and distributed time delays

$$\begin{cases} \dot{x}(t) = x(t)(a_1 - b \int_{-\infty}^t f(t-s)x(s)ds - \frac{c_1y(t)}{x(t) + k_1}), \\ \dot{y}(t) = y(t)(a_2 - \frac{c_2y(t-\tau)}{x(t-\tau) + k_2}), \end{cases} \quad (1.6)$$

where $x(t)$ is the population density of the prey and $y(t)$ is the population density of the predator at time t . The parameters $a_1, a_2, b, c_1, c_2, k_1, k_2$ are positive constants: a_1, a_2 are the intrinsic growth rate of prey $x(t)$ and predator $y(t)$, respectively, b measures the strength of competition among individuals of species $x(t)$, $\frac{a_1}{b}$ is the environmental carrying capacity for the prey, $\frac{a_2(x(t-\tau)+k_2)}{c_2}$ denote the prey-dependent carrying capacity for the predator, c_1 is the maximum value of the per capita reduction rate of $x(t)$ due to $y(t)$, and c_2 has a similar meaning as c_1 . k_1, k_2 measure the extent to which environment provides protection to prey $x(t)$ and to predator $y(t)$, respectively. the nonnegative constant τ can be interpreted as the hunting delay of the predator population. We consider that past prey $x(s)$ has negative effect on the present prey $x(t)(t \geq s)$, such as consumption of food resources, the delay kernel function $f(t)$ is the weight given to the population $x(t)$ at t time units ago. Under the assumption that $f(t) \geq 0$ for all $t \geq 0$ and the normalized condition

$$\int_0^{+\infty} f(t)dt = 1.$$

The rest of this paper is organized as follows. In Section 2, by analyzing the associated characteristic equation, the asymptotic stability of the equilibria and the existence Hopf bifurcations at the positive equilibrium are investigated. In Section 3, the formulae for determining the direction of the Hopf bifurcation and the stability of bifurcation periodic solutions are given by using the normal form method and center manifold theorem introduced by [12]. In Section 4, numerical simulations are performed to support our theoretical results. A brief conclusion is given in last section.

2. Stability of equilibria and existence of local Hopf bifurcation

For system (1.6), the delay kernel function $f(t)$ take the weak generic kernel function $f(t) = \sigma e^{-\sigma t}(\sigma > 0)$, where weak generic kernel function implies that the importance of the event decrease exponentially with t . we define a new variable

$$u(t) = \int_{-\infty}^t \sigma e^{-\sigma(t-s)} x(s) ds. \quad (2.1)$$

Taking the derivative of $u(t)$ with respect to t in (2.1), we have

$$u(t) = \sigma x(t) - \sigma u(t). \quad (2.2)$$

Then the system (1.6) is equivalent to the following system

$$\begin{cases} \dot{x}(t) = x(t)(a_1 - bu(t) - \frac{c_1 y(t)}{x(t) + k_1}), \\ \dot{u}(t) = \sigma x(t) - \sigma u(t), \\ \dot{y}(t) = y(t)(a_2 - \frac{c_2 y(t-\tau)}{x(t-\tau) + k_2}). \end{cases} \quad (2.3)$$

The initial conditions for systems (2.3) take the form

$$\begin{aligned} x(\theta) &= \phi(\theta), u(\theta) = \varphi(\theta), y(\theta) = \omega(\theta), \\ \phi(\theta) &\geq 0, \varphi(\theta) \geq 0, \omega(\theta) \geq 0, \theta \in [-\tau, 0), \\ \phi(0) &> 0, \varphi(0) > 0, \omega(0) > 0, \end{aligned} \quad (2.4)$$

where $\phi(\theta), \varphi(\theta), \omega(\theta)$ are continuous and bounded functions in $[-\tau, 0)$.

It is easy to see that the system (2.3) has a unique solution $(x(t), u(t), y(t))$ satisfying initial conditions (2.4), and we know from [13] that the solutions of system (2.3) corresponding to initial conditions (2.4) remain positive for all $t \geq 0$. Furthermore, for any positive solution $(x(t), u(t), y(t))$ of system (2.3), there exist a $T \geq 0$ such that

$$0 \leq x(t) \leq \frac{a_1}{b}, \quad 0 \leq u(t) \leq \frac{a_1}{b}, \quad 0 \leq y(t) \leq \frac{a_2(k_2 + \frac{a_1}{b})}{c_2} e^{a_2 t}, \quad t \geq T,$$

similarly to the proofs in [18].

Obviously, system (2.3) always has three feasible boundary equilibria $E_1(0, 0, 0)$, $E_2(\frac{a_1}{b}, \frac{a_1}{b}, 0)$, $E_3(0, 0, \frac{a_2 k_2}{c_2})$. From the biological viewpoint, we are more interested in the positive equilibrium. To guarantee that system (2.3) has always a positive equilibrium, we assume that the coefficients of system (2.3) satisfy the following condition

$$(H1) \quad \frac{a_1 k_1}{c_1} > \frac{a_2 k_2}{c_2}.$$

Under the hypothesis (H1), system (2.3) has a unique positive equilibrium $E(x^*, x^*, y^*)$, where $y^* = \frac{a_2}{c_2}(x^* + k_2)$ and x^* is the positive root of the following equation

$$bx^2 + (bk_1 - a_1 + \frac{a_2 c_1}{c_2})x + (\frac{a_2 c_1 k_2}{c_2} - a_1 k_1) = 0.$$

2.1. Stability of the boundary equilibria

Proposition 2.1. *The boundary equilibria E_1, E_2 are unstable for all $\tau \geq 0$.*

Proof. Linearizing system (2.3) at E_1, E_2 , we obtain characteristic equations

$$(\lambda - a_1)(\lambda + \sigma)(\lambda - a_2) = 0$$

and

$$(\lambda^2 + \sigma\lambda + a_1\sigma)(\lambda - a_2) = 0,$$

respectively, the characteristic equations exist at least one positive root, respectively, thus E_1, E_2 are unstable for all $\tau \geq 0$. \square

Proposition 2.2. (i) *If*

$$\frac{a_1 k_1}{c_1} > \frac{a_2 k_2}{c_2},$$

then E_3 is unstable for all $\tau \geq 0$;

(ii) *If*

$$\frac{a_1 k_1}{c_1} < \frac{a_2 k_2}{c_2},$$

then E_3 is asymptotically stable for $\tau \in [0, \tau_s)$, where $\tau_s = \frac{\pi}{2a_2}$.

