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Cocycle Perturbation on Banach Algebras

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Abstract: Let α be a flow on a Banach algebra $\mathfrak{B}$, and $t \mapsto u_t$ a continuous function from $\mathbf{R}$ into the group of invertible elements of $\mathfrak{B}$ such that $u_s\alpha_s(u_t) = u_{s+t}$, $s, t \in \mathbf{R}$. Then $\beta_t = \mathrm{Ad}u_t \circ \alpha_t$, $t \in \mathbf{R}$ is also a flow on $\mathfrak{B}$, where $\mathrm{Ad}u_t(B) \triangleq u_t B u_t^{-1}$ for any $B \in \mathfrak{B}$. β is said to be a cocycle perturbation of α. We show that if α, β are two flows on a nest algebra (or quasi-triangular algebra), then β is a cocycle perturbation of α. And the flows on a nest algebra (or quasi-triangular algebra) are all uniformly continuous.

Key words: cocycle perturbation, inner perturbation, nest algebra, quasi-triangular algebra

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1 Introduction

In the quantum mechanics of particle systems with an infinite number of degrees of freedom, an important problem is to study the differential equation

$$\frac{\mathrm{d}\alpha_t(A)}{\mathrm{d}t} = S\alpha_t(A)$$

under variety of circumstances and assumptions. In each instance the A corresponds to an observable, or state, of the system and is represented by an element of some suitable Banach algebra $\mathfrak{B}$. S is an operator on $\mathfrak{B}$, and $\{\alpha_t\}_{t\in\mathbf{R}}$ is a group of bounded automorphisms on $\mathfrak{B}$. The function

$$t \in \mathbf{R} \mapsto \alpha_t(A) \in \mathfrak{B}$$

describes the motion of A. The dynamics are given by solutions of the differential equation subject to certain supplementary conditions of continuity. Thus it is worth to study the group of bounded automorphisms on $\mathfrak{B}$. For more details see [1].

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A flow α on $\mathfrak{B}$ is a group homomorphism of the real line $\mathbf{R}$ into the group of bounded automorphisms on $\mathfrak{B}$ (i.e., $t \mapsto \alpha_t$) such that

$$\lim_{t\to t_0} \|\alpha_t(B) - \alpha_{t_0}(B)\| = 0, \qquad t_0 \in \mathbf{R},\ B \in \mathfrak{B}.$$

If there exists an $h \in \mathfrak{B}$ such that

$$\alpha_t(B) = \mathrm{e}^{th} B \mathrm{e}^{-th}, \qquad B \in \mathfrak{B},\ t \in \mathbf{R},$$

then we call α an inner flow. We say that a flow α is uniformly continuous if

$$\lim_{t\to t_0} \|\alpha_t - \alpha_{t_0}\| = 0, \qquad t_0 \in \mathbf{R}.$$

If α is a flow on $\mathfrak{B}$ and if u is a continuous map of $\mathbf{R}$ into the group of invertible elements $G(\mathfrak{B})$ of $\mathfrak{B}$ such that

$$u_s \alpha_s(u_t) = u_{s+t}, \qquad s, t \in \mathbf{R},$$

then we call $u = (u_t)_{t\in\mathbf{R}}$ an α-cocycle for $(\mathfrak{B},\ \mathbf{R},\ \alpha)$. Let

$$\beta_t = \mathrm{Ad}u_t \circ \alpha_t, \qquad t \in \mathbf{R},$$

where

$$\mathrm{Ad}u_t(B) \triangleq u_t B u_t^{-1},$$

i.e.,

$$\beta_t(B) = u_t \alpha_t(B) u_t^{-1}, \qquad B \in \mathfrak{B}.$$

Then β is also a flow on $\mathfrak{B}$, and is said to be a cocycle perturbation of α.

