Communications
in
Mathematical
Research
30(1)(2014), 51–59

T^*-extension of Lie Supertriple Systems

Feng Jian-qiang

(*Academy of Mathematical and Computer Sciences, Hebei University, Baoding, Hebei, 071002*)

Communicated by Du Xian-kun

Abstract: In this article, we study the Lie supertriple system (LSTS) T over a field $\mathbb{K}$ admitting a nondegenerate invariant supersymmetric bilinear form (call such a T metrisable). We give the definition of T^*_ω-extension of an LSTS T, prove a necessary and sufficient condition for a metrised LSTS (T, ϕ) to be isometric to a T^*-extension of some LSTS, and determine when two T^*-extensions of an LSTS are "same", i.e., they are equivalent or isometrically equivalent.

Key words: pseudo-metrised Lie supertriple system, metrised Lie supertriple system, T^*-extension

2000 MR subject classification: 17A40, 17B05

Document code: A

Article ID: 1674-5647(2014)01-0051-09

1 Introduction

A Lie (super)triple system over a field $\mathbb{K}$ is called pseudo-metrisable if it admits an invariant nondegenerate bilinear form, and if further, the bilinear form can be chosen to be (super)symmetric, then T is called metrisable. Recently, metrisable Lie (super)triple systems have attracted a lot of attention due to its applications in the areas of mathematics and physics (see, for example, [1–6]).

The method of T^*-extension of Lie algebras was first introduced by Bordemann[7] in 1997 and this method is an important method for studying algebraic structures. In our early paper, we investigated the T^*-extension of Lie triple systems (see [6]). This paper is devoted to transfer the T^*-extension method to Lie supertriple systems.

Throughout this paper, all Lie supertriple systems considered are assumed to be of finite dimension over a field $\mathbb{K}$.

Received date: May 18, 2011.
Foundation item: The NSF (A2010000194) of Hebei Province.
E-mail address: vonjacky@126.com (Feng J Q).

2 Lie Supertriple Systems

In this section, we first briefly sketch the notion of a (pseudo-)metrisable Lie supertriple system.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space over $\mathbb{K}$, where $V_{\bar{0}}$ and $V_{\bar{1}}$ are called bosonic and fermionic space, respectively, in physics literature. We denote the degree by

$$\deg(x) = \begin{cases} 0, & \text{if } x \in V_{\bar{0}}; \\ 1, & \text{if } x \in V_{\bar{1}}. \end{cases}$$

and write $(-1)^{xy} := (-1)^{\deg(x)\deg(y)}$.

Any element considered in this article is always assumed to be homogeneous, i.e., either $x \in V_{\bar{0}}$ or $x \in V_{\bar{1}}$.

Notice that the associate algebra $\mathrm{End}V$ is a superalgebra $\mathrm{End}V = \mathrm{End}_{\bar{0}}V \oplus \mathrm{End}_{\bar{1}}V$,

$$\mathrm{End}_\alpha V = \{a \in \mathrm{End}V \mid aV_s \subseteq V_{s+\alpha},\ s = \bar{0}, \bar{1}\}, \qquad \alpha = \bar{0}, \bar{1}.$$

Definition 2.1 *A Lie supertriple system (LSTS) is a $\mathbf{Z}_2$-graded space $T = T_{\bar{0}} \oplus T_{\bar{1}}$ over $\mathbb{K}$ with a trilinear composition $[\cdot, \cdot, \cdot]$, satisfying the following conditions*:

(1) $\deg([xyz]) = (\deg(x) + \deg(y) + \deg(z))(\mathrm{mod}\ 2)$;

(2) $[yxz] = -(-1)^{xy}[xyz]$;

(3) $(-1)^{xz}[xyz] + (-1)^{yx}[yzx] + (-1)^{zy}[zxy] = 0$;

(4) $[uv[xyz]] = [[uvx]yz] + (-1)^{(u+v)x}[x[uvy]z] + (-1)^{(u+v)(x+y)}[xy[uvz]]$.

An ideal of an LSTS T is a graded subspace I for which $[I, T, T] \subseteq I$. Moreover, if $[TII] = 0$, then I is called an abelian ideal of T. T is called abelian if it is an abelian ideal of itself. For any graded subspace V in T, the centralizer $Z_T(V)$ of V in T is defined by

$$Z_T(V) = \{x \in T \mid [xvt] = [xtv] = 0, \text{ for all } t \in T,\ v \in V\}.$$

In particular, $Z_T(T)$ is called the center of T and denoted simply by $Z(T)$. If T is an LSTS, define the lower central series for T by $T^0 := T$ and $T^{n+1} := [T^nTT]$ for $n \geq 0$. T is called nilpotent (of nilindex m) if there is a (smallest) positive integer m such that $T^m = 0$. Put $T^{(0)} := T$ and $T^{(n+1)} := [T^{(n)}TT^{(n)}]$. Then T is called solvable (of length k) if there is a (smallest) positive integer k such that $T^{(k)} = 0$.