Proof. Linearizing system (2.3) at E_3 , we obtain characteristic equation

$$(\lambda - a_1 + \frac{c_1 a_2 k_2}{k_1 c_2})(\lambda + \sigma)(\lambda + a_2 e^{-\lambda \tau}) = 0.$$

(i) If

$$\frac{a_1 k_1}{c_1} > \frac{a_2 k_2}{c_2},$$

then this characteristic equations exists at lest one positive root, thus E_3 is unstable for all $\tau \geq 0$.

(ii) If

$$\frac{a_1 k_1}{c_1} < \frac{a_2 k_2}{c_2},$$

in the case of $\tau = 0$, all roots of this characteristic equation are negative. Then E_3 is asymptotically stable. Now, we suppose $\tau > 0$ and let $\lambda = i\omega$ ($\omega > 0$) is a purely imaginary root of equation

$$\lambda + a_2 e^{-\lambda \tau} = 0,$$

which implies that

$$\begin{cases} \cos(\omega \tau) = 0, \\ \omega - a_2 \sin(\omega \tau) = 0, \end{cases}$$

then

$$\begin{cases} \omega = a_2, \\ \tau = \frac{\pi}{2a_2} + \frac{2k\pi}{a_2}, k = 0, 1, 2, \dots \end{cases}$$

Define $\tau_s = \min\{\tau\}$, that is, $\tau_s = \frac{\pi}{2a_2}$, thus E_3 is asymptotically stable for $\tau \in [0, \tau_s)$. This completes the proof. \square

2.2. Stability of the positive equilibrium and existence of local Hopf bifurcation

Let $\bar{x} = x - x^*$, $\bar{u} = u - x^*$, $\bar{y} = y - y^*$. Dropping the bars, ignoring the higher order terms, then from system (2.3) we obtain the following linear system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{u}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ u(t) \\ y(t) \end{pmatrix} + B \begin{pmatrix} x(t - \tau) \\ u(t - \tau) \\ y(t - \tau) \end{pmatrix}, \quad (2.5)$$

where

$$A = \begin{pmatrix} \frac{c_1 x^* y^*}{(x^* + k_1)^2} & -bx^* & -\frac{c_1 x^*}{x^* + k_1} \\ \sigma & -\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a_2^2}{c_2} & 0 & -a_2 \end{pmatrix}.$$

The characteristic equation of system (2.5) is $\det(\lambda I - A - B e^{-\lambda \tau}) = 0$, i.e.

$$\lambda^3 + d_1 \lambda^2 + d_2 \lambda + (d_3 \lambda^2 + d_4 \lambda + d_5) e^{-\lambda \tau} = 0, \quad (2.6)$$

where $d_1 = \sigma - \frac{c_1 x^* y^*}{(x^* + k_1)^2}$, $d_2 = -\frac{\sigma c_1 x^* y^*}{(x^* + k_1)^2} + b x^* \sigma$, $d_3 = a_2$, $d_4 = \sigma a_2 - a_2 \frac{c_1 x^* y^*}{(x^* + k_1)^2} + \frac{c_1 x^*}{(x^* + k_1)} \frac{a_2^2}{c_2}$, $d_5 = b x^* \sigma a_2 - a_2 \sigma \frac{c_1 x^* y^*}{(x^* + k_1)^2} + \sigma \frac{c_1 x^*}{(x^* + k_1)} \frac{a_2^2}{c_2}$.

The characteristic equation of system (2.5) without time delay is

$$\lambda^3 + (d_1 + d_3)\lambda^2 + (d_4 + d_5)\lambda + d_5 = 0. \quad (2.7)$$

(H2) $(d_1 + d_3) > 0$, $d_5 > 0$, $(d_1 + d_3)(d_4 + d_2) - d_5 > 0$.

By (H2) and the Routh-Hurwitz criterion, we know that all the roots of Eq.(2.7) have negative real parts, i.e., positive equilibrium $E(x^*, x^*, y^*)$ is locally asymptotically stable.

When $\tau > 0$, noting that $i\omega (\omega > 0)$ is a root of Eq.(2.6), separating real and imaginary parts, we have

$$\begin{aligned} (d_5 - d_3\omega^2) \cos \omega\tau + d_4\omega \sin \omega\tau &= d_1\omega^2, \\ -(d_5 - d_3\omega^2) \sin \omega\tau + d_4\omega \cos \omega\tau &= \omega^3 - d_2\omega. \end{aligned} \quad (2.8)$$

Squaring and adding the two Eqs.(2.8) we obtain

$$\omega^6 + (-2d_2 + d_1^2 - d_3^2)\omega^4 + (d_2^2 - d_4^2 + 2d_3d_5)\omega^2 - d_5^2 = 0. \quad (2.9)$$

Obviously, (2.9) has at least one positive real root. Without loss of generality, assuming that it has six positive real roots, denoted by $\omega_i (i = 1, 2, \dots, 6)$ respectively. So we have

$$\cos \omega_i \tau = \frac{d_1\omega_i^2(d_5 - d_3\omega_i^2) + d_4\omega(\omega_i^3 - d_2\omega_i)}{(d_5 - d_3\omega_i^2)^2 + (d_4\omega_i)^2} \quad (i = 1, 2, \dots, 6).$$

Thus, denoting

$$\tau_i^j = \frac{1}{\omega_i} \arccos\left\{ \frac{d_1\omega_i^2(d_5 - d_3\omega_i^2) + d_4\omega(\omega_i^3 - d_2\omega_i)}{(d_5 - d_3\omega_i^2)^2 + (d_4\omega_i)^2} + 2\pi j \right\} \quad (j = 0, 1, \dots).$$

Then $\pm i\omega_i$ is a pair of purely imaginary roots of (2.6) with $\tau = \tau_i^j$. Define

$$\tau_0 = \tau_{i0}^0 = \min_{i \in \{1, 2, 3, \dots, 6\}} \{\tau_i^0\}, \quad \omega_0 = \omega_{i0}.$$

Taking the derivative of λ with respect to τ in (2.6), it is easy to have

$$\begin{aligned} (3\lambda^2 + 2d_1\lambda + d_2) \frac{d\lambda}{d\tau} + (2d_3\lambda + d_4)e^{-\lambda\tau} \frac{d\lambda}{d\tau} - (d_3\lambda^2 + d_4\lambda + d_5)e^{-\lambda\tau} \left(\tau \frac{d\lambda}{d\tau} + \lambda \right) &= 0, \\ \left(\frac{d\lambda}{d\tau} \right)^{-1} &= \frac{(3\lambda^2 + 2d_1\lambda + d_2) + (2d_3\lambda + d_4)e^{-\lambda\tau}}{(d_3\lambda^2 + d_4\lambda + d_5)e^{-\lambda\tau}\lambda} - \frac{\tau}{\lambda}. \end{aligned}$$