If α is a flow on $\mathfrak{B}$, let $D(\delta_\alpha)$ be composed of those $B \in \mathfrak{B}$ for which there exists an $A \in \mathfrak{B}$ with the property that

$$A = \lim_{t\to 0} \frac{\alpha_t(B) - B}{t}.$$

Then δ_α is a linear operator on $D(\delta_\alpha)$ defined by

$$\delta_\alpha(B) = A.$$

We call δ_α the infinitesimal generator of α. By Proposition 3.1.6 of [1], δ_α is a closed derivation, i.e., the domain $D(\delta_\alpha)$ is a dense subalgebra of $\mathfrak{B}$ and δ_α is closed as a linear operator on $D(\delta_\alpha)$ and satisfies

$$\delta_\alpha(AB) = \delta_\alpha(A)B + A\delta_\alpha(B), \qquad A, B \in D(\delta_\alpha).$$

We call β an inner perturbation of α if α, β are two flows on $\mathfrak{B}$,

$$D(\delta_\alpha) = D(\delta_\beta),$$

and there exists an $h \in \mathfrak{B}$ such that

$$\delta_\beta = \delta_\alpha + \mathrm{adi}h,$$

where i is the imaginary unit, and

$$\mathrm{adi}h(B) \triangleq \mathrm{i}(hB - Bh), \qquad B \in \mathfrak{B}.$$

Moreover,

$$D(\delta_\alpha) = \mathfrak{B}$$

if and only if α is uniformly continuous. For more details see [1–2].

The problem we consider here is classifying cocycle of flows on Banach algebras. Such a problem has been considered in the C^*-algebra cases, notably by Kishimoto[3–10]. We refer the reader to [3] for a detailed study of the general results concerning cocycles and invariants for cocycle perturbation of flows on C^*-algebras. In particular, it is shown that for a flow α on a C^*-algebra $\mathfrak{A}$, if $u = (u_t)_{t\in\mathbf{R}}$ is an α-cocycle for $(\mathfrak{A},\ \mathbf{R},\ \alpha)$, and u is differentiable, i.e., $\lim\limits_{t\to t_0} \dfrac{u_t - u_{t_0}}{t - t_0}$ exists for any $t_0 \in \mathbf{R}$, and

$$h = -\mathrm{i}\frac{\mathrm{d}u_t}{\mathrm{d}t}\Big|_{t=0} \in \mathfrak{A},$$

then the infinitesimal generator of the flow

$$\beta_t = \mathrm{Ad}u_t \circ \alpha_t$$

is given by

$$\delta_\beta = \delta_\alpha + \mathrm{adi}h,$$

i.e., β is an inner perturbation of α. Moreover, for any α-cocycle $u = (u_t)_{t\in\mathbf{R}}$, there is a $w \in G(\mathfrak{A})$ and a differentiable α-cocycle $v = (v_t)_{t\in\mathbf{R}}$, i.e., v_t is an α-cocycle and differentiable such that

$$u_t = wv_t\alpha_t(w^{-1}).$$

In Section 2, we consider the cocycle of flows on Banach algebras and obtain some similar results to [3].

It is well-known that a flow α on $\mathfrak{B}$ may not be uniformly continuous even if $\mathfrak{B}$ is a C^*-algebra (see [1–2]). In Section 3, we study the flows on a nest algebra $\tau(\mathcal{N})$ and the quasi-triangular algebra

$$Q\tau(\mathcal{N}) = \tau(\mathcal{N}) + K$$

(see [11]). We recall that a nest $\mathcal{N}$ is a chain of closed subspaces of a Hilbert space $\mathfrak{H}$ containing $\{0\}$ and $\mathfrak{H}$ which is, in addition, closed under taking arbitrary intersections and closed spans. The nest algebra $\mathcal{T}(\mathcal{N})$ associated with $\mathcal{N}$ is the set of all $T \in B(\mathfrak{H})$ which leave each element of the nest invariant. For instance, if $\mathfrak{H}$ is separable with orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $\mathfrak{H}_n = \mathrm{span}\{e_1, \cdots, e_n\}$, then

$$\mathcal{N} = \{0, \mathfrak{H}\} \cup \{\mathfrak{H}_n\}_{n=1}^{\infty}$$

is a nest. In this case, $\mathcal{T}(\mathcal{N})$ is simply the set of all operators whose matrix representation with respect to this basis is upper triangular. It is obvious that $\tau(\mathcal{N})$ and $Q\tau(\mathcal{N})$ are typical Banach algebras. We obtain that all of the flows on $\tau(\mathcal{N})$ (or $Q\tau(\mathcal{N})$) are uniformly continuous. Moreover, all of the flows are cocycle perturbation to each other.