Definition 2.2 *If an LSTS T admits a nondegenerate bilinear form b satisfying conditions*

(1) $b(x, y) = 0$ *unless* $d(x) = d(y)$; *(consistence)*

(2) $b([x, y, u], v) = -(-1)^{(x+y)u}b(u, [x, y, v])$, *(invariance)*

then we call T pseudo-metrisable and the pair (T, b) a pseudo-metrised LSTS. If, in addition, b satisfies also;

(3) $b(x, y) = (-1)^{xy}b(y, x)$, *(supersymmetry)*

then we call T metrisable and the pair (T, b) a metrised LSTS.

Proposition 2.1[1] *The following conditions are equivalent*:

(1) $b([x, y, u], v) = -(-1)^{(x+y)u}b(u, [x, y, v])$;

(2) $b([x,y,u],\ v) = -(-1)^{(u+v)y} b(x,\ [u,v,y])$;

(3) $b(x,\ [y,u,v]) = (-1)^{xy+uv} b(y,\ [x,v,u])$.

Define multiplication operators $L(\cdot\,,\cdot),\ P(\cdot\,,\cdot),\ R(\cdot\,,\cdot)$ on T by

$$L(x,y)z := [x,y,z], \qquad P(x,y)z := (-1)^{yz}[xzy], \qquad R(x,y)z := (-1)^{(x+y)z}[z,x,y].$$

Definition 2.3 *For $x,y,z \in T,\ f \in T^*$, define the following dual multiplication operators on T^* by*

(1) $(L^*(x,y)f)(z) := (-1)^{xy} f(L(y,x)(z))$;

(2) $(P^*(x,y)f)(z) := (-1)^{xy} f(P(y,x)(z))$;

(3) $(R^*(x,y)f)(z) := (-1)^{xy} f(R(y,x)(z))$.

Noticing that for any $x,y,z \in T,\ f \in T^*$,

$$\begin{aligned} L^*(x,y)f)(z) &= (-1)^{xy} f([yxz]) = (-1)^{(x+y)z} f([zxy]) - (-1)^{xy+(x+y)z} f([zyx]) \\ &= f(R(x,y)(z)) - (-1)^{xy} f(R(y,x)(z)) = ((-1)^{xy} R^*(y,x) - R^*(x,y))f(z) \end{aligned}$$

and

$$\begin{aligned} (P^*(x,y)f)(z) &= (-1)^{xy} f(P(y,x)(z)) = (-1)^{x(y+z)} f([yzx]) \\ &= -(-1)^{xy+xz+yz} f([zyx]) = -(-1)^{xy} f(R(y,x)(z)) = (-R^*(xy)f)(z), \end{aligned}$$

we have

$$L^*(x,y) = (-1)^{xy} R^*(y,x) - R^*(x,y) \qquad \text{and} \qquad P^*(x,y) = -R^*(x,y). \tag{2.1}$$

Definition 2.4 *A trilinear mapping $\omega : T \times T \times T \to T^*$ is called a* 3-supercocycle *if it satisfies the following conditions*:

(1) $\omega(y,x,z) = -(-1)^{xy}\omega(x,y,z)$;

(2) $(-1)^{xz}\omega(x,y,z) + (-1)^{yx}\omega(y,z,x) + (-1)^{zy}\omega(z,x,y) = 0$;

(3) $$\begin{aligned} &(-1)^{(u+v)(x+y+z)} L^*(u,v)\omega(x,y,z) + \omega(u,v,\ [xyz]) \\ &= R^*(y,z)\omega(u,v,x) + (-1)^{xy} P^*(x,z)\omega(u,v,y) + (-1)^{(x+y)z} L^*(x,y)\omega(u,v,z) \\ &\quad + \omega([uvx],\ y,z) + (-1)^{(u+v)x}\omega(x,[uvy],z) + (-1)^{(u+v)(x+y)}\omega(x,y,\ [uvz]). \end{aligned}$$

3 T^*_ω-extension

Recall that if ϕ is a bilinear form on a vector space V, and W is a subspace of V, then the right orthogonal space (resp. left orthogonal space) of W is given by $W^\perp := \{v \in V \mid \phi(w,v) = 0,\ \forall w \in W\}$ (resp. ${}^\perp W := \{v \in V \mid \phi(v,w) = 0,\ \forall w \in W\}$). The intersection of ${}^\perp V$ and $V^\perp$ is called the kernel N_ϕ of ϕ. The following lemma gives the basic results of pseudo-metrised LSTS.