To simplify our expression, we denote ω_0, τ_0 by ω, τ respectively, then

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{(-3\omega^2 + 2d_1i\omega + d_2) + (2d_3i\omega + d_4)(\cos \omega\tau - i \sin \omega\tau)}{(-d_3\omega^2 + d_4i\omega + d_5)(\cos \omega\tau - i \sin \omega\tau)i\omega} - \frac{\tau}{i\omega}.$$

Let $A = -3\omega^2 + d_2 + 2d_3\omega \sin \omega\tau + d_4 \cos \omega\tau$, $B = 2d_1\omega + 2d_3\omega \cos \omega\tau - d_4 \sin \omega\tau$, $C = \omega((-d_3\omega^2 + d_5) \sin \omega\tau - d_4\omega \cos \omega\tau)$, $D = \omega((-d_3\omega^2 + d_5) \cos \omega\tau + d_4\omega \sin \omega\tau)$, then

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{A + iB}{C + iD} - \frac{\tau}{i\omega}.$$

Let $Q = C^2 + D^2 > 0$, then $Q \operatorname{Re}(\frac{d\lambda}{d\tau})^{-1} = AC + BD$. Noting that

$$\operatorname{sign}[\operatorname{Re}(\frac{d\lambda}{d\tau}) |_{\tau=\tau_0}] = \operatorname{sign}[\operatorname{Re}(\frac{d\lambda}{d\tau})^{-1} |_{\tau=\tau_0}].$$

To obtain the transversal condition, we also need the condition as following

$$(H3) \operatorname{sign}[\operatorname{Re}(\frac{d\lambda}{d\tau}) |_{\tau=\tau_0}] > 0, \text{ i.e., } AC + BD > 0.$$

From Corollary 2.4 [21], it is easy to obtain the following theorem.

Theorem 2.1. *Assume that (H1) – (H3) hold, then the following results hold:*

(i) *The positive equilibrium $E(x^*, x^*, y^*)$ of system (2.3) is locally asymptotically stable for $\tau \in [0, \tau_0)$;*

(ii) *System (2.3) undergoes a Hopf bifurcation when $\tau = \tau_0$. That is, system (2.3) has a branch of periodic solutions bifurcating from the positive equilibrium $E(x^*, x^*, y^*)$ near $\tau = \tau_0$.*

3. Direction and stability of Hopf bifurcation

In the previous section, we studied mainly the stability of the positive equilibrium $E(x^*, x^*, y^*)$ and the existence of Hopf bifurcation at the positive equilibrium $E(x^*, x^*, y^*)$. In this section, we study the direction of bifurcation and stability of bifurcating periodic solutions arising through Hopf bifurcation by applying the normal form theory and center manifold theorem introduced by [12]. Throughout this section, we always assume that system (2.3) undergoes Hopf bifurcation at the positive equilibrium $E(x^*, x^*, y^*)$ for $\tau = \tau_0$, at which the characteristic Eq.(2.6) has a pair of imaginary roots $\pm i\omega_0$.

Now we re-scale the time by $t = s\tau$, $\hat{x}(s) = x(s\tau) - x^*$, $\hat{u}(s) = u(s\tau) - x^*$, $\hat{y}(s) = y(s\tau) - y^*$, $\tau = \tau_0 + \mu$, and still denote by $x(t) = \hat{x}(s)$, $u(t) = \hat{u}(s)$, $y(t) = \hat{y}(s)$, then system (2.3) can be written as

$$\dot{\varphi}(t) = L_\mu(\varphi_t) + f(\mu, \varphi_t), \quad (3.1)$$

where $\varphi(t) = (x, u, y)^T$, $\varphi_t(\theta) = \varphi(t + \theta)$, $\theta \in [-1, 0]$, for $\phi = (\phi_1, \phi_2, \phi_3) \in C([-1, 0], \mathbb{R}^3)$,

$$\begin{aligned} & L_\mu(\phi) \\ = & (\tau_0 + \mu) \left(\left(\begin{array}{ccc} \frac{c_1 x^* y^*}{(x^* + k_1)^2} & -bx^* & -\frac{c_1 x^*}{x^* + k_1} \\ \sigma & -\sigma & 0 \\ 0 & 0 & 0 \end{array} \right) \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a_2^2}{c_2} & 0 & -a_2 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \right), \end{aligned}$$

and

$$f(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f_{ijk}^{(1)} \phi_1^i(0) \phi_2^j(0) \phi_3^k(0) \\ 0 \\ \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f_{ijk}^{(2)} \phi_1^i(-1) \phi_2^j(-1) \phi_3^k(0) \end{pmatrix},$$

where $f^{(1)} = x(t)(a_1 - bu(t) - \frac{c_1 y(t)}{x(t) + k_1})$, $f^{(2)} = y(t)(a_2 - \frac{c_2 y(t - \tau)}{x(t - \tau) + k_2})$, $f_{ijk}^{(1)} = \frac{\partial^{i+j+k} f^{(1)}}{\partial x^i \partial u^j \partial y^k} \Big|_{(x^*, x^*, y^*)}$, $f_{ijk}^{(2)} = \frac{\partial^{i+j+k} f^{(2)}}{\partial x^i \partial u^j \partial y^k} \Big|_{(x^*, x^*, y^*)}$. By the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta, \mu)$, $\theta \in [-1, 0]$, whose elements are functions of bounded variation, such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta).$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu) \left(\begin{pmatrix} \frac{c_1 x^* y^*}{(x^* + k_1)^2} & -bx^* & -\frac{c_1 x^*}{x^* + k_1} \\ \sigma & -\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta) - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a_2^2}{c_2} & 0 & -a_2 \end{pmatrix} \delta(\theta + 1) \right).$$

For $\phi \in C^1([-1, 0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi}{d\theta}, \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), \theta = 0, \end{cases} \quad \text{and} \quad R(\mu)\phi = \begin{cases} 0, \theta \in [-1, 0), \\ f(\mu, \phi), \theta = 0. \end{cases} \quad (3.2)$$

Then system (3.1) is equivalent to the following abstract ODE

$$\dot{\varphi}_t = A(\mu)\varphi_t + R(\mu)\varphi_t, \quad (3.3)$$

where $\varphi_t(\theta) = \varphi(t + \theta)$ for $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (C^3)^*)$, where $(C^3)^*$ is 3-dimensional complex row vector. The adjoint operator A^* of $A(0)$ is defined as

$$A^*\psi(s) = \begin{cases} -\frac{d\psi}{ds}, s \in (0, 1], \\ \int_{-1}^0 \psi(-t)d\eta(t, 0), s = 0. \end{cases} \quad (3.4)$$