2 Cocycle Perturbations

Let $\mathfrak{B}$ be a Banach algebra, α be a flow on $\mathfrak{B}$ and u be an α-cocycle. Then

$$\beta_t = \mathrm{Ad}u_t \circ \alpha_t, \qquad t \in \mathbf{R}$$

is a cocycle perturbation of α. In this section, we obtain that β is an inner perturbation of α if and only if u is differentiable (see Theorem 2.1). Moreover, for any α-cocycle u, there is a differentiable α-cocycle v and an inventible element w in $\mathfrak{B}$ such that

$$u_t = wv_t\alpha_t(w^{-1})$$

(see Theorem 2.2).

The following lemmas are useful for this paper.

Lemma 2.1 ([1], Proposition 3.1.3) *Let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a flow on the Banach algebra $\mathfrak{B}$. Then there exist $M \geq 1$ and $\xi \geq \inf\limits_{t\neq 0}(t^{-1}\log\|\alpha_t\|)$ such that $\|\alpha_t\| \leq M\mathrm{e}^{\xi|t|}$.*

Lemma 2.2 ([1], Proposition 3.1.33) *Let α be a flow on a Banach algebra $\mathfrak{B}$ with infinitesimal generator δ_α. For each $P \in \mathfrak{B}$ define the bounded derivation δ_P by*

$$D(\delta_P) = \mathfrak{B}$$

and

$$\delta_P(B) = \mathrm{i}[P,\ B] \triangleq \mathrm{i}(PB - BP), \qquad B \in \mathfrak{B}.$$

Then $\delta + \delta_P$ generates a flow on $\mathfrak{B}$ given by

$$\alpha_t^P(B) = \alpha_t(B) + \sum_{n\geq 1}\mathrm{i}^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n[\alpha_{t_n}(P),\ [\cdots[\alpha_{t_1}(P),\ \alpha_t(B)]]],$$

$$B \in \mathfrak{B},\ t \in \mathbf{R}.$$

Lemma 2.3 *Let $\mathfrak{B}$ be a Banach algebra with unit 1, α be a flow on $\mathfrak{B}$ and δ denote the infinitesimal generator of α. Furthermore, for each $P \in \mathfrak{B}$, define δ_P as in Lemma 2.2. Then $\delta + \delta_P$ generates a flow on $\mathfrak{B}$ given by*

$$\alpha_t^P(B) = \alpha_t(B) + \sum_{n\geq 1}\mathrm{i}^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n[\alpha_{t_n}(P),\ [\cdots[\alpha_{t_1}(P),\ \alpha_t(B)]]].$$

Moreover,

$$\alpha_t^P(B) = u_t^P\alpha_t(B)(u_t^P)^{-1},$$

where u_t^P is a one-parameter family of invertible elements, determined by

$$u_t^P = 1 + \sum_{n\geq 1}\mathrm{i}^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\alpha_{t_n}(P)\cdots\alpha_{t_1}(P),$$

which satisfies the α-cocycle relation

$$u_{t+s}^P = u_t^P\alpha_t(u_s^P).$$

All integrals converge in the strong topology. The integrals define norm-convergent series of bounded operators and there exist $M \geq 1$ and $\xi \geq \inf\limits_{t\neq 0}(t^{-1}\log\|\alpha_t\|)$ such that

$$\|\alpha_t^P(B) - \alpha_t(B)\| \leq M\mathrm{e}^{\xi|t|}(\mathrm{e}^{M|t|\|P\|} - 1), \qquad \|u_t^P - 1\| \leq M\mathrm{e}^{\xi|t|}(\mathrm{e}^{M|t|\|P\|} - 1).$$

Proof. The first statement of the lemma can be obtained from Lemma 2.2. We just give the proof of the last statement of this lemma.

We consider u_t^P defined by the series.