Lemma 3.1 *Let (T,ϕ) be a pseudo-metrised LSTS over a field $\mathbb{K}$, and V be an arbitrary vector subspace of T.*

(i) *Let I be an ideal of T. Then ${}^\perp I$ and $I^\perp$ are ideals of T and $I^\perp,\ {}^\perp I \subset Z_T(I)$;*

(ii) *For arbitrary subspace V, $Z_T(V) = [VTT]^\perp = {}^\perp[VTT]$. If V is an ideal, then $Z_T(V)$ is an ideal*;

(iii) *In particular, $Z(T) = (T^{(1)})^\perp = {}^\perp(T^{(1)})$ for $T^{(1)} = [TTT]$.*

Now we consider the transfer of invariant bilinear forms from one LSTS to another. Let T (resp. T') be an LSTS over a field $\mathbb{K}$, f (resp. g) be an invariant bilinear form on T (resp. T'), and $m : T \to T'$ be a homomorphism of LSTS. Then we have the following lemma.

Lemma 3.2 *Under the above assumptions, we have*

(i) *The pull back m^*g of g is again an invariant bilinear form on T;*

(ii) *Suppose that m is surjective and* $\ker m$ *is contained in the kernel of f. Then the projection f^m of f is an invariant bilinear form on T';*

(iii) *If U is a subsystem of T, then $U \cap U^{\perp}$ is an ideal of U. Let $p : U \to U/(U \cap U^{\perp})$ be the projection and f_U be the restriction of f to $U \times U$. Then the projection $(f_U)^p$ is a nondegenerate invariant bilinear form on the factor system $U/(U \cap U^{\perp})$;*

(iv) *The bilinear form $f \perp g$ is invariant on the direct sum $T \oplus T'$. Moreover, $f \perp g$ is nondegenerate if and only if f and g are nondegenerate.*

The proofs of both Lemmas 3.1 and 3.2 are similar to that of Lie triple systems, which can be found in [6].

Now we generalize the notion of T^*-extension of a Lie triple system to that of a Lie supertriple system.

Definition 3.1 *Let T be an LSTS, T^* be the dual space of T, and ω be a 3-supercocycle. Define a ternary multiplication on $T^*_\omega T = T \oplus T^*$ by*

$$\begin{aligned}&[x+f,\, y+g,\, z+h]\\ &= [xyz]_T + \omega(x,y,z) + (-1)^{(x+y)z} L^*(x,y)h + (-1)^{xy} P^*(x,z)g + R^*(y,z)f\end{aligned}$$

for all $x, y, z \in T$, and $f, g, h \in T^*$, where $x+f$ (resp. $y+g$, $z+h$) is homogeneous of degree $\deg(x)$ *(resp.* $\deg(y)$, $\deg(z)$*), and $[xyz]_T$ is the Lie superbracket in T.*

Lemma 3.3 *Under the above definition, if* $\deg(\omega) = 0$*, then $T^*_\omega T$ is an LSTS, which is called the T^*-extension of the LSTS T by means of ω. In particular, if $\omega = 0$, then $T^*_0 T$ is called the trivial T^*-extension of T.*

Proof. Here we only consider the last equation in the definition of LSTS. We need to verify

$$\begin{aligned}&[u+i,\, v+j,\, [x+f,\, y+g,\, z+h]]\\ &= [[u+i,\, v+j,\, x+f],\ y+g,\, z+h] + (-1)^{(u+v)x}[x+f,\ [u+i,\, v+j,\, y+g],\, z+h]\\ &\quad + (-1)^{(u+v)(x+y)}[x+f,\, y+g,\, [u+i,\, v+j,\, z+h]]\end{aligned}$$

for $u, v, x, y, z \in T$, $i, j, f, g, h \in T^*$. Expand this equation by Definition 3.1. Then all items consist of the ternary compositions in T and the 3-supercocycle ω are canceled by the definitions of an LSTS and a 3-supercocycle. The items consisting of h reads