For $\psi \in C^1([0, 1], (C^3)^*)$, $\phi \in C^1([-1, 0], C^3)$, we define a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \quad (3.5)$$

where $\eta(\theta) = \eta(\theta, 0)$, which meets $\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle$. By the discussion in Section 2 and transformation $t = s\tau$, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^* . Next we need to calculate the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. Suppose that $q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_0\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_0$, from the definition of $A(0)$, we have

$$\tau_0 \begin{pmatrix} i\omega_0 - \frac{c_1 x^* y^*}{(x^* + k_1)^2} & bx^* & \frac{c_1 x^*}{x^* + k_1} \\ -\sigma & i\omega_0 + \sigma & 0 \\ -\frac{a_2^2 e^{-i\omega_0\tau_0}}{c_2} & 0 & i\omega_0 + a_2 e^{-i\omega_0\tau_0} \end{pmatrix} q(0) = 0.$$

Thus, we get

$$\alpha = \frac{\sigma}{i\omega + \sigma}, \beta = \frac{a_2^2 e^{-i\omega_0 \tau_0}}{c_2(i\omega_0 + a_2 e^{-i\omega_0 \tau_0})}.$$

Suppose that $q^*(\theta) = D(1, \alpha^*, \beta^*)e^{i\omega_0 \tau_0 \theta}$ is the eigenvector of A^* corresponding to $-i\omega_0 \tau_0$, from the definition of A^* , we have

$$\tau_0 \begin{pmatrix} -i\omega_0 - \frac{c_1 x^* y^*}{(x^* + k_1)^2} & -\sigma & -\frac{a_2^2 e^{i\omega_0 \tau_0}}{c_2} \\ b x^* & -i\omega_0 + \sigma & 0 \\ \frac{c_1 x^*}{x^* + k_1} & 0 & -i\omega_0 + a_2 e^{i\omega_0 \tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha^* \\ \beta^* \end{pmatrix} = 0.$$

Thus, we obtain

$$\alpha^* = \frac{b x^*}{i\omega - \sigma}, \beta^* = \frac{\frac{c_1 x^*}{x^* + k_1}}{i\omega_0 - a_2 e^{i\omega_0 \tau_0}}.$$

Since

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*)(1, \alpha, \beta)^T \\ &\quad - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*) e^{-i\omega_0 \tau_0 (\xi - \theta)} d\eta(\theta) (1, \alpha, \beta)^T e^{i\omega_0 \tau_0 \xi} d\xi \\ &= \bar{D} \{1 + \alpha \bar{\alpha}^* + \beta \bar{\beta}^* - \int_{-1}^0 (1, \bar{\alpha}^*, \bar{\beta}^*) \theta e^{i\omega_0 \tau_0 \theta} d\eta(\theta) (1, \alpha, \beta)^T\} \\ &= \bar{D} \{1 + \alpha \bar{\alpha}^* + \beta \bar{\beta}^* + \tau_0 (\frac{a_2^2 \bar{\beta}^*}{c_2} - a_2 \bar{\beta}^* \beta) e^{-i\omega_0 \tau_0}\}. \end{aligned}$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1$ and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$, we choose

$$\bar{D} = \frac{1}{1 + \alpha \bar{\alpha}^* + \beta \bar{\beta}^* + \tau_0 (\frac{a_2^2 \bar{\beta}^*}{c_2} - a_2 \bar{\beta}^* \beta) e^{-i\omega_0 \tau_0}}.$$

In the following part of this section, we use the theory by [12] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$z(t) = \langle q^*, \varphi_t \rangle, W(t, \theta) = \varphi_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}. \quad (3.6)$$

On the center manifold C_0 , we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$ where

$$W(z(t), \bar{z}(t), \theta) = W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots, \quad (3.7)$$

z and \bar{z} are local coordinates for the center manifold C_0 in the direction of q^* and \bar{q}^* . Note that W is real if φ_t is real, we are only interested in real solutions. For the solution $\varphi_t \in C_0$ of (3.3), since $\mu = 0$, which together with (3.2) – (3.6) imply that

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{\varphi}_t \rangle = \langle q^*, A(0)\varphi_t \rangle + \langle q^*, R(0)\varphi_t \rangle \\ &= i\omega_0 \tau_0 z(t) + \bar{q}^*(0) f(0, W(z(t), \bar{z}(t), 0) + 2\text{Re}\{z(t)q(0)\}) \\ &= i\omega_0 \tau_0 z(t) + g(z, \bar{z}), \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f(0, W(z(t), \bar{z}(t), 0) + 2Re\{z(t)q(0)\}) \stackrel{def}{=} \bar{q}^*(0) f_0(z, \bar{z}) \\ &= g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots \end{aligned} \quad (3.9)$$

From (3.6) and (3.7), we know that $\varphi_t(\theta) = (\varphi_{1t}(\theta), \varphi_{2t}(\theta), \varphi_{3t}(\theta))^T = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$, $q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_0\theta}$, and

$$\begin{aligned} \varphi_{1t}(0) &= W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + z + \bar{z} + \dots, \\ \varphi_{2t}(0) &= W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \alpha z + \bar{\alpha} \bar{z} + \dots, \\ \varphi_{3t}(0) &= W_{20}^{(3)}(0) \frac{z^2}{2} + W_{11}^{(3)}(0) z\bar{z} + W_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + \beta z + \bar{\beta} \bar{z} + \dots, \\ \varphi_{1t}(-1) &= W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_0} z + e^{i\omega_0\tau_0} \bar{z} + \dots, \\ \varphi_{3t}(-1) &= W_{20}^{(3)}(-1) \frac{z^2}{2} + W_{11}^{(3)}(-1) z\bar{z} + W_{02}^{(3)}(-1) \frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_0} \beta z + e^{i\omega_0\tau_0} \bar{\beta} \bar{z} + \dots \end{aligned}$$