By Lemma 2.1, there exist $M \geq 1$ and $\xi \geq \inf\limits_{t\neq 0}(t^{-1}\log\|\alpha_t\|)$ such that

$$\|\alpha_t\| \leq M\mathrm{e}^{\xi|t|}.$$

Let

$$M_t = \begin{cases} M\mathrm{e}^{\xi|t|}, & |t| > 1; \\ M, & |t| \leq 1. \end{cases}$$

Then the n-th term in this series is well defined and has norm less than $\dfrac{|t|^n}{n!}M_t^n\|P\|^n$, and u_t^P is a norm-continuous one-parameter family of elements of $\mathfrak{A}$ with $u_0^P = 1$ and $\|u_t^P\| \leq \mathrm{e}^{|t|M_t\|P\|}$. Consequently, u_t^P is invertible for all $t \in [-t_0,\ t_0]$ where $t_0 > 0$, and $(u_t^P)^{-1}$ is a norm-continuous one-parameter family of elements of $\mathfrak{B}$ for all $t \in [-t_0,\ t_0]$. And one has

$$\frac{\mathrm{d}u_t^P}{\mathrm{d}t} = \mathrm{i}u_t^P\alpha_t(P),$$

and

$$\lim_{t\to 0}\frac{u_t^P - 1}{t} = \frac{\mathrm{d}u_t^P}{\mathrm{d}t}\Big|_{t=0} = \mathrm{i}P.$$

Hence,

$$\lim_{t\to 0}\frac{(u_t^P)^{-1} - 1}{t} = \lim_{t\to 0}(u_t^P)^{-1}\frac{1 - u_t^P}{t} = -\mathrm{i}P.$$

To establish the α-cocycle relation, we first note that

$$\frac{\mathrm{d}u_{t+s}^P}{\mathrm{d}s} = \mathrm{i}u_{t+s}^P\alpha_{t+s}(P)$$

and

$$u_{t+s}^P|_{s=0} = u_t^P.$$

Since

$$\alpha_t(u_s^P) = u_s^{\alpha_t(P)},$$

we can get

$$\frac{\mathrm{d}}{\mathrm{d}s}u_t^P\alpha_t(u_s^P) = \mathrm{i}u_t^P u_s^{\alpha_t(P)}\alpha_s(\alpha_t(P)) = \mathrm{i}u_t^P\alpha_t(u_s^P)\alpha_{t+s}(P).$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}s}u_t^P\alpha_t(u_s^P)|_{s=0} = \mathrm{i}u_t^P\alpha_t(p).$$

Thus $s \mapsto u_{t+s}^P$ and $s \mapsto u_t^P\alpha_t(u_s^P)$ satisfy the same first-order differential equation and the boundary condition for each $t \in \mathbf{R}$. Therefore, the two functions are equal and can be obtained by iteration of the integral equation

$$X_t(s) = u_t^P + \mathrm{i}\int_0^s \mathrm{d}s' X_t(s')\alpha_{t+s'}(P).$$

Hence,

$$u_{t+s}^P = u_t^P\alpha_t(u_s^P), \qquad t, s \in \mathbf{R}.$$

Since u_t^P is invertible for all $t \in [-t_0,\ t_0]$ with $t_0 > 0$, we know that u_t^P is a norm-continuous one-parameter family of inventible elements. Thus $t \mapsto u_t^P\alpha_t(B)(u_t^P)^{-1}$ defines a flow β_t on $\mathfrak{B}$.

Let $\tilde{\delta}$ denote the infinitesimal generator of β. We prove

$$\tilde{\delta} = \delta + \delta_P.$$

Choosing $A \in D(\delta + \delta_P)$, one has

$$\begin{aligned}
\delta(A) =& \lim_{t\to 0}\frac{\alpha_t(A) - A}{t},\\
\tilde{\delta}(A) =& \lim_{t\to 0}\frac{\beta_t(A) - A}{t}\\
&= \lim_{t\to 0}\Big(\frac{u_t^P\alpha_t(A)(u_t^P)^{-1} - u_t^PA(u_t^P)^{-1}}{t} + \frac{u_t^PA(u_t^P)^{-1} - u_t^PA}{t} + \frac{u_t^PA - A}{t}\Big)\\
&= (\delta + \delta_P)(A).
\end{aligned}$$

Similarly, if $A \in D(\tilde{\delta})$, we obtain that

$$\tilde{\delta}(A) = (\delta + \delta_P)(A),$$

and then

$$\tilde{\delta} = \delta + \delta_P.$$

Thus, by Theorem 3.1.26 in [1], one must have

$$\alpha_t^P(B) = \beta_t(B) = u_t^P \alpha_t(B)(u_t^P)^{-1}, \qquad B \in \mathfrak{B}.$$

Finally, the estimates on $\alpha_t^P(B) - \alpha_t(B)$ and $u_t^P - 1$ are straightforward.

