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Cofiniteness of Local Cohomology Modules with Respect to a Pair of Ideals

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Abstract: Let R be a commutative Noetherian ring, I and J be two ideals of R, and M be an R-module. We study the cofiniteness and finiteness of the local cohomology module $H^i_{I,J}(M)$ and give some conditions for the finiteness of $\mathrm{Hom}_R(R/I,\ H^s_{I,J}(M))$ and $\mathrm{Ext}^1_R(R/I,\ H^s_{I,J}(M))$. Also, we get some results on the attached primes of $H^{\dim M}_{I,J}(M)$.

Key words: local cohomology, cofinite module, attached prime

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1 Introduction

Throughout this paper, we always assume that R is a commutative Noetherian ring, I and J are two ideals of R, and M is an R-module. Takahashi *et al.*[1] introduced the concept of local cohomology module $H^i_{I,J}(M)$ with respect to a pair of ideals (I, J). The set of elements x of M such that $I^n \subseteq \mathrm{Ann}(x) + J$ for some integer $n \gg 1$ is said to be (I, J)-torsion submodule of M and is denoted by $\varGamma_{I,J}(M)$. For an integer $i \geq 0$, the local cohomology functor $H^i_{I,J}$ with respect to (I, J) is defined to be the i-th right derived functor of $\varGamma_{I,J}$. Note that, if $J = 0$, then $H^i_{I,J}(\cdot)$ coincides with $H^i_I(\cdot)$. When M is finitely generated, we know that $H^i_{I,J}(M) = 0$ for $i > \dim M$ from Theorem 4.7 in [1].

Hartshorne[2] defined an R-module M to be I-cofinite if

$$\mathrm{Supp} M \subseteq V(I) \qquad \text{and} \qquad \mathrm{Ext}^i_R(R/I,\ M)$$

is finitely generated for all $i \geq 0$. Also, he asked the following question:

Question If M is finitely generated, when is $\mathrm{Ext}^j_R(R/I,\ H^i_I(M))$ finitely generated for

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all $i \geq 0$ and $j \geq 0$ (considering $\mathrm{Supp}(H_I^i(M)) \subseteq V(I)$, so $\mathrm{Ext}_R^j(R/I,\ H_I^i(M))$ is finitely generated if and only if $H_I^i(M)$ is I-cofinite).

Hartshorne[2] showed that if $(R, \mathfrak{m})$ is a complete regular local ring and M is finitely generated, then $H_I^i(M)$ is I-cofinite in two cases:

(a) I is a non-zero principal ideal;

(b) I is a prime ideal with $\dim R/I = 1$.

Yoshida[3], Delfino and Marley[4] extended (b) to all dimension one ideals I of any local ring R, and Kawasaki[5] proved (a) for any ring R.

Let

$$W(I, J) = \{p \in \mathrm{Spec}(R) \mid I^n \subseteq J + p \text{ for an integer } n \gg 1\}.$$

As a generalization of I-cofinite module, we give the following definition:

Definition 1.1 *An R-module M is said to be (I, J)-cofinite if* $\mathrm{Supp}M \subseteq W(I, J)$ *and* $\mathrm{Ext}_R^i(R/I,\ M)$ *is finitely generated for all $i \geq 0$.*

For an R-module M, the cohomological dimension of M with respect to I and J is defined as

$$\mathrm{cd}(I, J, M) = \sup\{i \in \mathbf{Z} \mid H_{I,J}^i(M) \neq 0\}.$$

When $J = 0$, then $\mathrm{cd}(I, J, M)$ coincides with $\mathrm{cd}(I, M)$.

In this paper, we mainly consider the (I, J)-cofiniteness of $H_{I,J}^i(M)$. Since

$$\mathrm{Supp}(H_{I,J}^i(M)) \subseteq W(I, J),$$

we focus on the finiteness of $\mathrm{Ext}_R^j(R/I,\ H_{I,J}^i(M))$.