According to (3.8) and (3.9), we obtain

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f_0(z, \bar{z}) \\ &= \tau_0 \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*) \begin{pmatrix} \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f_{ijk}^{(1)} \varphi_{1t}^i(0) \varphi_{2t}^j(0) \varphi_{3t}^k(0) \\ 0 \\ \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f_{ijk}^{(2)} \varphi_{1t}^i(-1) \varphi_{3t}^j(-1) \varphi_{3t}^k(0) \end{pmatrix} \\ &= \tau_0 \bar{D} \left\{ z^2 \left[\frac{f_{200}^{(1)}}{2} + f_{110}^{(1)} \alpha + f_{101}^{(1)} \beta + f_{011}^{(1)} \alpha \beta + \bar{\beta}^* \left(\frac{f_{200}^{(2)}}{2} e^{-2i\omega_0\tau_0} + f_{110}^{(2)} \beta e^{-2i\omega_0\tau_0} \right. \right. \right. \\ &\quad \left. \left. + f_{101}^{(2)} \beta e^{-i\omega_0\tau_0} + f_{011}^{(2)} \beta^2 e^{-i\omega_0\tau_0} \right] \right. \\ &\quad \left. + z\bar{z} \left[f_{200}^{(1)} + f_{110}^{(1)} (\alpha + \bar{\alpha}) + f_{101}^{(1)} (\beta + \bar{\beta}) + f_{011}^{(1)} (\alpha\bar{\beta} + \bar{\alpha}\beta) + \bar{\beta}^* (f_{200}^{(2)} \right. \right. \\ &\quad \left. \left. + f_{110}^{(2)} (\beta + \bar{\beta}) + f_{101}^{(2)} (\bar{\beta} e^{-i\omega_0\tau_0} + \beta e^{i\omega_0\tau_0}) + f_{011}^{(2)} \beta\bar{\beta} (e^{i\omega_0\tau_0} + e^{-i\omega_0\tau_0}) \right] \right. \\ &\quad \left. + \bar{z}^2 \left[\frac{f_{200}^{(1)}}{2} + f_{110}^{(1)} \bar{\alpha} + f_{101}^{(1)} \bar{\beta} + f_{011}^{(1)} \bar{\alpha}\bar{\beta} + \bar{\beta}^* \left(\frac{f_{200}^{(2)}}{2} e^{2i\omega_0\tau_0} + f_{110}^{(2)} \bar{\beta} e^{2i\omega_0\tau_0} \right. \right. \right. \\ &\quad \left. \left. + f_{101}^{(2)} \bar{\beta} e^{i\omega_0\tau_0} + f_{011}^{(2)} \bar{\beta}^2 e^{i\omega_0\tau_0} \right] \right. \\ &\quad \left. + z^2 \bar{z} \left[\frac{f_{200}^{(1)}}{2} (W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) \right. \right. \\ &\quad \left. \left. + \frac{f_{110}^{(1)}}{2} (W_{20}^{(1)}(0) \bar{\alpha} + 2W_{11}^{(1)}(0) \alpha + W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)) \right. \right. \\ &\quad \left. \left. + \frac{f_{101}^{(1)}}{2} (W_{20}^{(1)}(0) \bar{\beta} + 2W_{11}^{(1)}(0) \beta + W_{20}^{(3)}(0) + 2W_{11}^{(3)}(0)) \right. \right. \\ &\quad \left. \left. + \frac{f_{011}^{(1)}}{2} (W_{20}^{(2)}(0) \bar{\beta} + W_{20}^{(3)}(0) \bar{\alpha} + 2W_{11}^{(2)}(0) \beta + 2W_{11}^{(3)}(0) \alpha) \right] \right. \\ &\quad \left. + \bar{\beta}^* \left[\frac{f_{200}^{(2)}}{2} (W_{20}^{(1)}(-1) e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(-1) e^{-i\omega_0\tau_0}) \right. \right. \\ &\quad \left. \left. + \frac{f_{110}^{(2)}}{2} (W_{20}^{(1)}(-1) \bar{\beta} e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(-1) \beta e^{-i\omega_0\tau_0} + W_{20}^{(3)}(-1) e^{i\omega_0\tau_0} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + 2W_{11}^{(3)}(-1)e^{-i\omega_0\tau_0} \\
& + \frac{f_{101}^{(2)}}{2}(W_{20}^{(1)}(-1)\bar{\beta} + 2W_{11}^{(1)}(-1)\beta + W_{20}^{(3)}(0)e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(0)e^{-i\omega_0\tau_0}) \\
& + \frac{f_{011}^{(2)}}{2}(W_{20}^{(3)}(-1)\bar{\beta} + W_{20}^{(3)}(0)\bar{\beta}e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(-1)\beta \\
& + 2W_{11}^{(3)}(0)\beta e^{-i\omega_0\tau_0})] + \frac{f_{300}^{(1)}}{2} + \frac{f_{201}^{(1)}}{2}(\bar{\beta} + 2\beta) \\
& + \frac{f_{210}^{(1)}}{2}(\bar{\alpha} + 2\alpha) + f_{111}^{(1)}(\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\beta) + \bar{\beta}^*[\frac{f_{300}^{(2)}}{2}e^{-i\omega_0\tau_0} \\
& + \frac{f_{201}^{(2)}}{2}(\bar{\beta}e^{-2i\omega_0\tau_0} + 2\beta) + \frac{f_{210}^{(2)}}{2}e^{-i\omega_0\tau_0}(\bar{\beta} + 2\beta) \\
& + f_{111}^{(2)}(\beta\bar{\beta} + \beta^2 + \bar{\beta}\beta e^{-2i\omega_0\tau_0})] + \dots \}.
\end{aligned}$$