$$\begin{aligned}&(-1)^{(x+y)z+(u+v)(x+y+z)} L^*(u,v)L^*(x,y)h\\ &= (-1)^{(u+v+x+y)z} L^*([uvx],\ y)h + (-1)^{(u+v)x}(-1)^{(u+v+x+y)z} L^*(x,\ [uvy])h\\ &\quad + (-1)^{(u+v)(x+y)+(u+v)z+(x+y)(u+v+z)} L^*(x,y)L^*(u,v)h,\end{aligned}$$

that is,

$$\begin{aligned}&h((-1)^{(u+v)(x+y)}L(y,x)L(v,u))\\ =\ &h(-(-1)^{(u+v)y}L(y,\ [vux])-L([vuy],\ x)+L(v,u)L(y,x)).\end{aligned}$$

The above equation holds due to the last equation in the definition of an LSTS. Other items consisting of i, j, f or g can be verified similarly. This completes the proof.

By this lemma, we always suppose that the 3-supercocycle ω satisfies $\deg(\omega)=0$.

It is clear from the definition that the subspace T^* is an abelian ideal of $T^*_\omega T$ and T is isomorphic to the factor supertriple system $T^*_\omega T/T^*$. Moreover, consider the following consistent supersymmetric bilinear form q_T on $T^*_\omega T$ defined for all $x, y\in T,\ f, g\in T^*$ by

$$q_T(x+f,\ y+g)=f(y)+(-1)^{xy}g(x). \tag{3.1}$$

We then have the following lemma.

Lemma 3.4 *Let $T,\ T^*,\ \omega$ and q_T be as above. Then q_T is a nondegenerate supersymmetric bilinear form on $T^*_\omega T$ and the following conditions are equivalent*:

(1) *q_T is invariant*;

(2) $\omega(x,y,u)(v)=-(-1)^{uv}w(x,y,v)(u)$;

(3) $\omega(x,y,u)(v)=-(-1)^{(u+v)(x+y)+xy}\omega(u,v,y)(x)$;

(4) $\omega(y,u,v)(x)=(-1)^{(y+u+v)x+(y+u)v+yu}\omega(x,v,u)(y)$.

Hence $(T^*_\omega,\ q_T)$ is a metrised LSTS if and only if ω satisfies one of (2)–(4).

Proof. If $x+f$ is orthogonal to all elements of $T^*_\omega T$, then, in particular, $f(y)=0$ for all $y\in T$ and $g(x)=0$ for all $g\in T^*$, which implies that $f=0$ and $x=0$. So the supersymmetric bilinear form q_T is nondegenerate.

Now we consider the invariant property. Let $x, y, u, v\in T$ and $f, g, p, q\in T^*$. Then we have

$$\begin{aligned}&q_T([x+f,\ y+g,\ u+p],\ v+q)\\ =\ &q_T([xyu]+\omega(x,y,u)+(-1)^{(x+y)u}L^*(x,y)p+(-1)^{xy}P^*(x,u)g+R^*(y,u)f,\ v+q)\\ =\ &\omega(x,y,u)(v)+(-1)^{(x+y)u+xy}P(L(y,x)v)+(-1)^{x(y+u)}g(P(u,x)v)+(-1)^{yu}f(R(u,y)v)\\ &+(-1)^{(x+y+u)v}q([xyu])\\ =\ &\omega(x,y,u)(v)+(-1)^{(x+y)u+xy}P([yxv])+(-1)^{x(y+u+v)}g([uvx])+(-1)^{yu+(y+u)v}f([vuy])\\ &+(-1)^{(x+y+u)v}q([xyu]).\end{aligned}$$

On the other hand,

$$\begin{aligned}&-(-1)^{(x+y)u}q_T(u+p,\ [x+f,\ y+g,\ v+q])\\ =\ &-(-1)^{(x+y)u}q_T(u+p,\ [xyv]+\omega(x,y,v)+(-1)^{(x+y)v}L^*(x,y)q\\ &+(-1)^{xy}P^*(x,v)g+R^*(y,v)f)\\ =\ &-(-1)^{(x+y)u}P([xyv])-(-1)^{uv}\omega(x,y,v)(u)-(-1)^{(x+y+u)v}(L^*(x,y)q)(u)\\ &-(-1)^{xy+uv}(P^*(x,v)g)(u)-(-1)^{uv}(R^*(y,v)f)(u)\end{aligned}$$

$$= (-1)^{(x+y)u+xy}P([yxv]) - (-1)^{uv}\omega(x,y,v)(u) - (-1)^{(x+y+u)v+xy}q([yxu])$$
$$- (-1)^{x(y+v+u)+uv}g([vux]) - (-1)^{y(u+v)}f([uvy])$$
$$= (-1)^{(x+y)u+xy}P([yxv]) - (-1)^{uv}\omega(x,y,v)(u) + (-1)^{(x+y+u)v}q([xyu])$$
$$+ (-1)^{x(y+v+u)}g([uvx]) + (-1)^{y(u+v)+uv}f([vuy]).$$