Theorem 2.1 *Let α be a flow on $\mathfrak{B}$, $(u_t)_{t\in\boldsymbol{R}}$ be an α-cocycle, and $\beta_t = \mathrm{Ad}u_t \circ \alpha_t$. Then β is an inner perturbation of α if and only if u_t is differentiable.*

Proof. Sufficiency. It follows immediately from Lemma 2.3.

Necessity. If u_t is an α-cocycle and differentiable with $h = -\mathrm{i}\dfrac{\mathrm{d}u_t}{\mathrm{d}t}\Big|_{t=0} \in \mathfrak{B}$, then u_t is given by

$$u_t = 1 + \sum_{n\geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \alpha_{t_n}(P)\cdots\alpha_{t_1}(P).$$

Hence, β is an inner perturbation of α by Lemma 2.3.

Corollary 2.1 *Adopt the assumptions of Lemma 2.3 and also assume that α_t is an inner flow, i.e., there exists an $h \in \mathfrak{B}$ such that*

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}th} A \mathrm{e}^{-\mathrm{i}th}, \qquad A \in \mathfrak{B},\ t \in \mathbf{R}.$$

Then

$$\alpha_t^P(A) = \varGamma_t^P A (\varGamma_t^P)^{-1}, \qquad u_t^P = \varGamma_t^P \mathrm{e}^{-\mathrm{i}th},$$

where u_t^P is defined as in Lemma 2.3, and

$$\varGamma_t^P = \mathrm{e}^{\mathrm{i}t(h+P)},$$

i.e., α_t^P is an inner flow.

Proof. If

$$\varGamma_t^P = \mathrm{e}^{\mathrm{i}t(h+P)}, \qquad X_t = \varGamma_t^P \mathrm{e}^{-\mathrm{i}th},$$

then

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mathrm{i}\varGamma_t^P P \mathrm{e}^{-\mathrm{i}th} = \mathrm{i}X_t\alpha_t(P)$$

and $X_0 = 1$. Thus, X_t is the unique solution of the integral equation

$$X_t = 1 + \mathrm{i}\int_0^t \mathrm{d}s X_s \alpha_s(P).$$

This solution can be obtained by iteration and one finds $X_t = u_t^P$, where u_t^P is defined as in Lemma 2.3, and

$$\alpha_t^P(A) = u_t^P \alpha_t(A)(u_t^P)^{-1} = \varGamma_t^P A (\varGamma_t^P)^{-1}.$$

The proof is completed.

In the following we show that every α-cocycle is similar to a differentiable α-cocycle.

Definition 2.1 *Let α be a flow on $\mathfrak{B}$. $A \in \mathfrak{B}$ is called an analytic element for α if there exists an analytic function $f:\ \mathbf{C} \to \mathfrak{B}$ such that*

$$f(t) = \alpha_t(A), \qquad t \in \mathbf{R}.$$

Lemma 2.4 *Let α be a flow on the Banach algebra $\mathfrak{B}$, and M, ξ be constants such that*

$$\|\alpha_t\| \le M\mathrm{e}^{\xi|t|}.$$

For $A \in \mathfrak{B}$, define

$$A_n = \sqrt{\frac{n}{\pi}} \int \alpha_t(A)\mathrm{e}^{-nt^2-\xi t}\mathrm{d}t, \qquad n = 1, 2, \cdots$$

Then each A_n is an entire analytic element for α_t, and there exists an N such that

$$\|A_n\| \le 2M\|A\|, \qquad n \ge N,$$

and $A_n \to A$ in the weak topology as $n \to \infty$. In particular, the α analytic elements form a normal-dense subspace of $\mathfrak{B}$.