Comparing the coefficients of (3.9) with the above equality, we get

$$\begin{aligned}
g_{20} &= 2\tau_0\bar{D}[\frac{f_{200}^{(1)}}{2} + f_{110}^{(1)}\alpha + f_{101}^{(1)}\beta + f_{011}^{(1)}\alpha\beta \\
& + \bar{\beta}^*(\frac{f_{200}^{(2)}}{2}e^{-2i\omega_0\tau_0} + f_{110}^{(2)}\beta e^{-2i\omega_0\tau_0} + f_{101}^{(2)}\beta e^{-i\omega_0\tau_0} + f_{011}^{(2)}\beta^2 e^{-i\omega_0\tau_0})], \\
g_{11} &= \tau_0\bar{D}[f_{200}^{(1)} + f_{110}^{(1)}(\alpha + \bar{\alpha}) + f_{101}^{(1)}(\beta + \bar{\beta}) + f_{011}^{(1)}(\alpha\bar{\beta} + \bar{\alpha}\beta) + \bar{\beta}^*(f_{200}^{(2)} \\
& + f_{110}^{(2)}(\beta + \bar{\beta}) + f_{101}^{(2)}(\bar{\beta}e^{-i\omega_0\tau_0} + \beta e^{i\omega_0\tau_0}) + f_{011}^{(2)}\beta\bar{\beta}(e^{-i\omega_0\tau_0} + e^{i\omega_0\tau_0}))], \\
g_{02} &= 2\tau_0\bar{D}[\frac{f_{200}^{(1)}}{2} + f_{110}^{(1)}\bar{\alpha} + f_{101}^{(1)}\bar{\beta} + f_{011}^{(1)}\bar{\alpha}\bar{\beta} + \bar{\beta}^*(\frac{f_{200}^{(2)}}{2}e^{2i\omega_0\tau_0} + f_{110}^{(2)}\bar{\beta}e^{2i\omega_0\tau_0} \\
& + f_{101}^{(2)}\bar{\beta}e^{i\omega_0\tau_0} + f_{011}^{(2)}\bar{\beta}^2 e^{i\omega_0\tau_0})], \\
g_{21} &= 2\tau_0\bar{D}\{[\frac{f_{200}^{(1)}}{2}(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + \frac{f_{110}^{(1)}}{2}(W_{20}^{(1)}(0)\bar{\alpha} + 2W_{11}^{(1)}(0)\alpha + W_{20}^{(2)}(0) \\
& + 2W_{11}^{(2)}(0)) + \frac{f_{101}^{(1)}}{2}(W_{20}^{(1)}(0)\bar{\beta} + 2W_{11}^{(1)}(0)\beta + W_{20}^{(3)}(0) + 2W_{11}^{(3)}(0)) \\
& + \frac{f_{011}^{(1)}}{2}(W_{20}^{(2)}(0)\bar{\beta} + W_{20}^{(3)}(0)\bar{\alpha} + 2W_{11}^{(2)}(0)\beta + 2W_{11}^{(3)}(0)\alpha)] \\
& + \bar{\beta}^*[\frac{f_{200}^{(2)}}{2}(W_{20}^{(1)}(-1)e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0}) + \frac{f_{110}^{(2)}}{2}(W_{20}^{(1)}(-1)\bar{\beta}e^{i\omega_0\tau_0} \\
& + 2W_{11}^{(1)}(-1)\beta e^{-i\omega_0\tau_0} + W_{20}^{(3)}(-1)e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(-1)e^{-i\omega_0\tau_0}) \\
& + \frac{f_{101}^{(2)}}{2}(W_{20}^{(1)}(-1)\bar{\beta} + 2W_{11}^{(1)}(-1)\beta + W_{20}^{(3)}(0)e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(0)e^{-i\omega_0\tau_0}) \\
& + \frac{f_{011}^{(2)}}{2}(W_{20}^{(3)}(-1)\bar{\beta} + W_{20}^{(3)}(0)\bar{\beta}e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(-1)\beta + 2W_{11}^{(3)}(0)\beta e^{-i\omega_0\tau_0})] \\
& + \frac{f_{300}^{(1)}}{2} + \frac{f_{201}^{(1)}}{2}(\bar{\beta} + 2\beta) + \frac{f_{210}^{(1)}}{2}(\bar{\alpha} + 2\alpha) + f_{111}^{(1)}(\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\beta) \\
& + \bar{\beta}^*[\frac{f_{300}^{(2)}}{2}e^{-i\omega_0\tau_0} + \frac{f_{201}^{(2)}}{2}(\bar{\beta}e^{-2i\omega_0\tau_0} + 2\beta) + \frac{f_{210}^{(2)}}{2}e^{-i\omega_0\tau_0}(\bar{\beta} + 2\beta) \\
& + f_{111}^{(2)}(\beta\bar{\beta} + \beta^2 + \bar{\beta}\beta e^{-2i\omega_0\tau_0})] \}.
\end{aligned}$$

Now we compute $W_{20}(\theta)$ and $W_{11}(\theta)$. In view of (3.3) and (3.6), we obtain

$$\dot{W} = \dot{\varphi}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2Re\{\bar{q}^*(0)f_0q(\theta)\}, \theta \in [-1, 0), \\ AW - 2Re\{\bar{q}^*(0)f_0q(\theta)\} + f_0, \theta = 0, \end{cases} \stackrel{def}{=} AW + h(z, \bar{z}, \theta), \quad (3.10)$$

where

$$h(z, \bar{z}, \theta) = h_{20}(\theta)\frac{z^2}{2} + h_{11}(\theta)z\bar{z} + h_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \quad (3.11)$$

Taking the derivative of W with respect to t , we have

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}.$$

By (3.7), (3.8) and (3.11), comparing the corresponding coefficient, we obtain

$$(A - 2i\omega_0\tau_0)W_{20}(\theta) = -h_{20}(\theta), \quad (3.12)$$

$$AW_{11}(\theta) = -h_{11}(\theta). \quad (3.13)$$

By (3.10) we know that for $\theta \in [-1, 0)$,

$$h(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \bar{g}\bar{q}(\theta).$$

Comparing the coefficients with (3.11) gives that

$$h_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad h_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (3.14)$$

According to (3.12), (3.13), (3.14) and the definition of A , we know

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \quad \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta).$$

Noting $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, so

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + Ee^{2i\omega_0\tau_0\theta}, \quad (3.15)$$

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + F, \quad (3.16)$$

where $F, E \in R^3$ is a constant vector. By (3.12), (3.13) and the definition of A in (3.2), we have

$$\int_{-1}^0 d\eta(\theta, 0)W_{20}(\theta) = 2i\omega_0\tau_0W_{20}(0) - h_{20}(0), \quad (3.17)$$

and

$$\int_{-1}^0 d\eta(\theta, 0)W_{11}(\theta) = -h_{11}(0). \quad (3.18)$$

From (3.10), (3.14) and the definition of f_0 , we have

$$h_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_0(d_1, 0, d_2)^T, \quad (3.19)$$

and

$$h_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_0(d_3, 0, d_4)^T, \quad (3.20)$$

where

$$\begin{aligned} d_1 &= \frac{f_{200}^{(1)}}{2} + f_{110}^{(1)}\alpha + f_{101}^{(1)}\beta + f_{011}^{(1)}\alpha\beta, \\ d_2 &= \left(\frac{f_{200}^{(2)}}{2} + f_{110}^{(2)}\beta\right)e^{-2i\omega_0\tau_0} + \left(f_{101}^{(2)}\beta + f_{110}^{(2)}\beta^2\right)e^{-i\omega_0\tau_0}, \\ d_3 &= f_{200}^{(1)} + f_{110}^{(1)}(\alpha + \bar{\alpha}) + f_{101}^{(1)}(\beta + \bar{\beta}) + f_{011}^{(1)}(\alpha\bar{\beta} + \bar{\alpha}\beta), \\ d_4 &= f_{200}^{(2)} + 2f_{110}^{(2)}Re(\beta) + 2f_{101}^{(2)}Re(\beta e^{i\omega_0\tau_0}) + 2f_{011}^{(2)}\beta\bar{\beta}Re(e^{i\omega_0\tau_0}). \end{aligned}$$