Comparing these results we get that q_T is invariant if and only if

$$\omega(x,y,u)(v) = -(-1)^{uv}\omega(x,y,v)(u).$$

In a similar way, by the equivalence condition of Proposition 2.1, we can obtain also that q_T is invariant if and only if

$$\omega(x,y,u)(v) = -(-1)^{(u+v)(x+y)+xy}\omega(u,v,y)(x)$$

and if and only if

$$\omega(y,u,v)(x) = (-1)^{(y+u+v)x+(y+u)v+yu}\omega(x,v,u)(y).$$

Thus the lemma is proved.

4 Metrisable LSTS

Lemma 4.1 *Let (T, ϕ) be a metrised LSTS of dimension n over a field $\mathbb{K}$, and I be an isotropic $\frac{n}{2}$-dimensional subspace of T. Then I is an ideal of T if and only if I satisfies $I^{(1)} := [TII] = 0$. Hence I is an ideal if and only if I is an abelian ideal of T.*

Proof. Since $\dim I + \dim I^{\perp} = n$ it follows that $I = I^{\perp}$. If I is an ideal of T, then

$$\phi([TIT],\ I) = \phi([TIT],\ I^{\perp}) = 0.$$

Hence $\phi(T,\ [TII]) = 0$, and the non-degeneracy property of ϕ implies $I^{(1)} = [TII] = 0$. Conversely, if $I^{(1)} = [TII] = 0$, then

$$\phi(I,\ [ITT]) = \phi([ITI],\ T) = \phi([TII],\ T) = 0.$$

Hence $[ITT] \subset I^{\perp} = I$. This implies that I is an ideal of T.

Theorem 4.1 *Let (T,ϕ) be a metrised LSTS of dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then (T,ϕ) is isometric to a T^*-extension $(T^*_\omega B,\ q_B)$ if and only if n is even and T contains an isotropic ideal I (i.e., $I \subset I^{\perp}$) of dimension $\frac{n}{2}$. In this case: $B \cong T/I$.*

Proof. Sufficiency. Since $\dim B = \dim B^*$, it is clear that $\dim T^*_\omega B$ is even. Moreover, it is clear from the definition of the multiplication in Definition 3.1 that B^* is an isotropic ideal of half the dimension of $T^*_\omega B$.

Necessity. Suppose that I is an $\frac{n}{2}$-dimensional isotropic ideal of T. Let B denote the factor supertriple system T/I and $p : T \to B$ the canonical projection. Now, since the characteristic $\mathbb{K}$ is not equal to 2, we can choose an isotropic complementary vector subspace B_0 to I in T, i.e., $T = B_0 \oplus I$ and $B_0^{\perp} = B_0$. Denote by p_0 (resp. p_1) the projection $T \to B_0$ (resp. $T \to I$) along I (resp. along B_0). Moreover, let ϕ^I denote the linear map $I \to B^* : i \to (px \to \phi(i,x))$. It is well-defined because $\phi(I,I) = 0$. Since ϕ is nondegenerate, $I^{\perp} = I$, and $\dim I = \dfrac{n}{2} = \dim B$. It follows that ϕ^I is a linear isomorphism. Furthermore, ϕ^I has the following intertwining property: Let $x, y, z \in T$ and $i \in I$. Then

$$\begin{aligned}\phi^I([xyi])(pz) &= \phi([xyi],\ z)\\ &= -(-1)^{(x+y)i}\phi(i,\ [xyz])\\ &= -(-1)^{(x+y)i}\phi^I(i)([px,\ py,\ pz])\\ &= -(-1)^{(x+y)i+xy}L^*(py,\ px)\phi^I(i)(pz)\\ &= -(-1)^{(x+y)i}L^*(px,\ py)\phi^I(i)(pz).\end{aligned}$$