Proof. Since

$$t \mapsto \mathrm{e}^{-n(t-z)^2} \in L^1(\mathbf{R}), \qquad z \in \mathbf{C},$$

we know that

$$f_n(z) = \sqrt{\frac{n}{\pi}} \int \alpha_t(A)\mathrm{e}^{-n(t-z)^2-\xi(t-z)}\mathrm{d}t$$

is well defined for all $z \in \mathbf{C}$. For $z = s \in \mathbf{R}$, we have

$$\begin{aligned} f_n(s) &= \sqrt{\frac{n}{\pi}} \int \alpha_t(A)\mathrm{e}^{-n(t-s)^2-\xi(t-s)}\mathrm{d}t \\ &= \sqrt{\frac{n}{\pi}} \int \alpha_{t+s}(A)\mathrm{e}^{-nt^2-\xi t}\mathrm{d}t \\ &= \alpha_s(A_n). \end{aligned}$$

But for $\eta \in \mathfrak{B}^*$ we have

$$\eta(f_n(z)) = \sqrt{\frac{n}{\pi}} \int \eta(\alpha_t(A))\mathrm{e}^{-n(t-z)^2-\xi(t-z)}\mathrm{d}t.$$

Since

$$|\eta(\alpha_t(A))| \le M\|\eta\|\|A\|\mathrm{e}^{\xi|t|},$$

it follows from the Lebesgue dominated convergence theorem that $t \mapsto \eta(f_n(z))$ is analytic. Hence each A_n is analytic for $\alpha_t(A)$. Furthermore, we can derive the estimate

$$\|A_n\| \le M\|A\|\sqrt{\frac{n}{\pi}} \int \mathrm{e}^{-n(t)^2-\xi t+\xi|t|}\mathrm{d}t \le M\|A\|(1+\mathrm{e}^{\frac{\xi^2}{4n}}).$$

Hence, there exists an N such that

$$\|A_n\| \le 2M\|A\|, \qquad n \ge N.$$

Noting that

$$\int \mathrm{e}^{-nt^2-\xi t}\mathrm{d}t = \mathrm{e}^{\frac{\xi^2}{4n}}\sqrt{\frac{\pi}{n}},$$

one has

$$\eta(A_n - A) = \sqrt{\frac{n}{\pi}} \int \mathrm{e}^{-nt^2-\xi t}\eta(\alpha_t(A) - \mathrm{e}^{-\frac{\xi^2}{4n}}A)\mathrm{d}t, \qquad \eta \in \mathfrak{B}^*.$$

For any $\varepsilon > 0$ we may choose a $\delta > 0$ such that $|t| < \delta$ implies

$$|\eta(\alpha_t(A) - A)| < \varepsilon.$$

Furthermore, we can choose an N large enough so that $\dfrac{\xi}{2N} < \dfrac{\delta}{2}$. It follows that if $n > N$, then

$$\begin{aligned}|\eta(A_n - A)| \leq & \sqrt{\frac{n}{\pi}} \int_{|t|<\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(\alpha_t(A) - \mathrm{e}^{-\frac{\xi^2}{4n}} A)| \mathrm{d}t \\ & + \sqrt{\frac{n}{\pi}} \int_{|t|\geq\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(\alpha_t(A) - \mathrm{e}^{-\frac{\xi^2}{4n}} A)| \mathrm{d}t.\end{aligned}$$

On the other hand,

$$\begin{aligned}& \sqrt{\frac{n}{\pi}} \int_{|t|<\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(\alpha_t(A) - \mathrm{e}^{-\frac{\xi^2}{4n}} A)| \mathrm{d}t \\ \leq & \sqrt{\frac{n}{\pi}} \int_{|t|<\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(\alpha_t(A) - A)| \mathrm{d}t + \sqrt{\frac{n}{\pi}} \int_{|t|<\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(A)(1 - \mathrm{e}^{-\frac{\xi^2}{4n}})| \mathrm{d}t \\ < & \varepsilon \mathrm{e}^{\frac{\xi^2}{4n}} + |1 - \mathrm{e}^{-\frac{\xi^2}{4n}}| \cdot \|\eta\| \cdot \|A\| M \mathrm{e}^{\frac{\xi^2}{4n}},\end{aligned}$$