Substituting (3.15) and (3.19) into (3.17) and noticing that

$$(i\omega_0\tau_0 I - \int_{-1}^0 e^{i\omega_0\tau_0\theta} d\eta(\theta))q(0) = 0 \text{ and } (-i\omega_0\tau_0 I - \int_{-1}^0 e^{-i\omega_0\tau_0\theta} d\eta(\theta))q(0) = 0,$$

we obtain

$$(2i\omega_0\tau_0 I - \int_{-1}^0 e^{2i\omega_0\tau_0\theta} d\eta(\theta))E = 2\tau_0(d_1, 0, d_2)^T,$$

which indicates that

$$\begin{pmatrix} 2i\omega_0 - \frac{c_1 x^* y^*}{(x^* + k_1)^2} & bx^* & \frac{c_1 x^*}{x^* + k_1} \\ -\sigma & 2i\omega_0 + \sigma & 0 \\ -\frac{a_2^2 e^{-2i\omega_0\tau_0}}{c_2} & 0 & 2i\omega_0 + a_2 e^{-2i\omega_0\tau_0} \end{pmatrix} E = 2(d_1, 0, d_2)^T,$$

it follows that

$$E = 2 \begin{pmatrix} 2i\omega_0 - \frac{c_1 x^* y^*}{(x^* + k_1)^2} & bx^* & \frac{c_1 x^*}{x^* + k_1} \\ -\sigma & 2i\omega_0 + \sigma & 0 \\ -\frac{a_2^2 e^{-2i\omega_0\tau_0}}{c_2} & 0 & 2i\omega_0 + a_2 e^{-2i\omega_0\tau_0} \end{pmatrix}^{-1} \begin{pmatrix} d_1 \\ 0 \\ d_2 \end{pmatrix}.$$

Similarly, Substituting (3.16) and (3.20) into (3.18), we have

$$-\int_{-1}^0 d\eta(\theta)F = \tau_0(d_3, 0, d_4)^T,$$

leads to

$$\begin{pmatrix} -\frac{c_1 x^* y^*}{(x^* + k_1)^2} & bx^* & \frac{c_1 x^*}{x^* + k_1} \\ -\sigma & \sigma & 0 \\ -\frac{a_2^2}{c_2} & 0 & a_2 \end{pmatrix} F = \begin{pmatrix} d_3 \\ 0 \\ d_4 \end{pmatrix},$$

it follows that

$$F = \begin{pmatrix} -\frac{c_1 x^* y^*}{(x^* + k_1)^2} & bx^* & \frac{c_1 x^*}{x^* + k_1} \\ -\sigma & \sigma & 0 \\ -\frac{a_2^2}{c_2} & 0 & a_2 \end{pmatrix}^{-1} \begin{pmatrix} d_3 \\ 0 \\ d_4 \end{pmatrix}.$$

Thus, we can determine $W_{20}(\theta), W_{11}(\theta)$, Therefore, each g_{ij} in (3.9) is determined by the parameters of system (2.3). Thus, we can evaluate the following values

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Rec_1(0)}{Re\lambda'(\tau_0)}, \\ \tau_2 &= -\frac{Imc_1(0) + \mu_2 Im\lambda'(\tau_0)}{\tau_0\omega_0}, \\ \beta_2 &= 2Rec_1(0), \end{aligned}$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical value τ_0 ; such that μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical(subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0$ ($\tau < \tau_0$); β_2 determines the stability of the bifurcating periodic solution: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solution is stable(unstable); and τ_2 determines the period of the bifurcating periodic solution: if $\tau_2 > 0$ ($\tau_2 < 0$), the period increases(decreases).

4. Numerical simulations

In this section, we present some numerical simulation to verify the main results by using MATLAB programming. We simulate the system (2.3) by choosing the parameters $a_1 = 1.5, a_2 = 0.5, b = 10, c_1 = 1.5, c_2 = 1.5, k_1 = 10, k_2 = 10, \sigma = 2$, some of the parameters are taken from the published theoretical results (see [4, 28]), then the system (2.3) has boundary equilibria $E_1(0, 0, 0), E_2(0.15, 0.15, 0), E_3(0, 0, 3.3333)$ and one positive equilibrium $E(0.1000, 0.1000, 3.3667)$, since $10 = \frac{k_1 a_1}{c_1} > \frac{k_2 a_2}{c_2} = \frac{10}{3}$, then E_1, E_2, E_3 are unstable (see Fig1, Fig2, Fig3). By algorithms in the previous sections, we obtain $\omega_0 = 0.5024, \tau_0 = 3.1235, c_1(0) = -281.9900 - 258.2700i, \mu_2 = 1294.1000, \tau_2 = 236.0245, \beta_2 = -563.9860$, the hypothesis of (H1) – (H3) hold. Thus, E is stable when $\tau \in [0, \tau_0)$, as depicted in Fig.4 and Fig.5. When τ pass through the critical value τ_0 , E loss its stability and Hopf bifurcation occurs at $\tau = \tau_0 = 3.1235$, as depicted in Fig.6 and Fig.7. From Fig.8 and Fig.9, it is evident that the system (2.3) undergoes a Hopf bifurcation at E when $\tau = \tau_0 = 3.1235$. Since $\mu_2 > 0, \beta_2 < 0, \tau_2 > 0$ the Hopf bifurcation is supercritical and the direction of bifurcation is $\tau > \tau_0$ and the bifurcating periodic solution is stable and the period increases.

5. Conclusion

In this paper, a modified Leslie-Gower predator-prey system with the Holling-II functional response and discrete and distributed delays is investigated. By choosing the discrete delay τ as the bifurcation parameter and analyzing the corresponding characteristic equation, the sufficient conditions for the local stability of the positive equilibrium and the existence of Hopf bifurcation are obtained. By using the normal form method and center manifold theorem, the explicit formulas which determine the direction, stability, and other properties of bifurcating periodic solutions are derived. Finally, numerical simulations are given to verify the theoretical analysis.

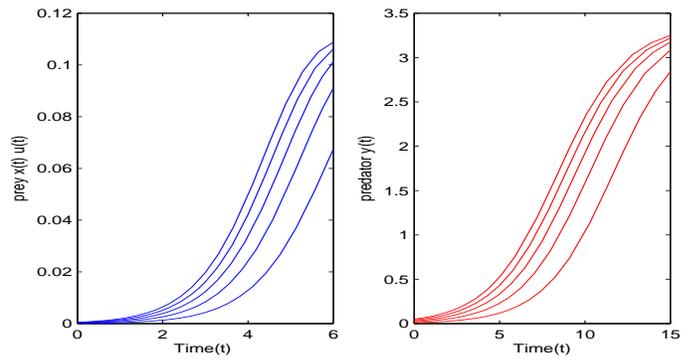


Figure 1. The trajectory of prey and predator density versus time with different initial conditions near E_1 . E_1 is unstable.