In Section 2, we discuss the finiteness of $\mathrm{Hom}_R(R/I,\ H_{I,J}^s(M))$ (see Theorem 2.1), which generalizes Theorem 2.1 in [6] and Theorem B(β) in [7]. In addition, when M is finitely generated and I is a principal ideal or $\mathrm{cd}(I, J, M) = 1$, we get the (I, J)-cofiniteness of $H_{I,J}^i(M)$ for all $i \geq 0$, which generalizes the corresponding results in [5] and [9], respectively. In Proposition 2.3(iii) of [10], it is proved that if

$$H_I^i(M) = 0, \qquad 0 \leq i < s,$$

then

$$\mathrm{Hom}(R/I,\ H_I^s(M)) \cong \mathrm{Ext}_R^s(R/I,\ M).$$

In this paper, we get the corresponding result for the local cohomology module with respect to (I, J). In Section 3, we prove the (I, J)-cofiniteness of $H_{I,J}^{\dim M}(M)$, which is a generalization of Theorem 3 in [4].

2 The Cofiniteness of $H_{I,J}^s(M)$

First, we give a theorem which is a generalization of Theorem 2.1 in [6] and Theorem B(β) in [7]. It is also a main result of this paper.

Theorem 2.1 *Assume that M is an R-module, and s is a non-negative integer, such that $\mathrm{Ext}^s_R(R/I,\ M)$ is finitely generated. If $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i<s$, then $\mathrm{Hom}_R(R/I,\ H^s_{I,J}(M))$ is finitely generated. In particular, $\mathrm{Ass}(H^s_{I,J}(M))\cap V(I)$ is a finite set.*

Proof. We use induction on s. Let $s=0$. Then

$$\mathrm{Hom}_R(R/I, H^0_{I,J}(M))\cong \mathrm{Hom}_R(R/I, M).$$

This result is clear. Now we assume that $s>0$, and the result has been proved for smaller values of s. Since $H^0_{I,J}(M)$ is (I,J)-cofinite, $\mathrm{Ext}^i_R(R/I,\ H^0_{I,J}(M))$ is finitely generated for all $i\geq 0$. The short exact sequence

$$0\to H^0_{I,J}(M)\to M\to M/H^0_{I,J}(M)\to 0$$

induces the exact sequence

$$\mathrm{Ext}^s_R(R/I,\ M)\to \mathrm{Ext}^s_R(R/I,\ M/H^0_{I,J}(M))\to \mathrm{Ext}^{s+1}_R(R/I,\ H^0_{I,J}(M)).$$

Then we get that $\mathrm{Ext}^s_R(R/I,\ M/H^0_{I,J}(M))$ is finitely generated.

It is easy to see that

$$H^0_{I,J}(M/H^0_{I,J}(M))=0,\qquad H^i_{I,J}(M/H^0_{I,J}(M))\cong H^i_{I,J}(M),\quad i>0.$$

So, we may assume that

$$H^0_{I,J}(M)=0.$$

Then

$$H^0_I(M)=0.$$

Let E be an injective hull of M and put $N=E/M$. Then

$$H^0_{I,J}(E)=0\qquad\text{and}\qquad \mathrm{Hom}_R(R/I,E)=0.$$

From the short exact sequence

$$0\to M\to E\to N\to 0,$$

we have that

$$\mathrm{Ext}^i_R(R/I,\ N)\cong \mathrm{Ext}^{i+1}_R(R/I,\ M)$$

and

$$H^i_{I,J}(N)\cong H^{i+1}_{I,J}(M),\qquad i\geq 0.$$

Applying the inductive hypothesis to N, it yields the finiteness of $\mathrm{Hom}_R(R/I, H^{s-1}_{I,J}(N))$. Hence $\mathrm{Hom}_R(R/I,\ H^s_{I,J}(M))$ is finitely generated. It follows that $\mathrm{Ass}(H^s_{I,J}(M))\cap V(I)$ is a finite set.