and

$$\begin{aligned}& \sqrt{\frac{n}{\pi}} \int_{|t|\geq\delta} \mathrm{e}^{-nt^2-\xi t} |\eta(\alpha_t(A) - \mathrm{e}^{-\frac{\xi^2}{4n}} A)| \mathrm{d}t \\ \leq & \sqrt{\frac{n}{\pi}} \|\eta\| \cdot \|A\| M \int_{|t|\geq\delta} \mathrm{e}^{-nt^2} \mathrm{d}t + \sqrt{\frac{n}{\pi}} \mathrm{e}^{-\frac{\xi^2}{4n}} \|\eta\| \cdot \|A\| \int_{|t|\geq\delta} \mathrm{e}^{-nt^2-\xi t} \mathrm{d}t.\end{aligned}$$

So, for any $\eta \in \mathfrak{B}^*$,

$$|\eta(A_n - A)| \to 0 \qquad \text{as } n \to \infty.$$

Finally, note that the norm closure and the weak closure of a convex set are the same, so the α analytic elements form a normal-dense subspace of $\mathfrak{B}$.

Theorem 2.2 *If u is an α-cocycle for $\mathfrak{B}$, then for a given $\varepsilon > 0$ there exist a differentiable α-cocycle v and a $w \in G(\mathfrak{B})$ such that*

$$\|w - 1\| < \varepsilon, \qquad u_t = w v_t \alpha_t(w^{-1}).$$

Proof. The proof is similar to that of Lemma 1.1 in [3] and is omitted.

For given two flows α and β on a unitary Banach algebra $\mathfrak{B}$, we say that β is a conjugate to α if there exists a bounded automorphism σ of $\mathfrak{B}$ such that

$$\beta = \sigma\alpha\sigma^{-1}.$$

Conjugate, cocycle perturbation and inner perturbation define three equivalence relations. We say that β is cocycle-conjugate to α if there exists a bounded automorphism σ of $\mathfrak{B}$ such that β is a cocycle perturbation of $\sigma\alpha\sigma^{-1}$. This also defines an equivalence relation among the flows. We say that α is approximately inner if there is a sequence $\{h_n\}$ in $\mathfrak{B}$ such that

$$\alpha_t = \lim \mathrm{Ad}\mathrm{e}^{th_n},$$

i.e.,

$$\alpha_t(A) = \lim \mathrm{Ad}\mathrm{e}^{th_n}(A), \qquad t \in \mathbf{R},\ A \in \mathfrak{B},$$

or equivalently, uniformly continuous in t on every compact subset of $\mathbf{R}$ and every $A \in \mathfrak{B}$. A flow on a Banach algebra $\mathfrak{B}$ is said to be asymptotically inner if there is a continuous function h of $\mathbf{R}_+$ into $\mathfrak{B}$ such that

$$\lim_{s\to\infty} \max_{|t|\le 1} \|\alpha_t(A) - \mathrm{Ade}^{th(s)}(A)\| = 0, \qquad A \in \mathfrak{B}.$$

Corollary 2.2 *Let α and β be two flows on Banach algebra $\mathfrak{B}$. Then the following conditions are equivalent*:

(i) *β is cocycle-conjugate to α*;

(ii) *β is an inner perturbation of $\sigma\alpha\sigma^{-1}$ for some automorphism σ of $\mathfrak{B}$, where $\sigma\alpha\sigma^{-1}$ is the action $t \mapsto \sigma\alpha_t\sigma^{-1}$.*

If one of above conditions is satisfied and α is inner (*approximately or asymptotically inner*), *then so is β.*

Proof. The first statement of the proposition can be obtained from Corollary 1.3 of [3]. We just give the proof of the last statement of the corollary.

First we prove that if β is an inner perturbation of α, i.e.,

$$\delta_\beta = \delta\alpha + \mathrm{iad}P,$$

then it follows that if α is inner (approximately or asymptotically inner), so is β.