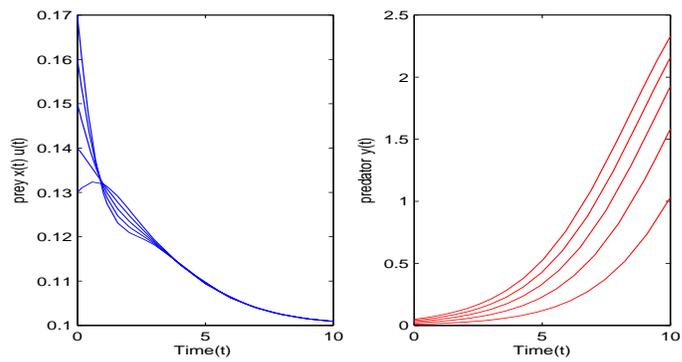


Figure 2. The trajectory of prey and predator density versus time with different initial conditions near E_2 . E_2 is unstable.

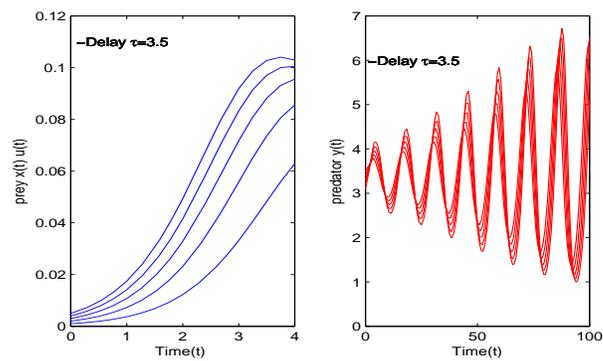


Figure 3. The trajectory of prey and predator density versus time with different initial conditions near E_3 . E_3 is unstable.

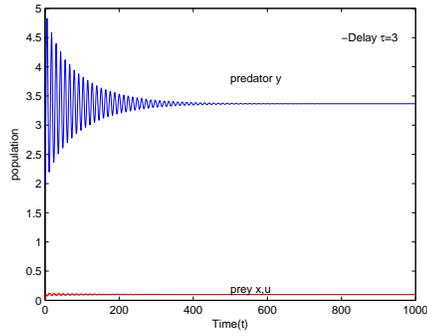


Figure 4. The trajectory of prey and predator density versus time with the initial condition $(x_0, u_0, y_0) = (0.01, 0.01, 2)$. When $\tau = 3 < \tau_0$, the equilibrium point E is asymptotically stable.

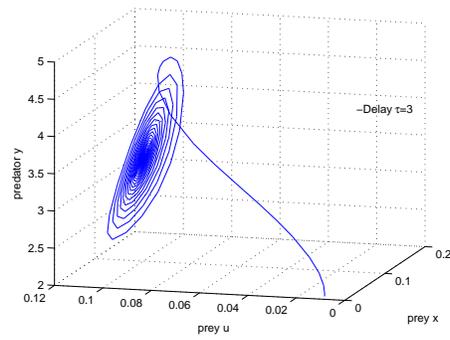


Figure 5. The phase portrait of predator density versus prey density with the initial condition $(x_0, u_0, y_0) = (0.01, 0.01, 2)$. When $\tau = 3 < \tau_0$, the equilibrium point E is asymptotically stable.

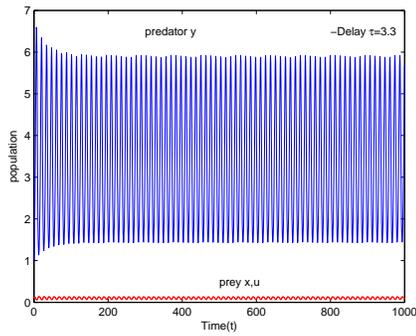


Figure 6. The trajectory of prey and predator density versus time with the initial condition $(x_0, u_0, y_0) = (0.1, 0.1, 1)$. When $\tau = 3.3 > \tau_0$, the system(2.3) is oscillatory.

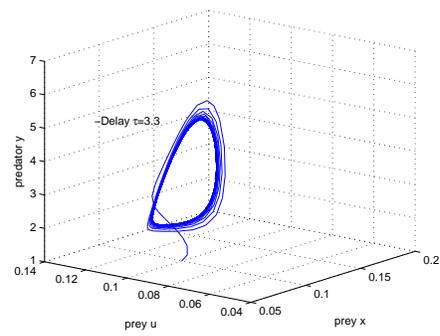


Figure 7. The phase portrait of predator density versus prey density with the initial condition $(x_0, u_0, y_0) = (0.1, 0.1, 1)$. When $\tau = 3.3 > \tau_0$, the system (2.3) show the bifurcating periodic solution from E .

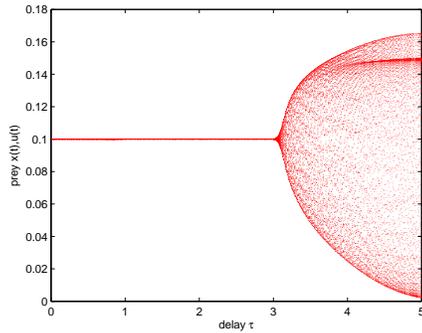


Figure 8. The relationship between ultimate oscillation range of prey x, u and delay τ .

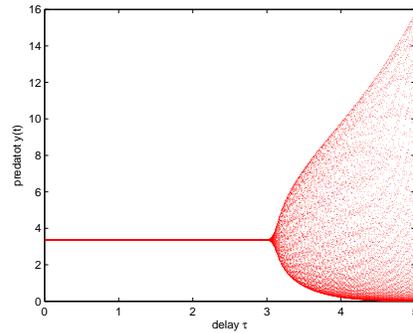


Figure 9. The relationship between ultimate oscillation range of predator y and delay τ .

From these waveforms and the phase trajectories above, it is shown that these results are in accord with the theoretical analysis.

From the research results of this paper, for the modified Leslie-Gower predator-prey system with discrete and distributed delays, we can predict the dynamical behavior of system by choosing parameter τ . If time delays are small enough, the interior equilibrium of system will keep stable in the long run, which indicates the state of ecological balance; if time delays are large enough, system becomes unstable, which suggests that the densities of the predator and prey population will periodic oscillation in a range and predator and prey population may become extinct, as depicted in Fig.6. Therefore, when the predator survival is threatened, his evolution may tend to shorten the hunting delay τ , such as predator run faster to keep the species survival.

In this paper, Holling type-II functional response is considered, if Holling type-II functional response is changed into Holling type-III, IV functional response, what will the dynamical behavior of system is? This is very valuable from the perspective of biological diversity, and we leave it as the future work.

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