Proposition 2.1 *Assume that M is an R-module, and s is a non-negative integer, such that $\mathrm{Ext}^{s+1}_R(R/I,\ M)$ is finitely generated. If $\mathrm{Ext}^{s+2-i}_R(R/I,\ H^i_{I,J}(M))$ is finitely generated for all $i<s$, then $\mathrm{Ext}^1_R(R/I,\ H^s_{I,J}(M))$ is finitely generated.*

Proof. We prove the result by induction on s. When $s=0$, from the short exact sequence

$$0\to H^0_{I,J}(M)\to M\to M/H^0_{I,J}(M)\to 0,$$

we get the exact sequence

$$0=\mathrm{Hom}_R(R/I,\ M/H^0_{I,J}(M))\to \mathrm{Ext}^1_R(R/I,\ H^0_{I,J}(M))\to \mathrm{Ext}^1_R(R/I,\ M)\to 0.$$

Then we have that $\mathrm{Ext}_R^1(R/I,\ H_{I,J}^0(M))$ is finitely generated.

Now we suppose that $s > 0$, and the claim has been proved for smaller values of s. From the exact sequence

$$\mathrm{Ext}_R^{s+1}(R/I,\ M) \to \mathrm{Ext}_R^{s+1}(R/I,\ M/H_{I,J}^0(M)) \to \mathrm{Ext}_R^{s+2}(R/I,\ H_{I,J}^0(M)),$$

we get that $\mathrm{Ext}_R^{s+1}(R/I, M/H_{I,J}^0(M))$ is finitely generated. By using the similar argument to that of Theorem 2.1 and setting $N = E(M)/M$, we get that $\mathrm{Ext}_R^s(R/I,\ N)$ is finitely generated and

$$\mathrm{Ext}_R^{s+1-i}(R/I,\ H_{I,J}^i(N)) \cong \mathrm{Ext}_R^{s+1-i}(R/I,\ H_{I,J}^{i+1}(M)), \qquad i < s-1.$$

So $\mathrm{Ext}_R^1(R/I,\ H_{I,J}^{s-1}(N))$ is finitely generated by the inductive hypothesis. Therefore, we get that $\mathrm{Ext}_R^1(R/I,\ H_{I,J}^s(M))$ is finitely generated.

Proposition 2.2 *Assume that M is finitely generated, and s is a non-negative integer, such that $H_{I,J}^i(M)$ is Artinian for all $i < s$. Then $H_{I,J}^i(M)$ is (I, J)-cofinite for all $i < s$.*

Proof. We use induction on i. When $i = 0$, since M is finitely generated, the result is clear. Now we suppose that $i > 0$, and the result has been proved for smaller values of i. By the inductive hypothesis, $H_{I,J}^j(M)$ is (I, J)-cofinite for all $j < i$. Hence, $\mathrm{Hom}_R(R/I, H_{I,J}^i(M))$ is finitely generated by Theorem 2.1, that is, $0 :_{H_{I,J}^i(M)} I$ is finitely generated, and thus

$$\lambda(0 :_{H_{I,J}^i(M)} I) < \infty.$$

Since $H_{I,J}^i(M)$ is an (I, J)-torsion, we have that

$$\lambda(H^k(a_1, a_2, \cdots, a_t,\ H_{I,J}^i(M))) < \infty,$$

where $I = (a_1, a_2, \cdots, a_t)$ for all $k \geq 0$ by Theorem 5.1 in [8]. By the same proof with Theorem 5.5 in [9], we deduce that

$$\lambda(\mathrm{Ext}_R^k(R/I, H_{I,J}^i(M)) < \infty, \qquad k \geq 0.$$

Hence $H_{I,J}^i(M)$ is (I, J)-cofinite. Therefore, we get that $H_{I,J}^i(M)$ is (I, J)-cofinite for all $i < s$.

By Theorem 2.1 and Proposition 2.2, we have the following results.

Proposition 2.3 *Assume that M is finitely generated, and s is a non-negative integer, such that $H_{I,J}^i(M)$ is Artinian for all $i < s$. Then $\mathrm{Hom}_R(R/I,\ H_{I,J}^s(M))$ is finitely generated. In particular, $\mathrm{Ass}(H_{I,J}^s(M)) \cap V(I)$ is a finite set.*

Proposition 2.4 *Assume that M is an R-module, and s is a non-negative integer, such that $\mathrm{Ext}_R^j(R/I,\ H_{I,J}^i(M))$ is finitely generated for all i and j (respectively for $i \leq s$ and all j). Then $\mathrm{Ext}_R^i(R/I,\ M)$ is finitely generated for all i (respectively for $i \leq s$).*