(a) If α is inner, then so is β by Corollary 2.1.

(b) If α is approximately inner, then there is a sequence $\{h_n\}$ in $\mathfrak{B}$ such that

$$\lim_{n\to\infty} \|\alpha_t(A) - \mathrm{Ade}^{th_n}(A)\| = 0.$$

For $\alpha_{n,t} \triangleq \mathrm{Ade}^{th_n}$, we construct an α_n-cocycle $u_{n,t}$ (resp. u) for $\alpha_{n,t}$ (resp. α) such that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{n,t}|_{t=0} = \mathrm{i}P$$

by means of Lemma 2.3. Since $\beta_t = \mathrm{Ad}u_t \circ \alpha_t$ and $u_{n,t} \to u_t$, we obtain that

$$\mathrm{Ad}u_{n,t} \circ \alpha_{n,t} \to \beta_t.$$

Besides, $\mathrm{Ad}u_{n,t} \circ \alpha_{n,t}$ is inner by Corollary 2.1. Then β is approximately inner.

(c) If α is asymptotically inner, then there is a continuous function h of $\mathbf{R}_+$ into $\mathfrak{B}$ such that

$$\lim_{s\to\infty} \|\alpha_t(A) - \mathrm{Ade}^{th(s)}(A)\| = 0, \qquad A \in \mathfrak{B}.$$

For $\alpha_{s,t} \triangleq \mathrm{Ade}^{th(s)}$, we construct an α_s-cocycle $u_{s,t}$ (resp. u) for $\alpha_{s,t}$ (resp. α) such that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{s,t}|_{t=0} = \mathrm{i}P$$

by means of Lemma 2.3. Since $\beta_t = \mathrm{Ad}u_t \circ \alpha_t$ and $u_{s,t} \to u_t$, we obtain that

$$\mathrm{Ad}u_{s,t} \circ \alpha_{s,t} \to \beta_t.$$

Besides, $\mathrm{Ad}u_{s,t} \circ \alpha_{s,t}$ is inner and

$$\mathrm{Ad}u_{s,t} \circ \alpha_{s,t} = \mathrm{Ade}^{t(h(s)+P)}$$

by Corollary 2.1. Then β is approximately inner. The proof is completed.

In the following we prove that if β is conjugate to α, i.e., there is a bounded automorphism σ of $\mathfrak{B}$ such that

$$\beta = \sigma\alpha\sigma^{-1},$$

then, if α is inner (approximately or asymptotically inner), then so is β.

(a)$'$ If α is inner, i.e.,

$$\alpha_t(A) = \mathrm{e}^{th} A \mathrm{e}^{-th},$$

then

$$\beta_t(A) = \sigma(\alpha_t(\sigma(A))) = \mathrm{e}^{t\sigma(h)} A \mathrm{e}^{-t\sigma(h)},$$

i.e., β is inner.

(b)$'$ If α is approximately inner, then there exits a sequence $\{h_n\}$ in $\mathfrak{A}$ such that

$$\lim_{n\to\infty} \|\alpha_t(A) - \mathrm{Ade}^{th_n}(A)\| = 0.$$

Because β is conjugate to α, i.e.,

$$\beta = \sigma\alpha\sigma^{-1},$$

σ is the bounded automorphism of $\mathfrak{A}$, so there exists an $M > 0$ such that

$$\|\sigma\| \leq M, \qquad \|\sigma^{-1}\| \leq M.$$

Therefore,

$$\|\beta_t(A) - \mathrm{Ade}^{t\sigma(h_n)}(A)\| = \|\sigma^{-1}(\beta_t(\sigma(A))) - \sigma^{-1}(\mathrm{e}^{t\sigma(h_n)}(\sigma(A)))\|,$$

i.e., β is asymptotically inner.

(c)$'$ If α is asymptotically inner, a similar argument shows that β is asymptotically inner.

Finally, if β is cocycle-conjugate to α, then there is a bounded automorphism σ of $\mathfrak{B}$ such that β is a cocycle perturbation of $\sigma\alpha\sigma^{-1}$. If α is inner (approximately or asymptotically inner), then so is $\sigma\alpha\sigma^{-1}$. Then so is β.

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