Proof. The case $s = 0$ is clear. Now we suppose that $s > 0$ and the case $s - 1$ is settled. Set $N = M/H_{I,J}^0(M)$. Then we have the long exact sequence

$$\cdots \to \mathrm{Ext}_R^i(R/I,\ H_{I,J}^0(M)) \to \mathrm{Ext}_R^i(R/I,\ M) \to \mathrm{Ext}_R^i(R/I,\ N) \to \cdots$$

and

$$H_{I,J}^i(N) \cong \begin{cases} 0, & i = 0; \\ H_{I,J}^i(M), & i > 0. \end{cases}$$

So we assume that $H^0_{I,J}(M) = 0$. By the same proof as in Theorem 2.1 and setting $L = E(M)/M$, we get

$$\mathrm{Ext}^i_R(R/I,\ L) \cong \mathrm{Ext}^{i+1}_R(R/I,\ M), \quad H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M), \qquad i \geq 0.$$

Furthermore, we get the finiteness of $\mathrm{Ext}^i_R(R/I,\ L)$ for all i (respectively for $i \leq s-1$) by the inductive hypothesis. So $\mathrm{Ext}^i_R(R/I,\ M)$ is finitely generated for all i (respectively for $i \leq s$). The proof is completed.

Corollary 2.1 *Assume that M is an R-module, and s is a non-negative integer, such that $H^i_{I,J}(M)$ is (I,J)-cofinite for all i (respectively for all $i \leq s$). Then $\mathrm{Ext}^i_R(R/I,\ M)$ is finitely generated for all i (respectively for all $i \leq s$).*

Proposition 2.5 *Assume that M is an R-module, s is a non-negative integer, such that $\mathrm{Ext}^i_R(R/I,\ M)$ is finitely generated for all $i \geq 0$, and $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i \neq s$. Then $H^s_{I,J}(M)$ is (I,J)-cofinite.*

Proof. We prove the result by induction on s. Let $N = M/H^0_{I,J}(M)$. Then we have that

$$H^i_{I,J}(N) \cong \begin{cases} 0, & i = 0; \\ H^i_{I,J}(M), & i > 0. \end{cases}$$

When $s = 0$, $H^i_{I,J}(N)$ is (I,J)-cofinite for all $i \geq 0$. Then $\mathrm{Ext}^i_R(R/I,\ N)$ is finitely generated for all $i \geq 0$ by Corollary 2.1. From the short exact sequence

$$0 \to H^0_{I,J}(M) \to M \to N \to 0,$$

we get the following long exact sequence:

$$\cdots \to \mathrm{Ext}^i_R(R/I,\ H^0_{I,J}(M)) \to \mathrm{Ext}^i_R(R/I,\ M) \to \mathrm{Ext}^i_R(R/I,\ N) \to \cdots,$$

which implies that $\mathrm{Ext}^i_R(R/I,\ H^0_{I,J}(M))$ is finitely generated for all $i \geq 0$, that is, $H^0_{I,J}(M)$ is (I,J)-cofinite.

Now we assume that $s > 0$ and the claim holds for $s-1$. We see that $\mathrm{Ext}^i_R(R/I,\ N)$ is finitely generated for all $i \geq 0$. By using the above long exact sequence and similar argument to that of Theorem 2.1 and letting $L = E(M)/M$, we can get

$$\mathrm{Ext}^i_R(R/I,\ L) \cong \mathrm{Ext}^{i+1}_R(R/I,\ M), \quad H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M), \qquad i \geq 0.$$

So $H^{s-1}_{I,J}(L)$ is (I,J)-cofinite by the inductive hypothesis. Then $H^s_{I,J}(M)$ is (I,J)-cofinite. The proof is completed.

Corollary 2.2 *Assume that M is a finitely generated R-module, and $\mathrm{cd}(I,J,M) = 1$. Then $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i \geq 0$.*

Proposition 2.6 *Assume that M is a finitely generated R-module, and I is a principal ideal. Then $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i \geq 0$.*

Proof. Note that $H^i_{I,J}(M) = 0$ for all $i > \mathrm{ara}(I\bar{R})$ by Proposition 4.11 in [1], where

$$\bar{R} = R/\sqrt{J + \mathrm{Ann}M}.$$

So when I is a principal ideal, $H^i_{I,J}(M) = 0$ for all $i > 1$. Since $H^0_{I,J}(M)$ is finitely generated, $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i \neq 1$. The result is clear.

Proposition 2.7 *Assume that M is an R-module. Then we have*
$$\mathrm{Hom}(R/I,\ H^1_{I,J}(M)) \cong \mathrm{Ext}^1_R(R/I,\ M/H^0_{I,J}(M)).$$

Proof. Let E be the injective hull of $M/H^0_{I,J}(M)$. Put $N = E/(M/H^0_{I,J}(M))$. Since
$$H^0_{I,J}(M/H^0_{I,J}(M)) = 0,$$
we have
$$H^0_{I,J}(E) = 0 \quad \text{and} \quad \mathrm{Hom}_R(R/I,\ E) = 0.$$
From the exact sequence
$$0 \to M/H^0_{I,J}(M) \to E \to N \to 0,$$
we have
$$\mathrm{Hom}(R/I,\ N) \cong \mathrm{Ext}^1_R(R/I,\ M/H^0_{I,J}(M)),$$
and
$$H^0_{I,J}(N) \cong H^1_{I,J}(M/H^0_{I,J}(M)).$$
Hence
$$\begin{aligned}\mathrm{Hom}(R/I, H^1_{I,J}(M)) &\cong \mathrm{Hom}(R/I,\ H^1_{I,J}(M/H^0_{I,J}(M)))\\ &\cong \mathrm{Hom}(R/I,\ H^0_{I,J}(N))\\ &\cong \mathrm{Hom}(R/I,\ N)\\ &\cong \mathrm{Ext}^1_R(R/I,\ M/H^0_{I,J}(M)).\end{aligned}$$

Next we give a proposition, which generalizes Proposition 2.3(iii) of [10].

Proposition 2.8 *Assume that M is an R-module, and s is a positive integer such that*
$$H^i_{I,J}(M) = 0, \qquad 0 \le i < s.$$
Then
$$\mathrm{Hom}(R/I,\ H^s_{I,J}(M)) \cong \mathrm{Ext}^s_R(R/I,\ M).$$

Proof. When $s = 1$, $H^0_{I,J}(M) = 0$. The result is clear, since
$$\mathrm{Hom}(R/I,\ H^0_{I,J}(M)) \cong \mathrm{Hom}_R(R/I,\ M).$$
Suppose that $s > 1$ and the claim holds for $s - 1$. Let E be an injective hull of M. Put $N = E/M$. Since $H^0_{I,J}(M) = 0$, we have $H^0_{I,J}(E) = 0$. By the exact sequence
$$0 \to M \to E \to N \to 0,$$
we have
$$\mathrm{Ext}^{i-1}_R(R/I,\ N) \cong \mathrm{Ext}^i_R(R/I,\ M), \quad H^{i-1}_{I,J}(N) \cong H^i_{I,J}(M), \qquad i > 0.$$
So
$$H^i_{I,J}(N) = 0, \qquad 0 \le i < s - 1.$$
Thus
$$\mathrm{Hom}(R/I,\ H^{s-1}_{I,J}(N)) \cong \mathrm{Ext}^{s-1}_R(R/I,\ N)$$
by the induction hypothesis. Hence
$$\mathrm{Hom}(R/I, H^s_{I,J}(M)) \cong \mathrm{Ext}^s_R(R/I, M).$$

3 The Cofiniteness of $H_{I,J}^{\dim M}(M)$

In this section, we always assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring, I and J are two ideals of R, and M is an R-module.

Assume that M is finitely generated. We know that $H_{I,J}^{\dim M}(M)$ is Artinian by Theorem 2.1 in [11]. Next, we discuss $\mathrm{Att}(H_{I,J}^{\dim M}(M))$ and the cofiniteness of $H_{I,J}^{\dim M}(M)$.

Lemma 3.1[12] *Suppose that M is a non-zero finitely generated R-module of dimension n. Then*

$$\mathrm{Att}(H_{I,J}^{n}(M)) = \{p \in \mathrm{Ass}M \mid \mathrm{cd}(I,\ R/p) = n,\ J \subseteq p\}.$$

Next, we show the cofiniteness of $H_{I,J}^{\dim M}(M)$. The following proposition is the key for this fact.

Proposition 3.1 *Assume that R is a complete ring and M is a finitely generated R-module of dimension n. Then*

$$\mathrm{Att}(H_{I,J}^{n}(M)) = \{p \in V(\mathrm{Ann}(M)) \mid \dim R/p = n,\ \sqrt{I+p} = \mathfrak{m},\ J \subseteq p\}.$$

Proof. Since $H_I^n(M)$ is Artinian, we have

$$\mathrm{Att}(H_I^n(M)) = \mathrm{Ass}D(H_I^n(M)) = \mathrm{Coass}(H_I^n(M)).$$

Hence

$$\{p \in \mathrm{Ass}M \mid \mathrm{cd}(I,\ R/p) = n\} = \{p \in V(\mathrm{Ann}(M)) \mid \dim R/p = n,\ \sqrt{I+p} = \mathfrak{m}\}$$

by Theorem A in [13] and Lemma 3 in [4], therefore

$$\begin{aligned}&\{p \in \mathrm{Ass}M \mid \mathrm{cd}(I, R/p) = n,\ J \subseteq p\}\\ =\ &\{p \in V(\mathrm{Ann}(M)) \mid \dim R/p = n, \sqrt{I+p} = \mathfrak{m}, J \subseteq p\}.\end{aligned}$$

Now the result is clear by Lemma 3.1.

Next, we give the main result of this section.

Theorem 3.1 *Assume that M is a finitely generated R-module of dimension n. Then $H_{I,J}^n(M)$ is (I,J)-cofinite. In fact, $\mathrm{Ext}_R^i(R/I,\ H_{I,J}^n(M))$ has finite length for all $i \geq 0$.*

Proof. Considering that $H_{I,J}^n(M)$ is Artinian, we know that $\mathrm{Ext}_R^i(R/I,\ H_{I,J}^n(M))$is Artinian and

$$\mathrm{Ext}_R^i(R/I,\ H_{I,J}^n(M)) \cong \mathrm{Ext}_{\hat{R}}^i(\hat{R}/I\hat{R},\ H_{I\hat{R},\ J\hat{R}}^n(\hat{M})), \qquad i \geq 0.$$

So we assume that R is complete. We know that $D(H_{I,J}^n(M))$ is finitely generated and $\mathrm{Att}(H_{I,J}^n(M))$ is a finite set. Suppose that

$$\mathrm{Att}(H_{I,J}^n(M)) = \{p_1, \cdots, p_k\}.$$

Then

$$\mathrm{Supp}D(H_{I,J}^n(M)) = V(p_1 \cap \cdots \cap p_k).$$

From Matlis Duality Theorem, we get

$$\lambda(\mathrm{Ext}_R^i(R/I,\ H_{I,J}^n(M))) < \infty \ \Leftrightarrow\ \lambda(D(\mathrm{Ext}_R^i(R/I,\ H_{I,J}^n(M)))) < \infty.$$

Since

$$D(\mathrm{Ext}^i_R(R/I,\ H^n_{I,J}(M))) \cong \mathrm{Tor}^R_i(R/I,\ D(H^n_{I,J}(M)))$$

by Theorem 11.57 in [14], it is enough for us to show that

$$\mathrm{Supp}\mathrm{Tor}^R_i(R/I,\ D(H^n_{I,J}(M))) \subseteq \{\mathfrak{m}\}.$$

In fact

$$\begin{aligned}\mathrm{Supp}\mathrm{Tor}^R_i(R/I,\ D(H^n_{I,J}(M))) &\subseteq V(I) \cap \mathrm{Supp}D(H^n_{I,J}(M))\\ &= V(I) \cap V(p_1 \cap \cdots \cap p_k)\\ &= V(I + p_1 \cap \cdots \cap p_k)\\ &= \{\mathfrak{m}\}\end{aligned}$$

by Proposition 3.1.

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