Hence after a completely analogous computation one has the following

$$\begin{cases}\phi^I([xyi]) = -(-1)^{(x+y)i}L^*(px,\ py)\phi^I(i),\\ \phi^I([xiy]) = (-1)^{ix}P^*(px,\ py)\phi^I(i),\\ \phi^I([ixy]) = R^*(px,\ py)\phi^I(i),\end{cases} \tag{4.1}$$

where $x, y \in T$ and $i \in I$. We define the following trilinear map:

$$\omega : B \times B \times B \to B^* : (pb_0,\ pb_0',\ pb_0'') \to \phi^I(p_1[b_0,\ b_0',\ b_0'']),$$

where b_0, b_0' and b_0'' are in B_0. This is well-defined since the restriction of the projection p to B_0 is a linear isomorphism. Now, let m denote the following linear map

$$T \to B \oplus B^* : b_0 + i \to pb_0 + \phi^I(i),$$

where $b_0 \in B$ and $i \in I$. Since p is restricted to B_0 and ϕ^I are linear isomorphisms, the map m is also a linear isomorphism. Moreover, m is an isomorphism of the metrised LSTS (T, ϕ) to the T^*-extension $(T^*_\omega B,\ q_B)$. Indeed, let $b_0, b_0', b_0'' \in B$ and $i, i', i'' \in I$. Then

$$\begin{aligned}&m([(b_0+i)(b_0'+i')(b_0''+i'')])\\ &= m(p_0([b_0, b_0', b_0'']) + p_1([b_0, b_0', b_0'']) + [b_0, b_0', i''] + [b_0, i', b_0''] + [i, b_0', b_0''])\\ &= p(p_0([b_0, b_0', b_0'']) + \phi^I(p_1([b_0, b_0', b_0'']) + [b_0, b_0', i''] + [b_0, i', b_0''] + [i, b_0', b_0''])\\ &= [pb_0,\ pb_0',\ pb_0''] + \omega(pb_0,\ pb_0',\ pb_0'') + (-1)^{(b_0+b_0')b_0''}L^*(pb_0,\ pb_0')\phi^I(i'')\\ &\quad + (-1)^{b_0b_0'}P^*(pb_0,\ pb_0'')\phi^I(i') + R^*(pb_0',\ pb_0'')\phi^I(i)\\ &= [pb_0 + \phi^I(i),\ pb_0' + \phi^I(i'),\ pb_0'' + \phi^I(i'')]\\ &= [m(b_0+i),\ m(b_0'+i'),\ m(b_0''+i'')],\end{aligned}$$

where we use the definition of ω, the intertwining properties of ϕ^I, the fact that p is a homomorphism, the definition of the product in $T^*_\omega B$, lemma 4.1 and (4.1). In addition, we have

$$\begin{aligned}(m^*q_B)(b_0+i,\ b_0'+i') &= q_B(pb_0 + \phi^I(i),\ pb_0' + \phi^I(i'))\\ &= \phi^I(i)(pb_0') + \phi^I(i')(pb_0)\\ &= \phi(i,\ b_0') + \phi(i',\ b)\\ &= \phi(b_0+i,\ b_0'+i'),\end{aligned}$$

where the fact that B_0 could be chosen to be isotropic entered in the last equation. Hence, $m^*q_B = \phi$ which implies that q_B is an invariant symmetric bilinear form on $T^*_\omega B$ or that

ω is cyclic. Therefore, $(T,\,\phi)$ and $(T^*_\omega B,\,q_B)$ are isomorphic as metrised algebras and the theorem is proved.

The proof of this theorem shows that the trilinear map ω depends on the choice of the isotropic subspace B_0 of T complementary to the ideal I. Therefore, there may be different T^*-extensions describing the "same" metrised LSTS.

Definition 4.1 *Let B_i, $i = 1, 2$, be two LSTS's over a field $\mathbb{K}$ and $\omega_i : B_i \times B_i \times B_i \to B_i^*$, $i = 1, 2$ be two different 3-supercocycles. The T^*-extension $T^*_{\omega_i} B_i$ of B_i are said to be equivalent if $B_1 = B_2 = B$ and there exists an isomorphism of LSTS $\Phi : T^*_{\omega_1} B_1 \to T^*_{\omega_2} B_2$ which is the identity on the ideal B^* and which induces the identity on the factor LSTS $T^*_{\omega_1} B_1/B^* = B = T^*_{\omega_2} B_2/B^*$. The two T^*-extensions $T^*_{\omega_i} B_i$ are said to be isometrically equivalent if they are equivalent and Φ is an isometry.*

Theorem 4.2 *Let B be an LSTS over a field of characteristic not equal to 2, and furthermore, let ω_i, $i = 1, 2$ be two 3-supercocycles: $B \times B \times B \to B^*$.*

(i) *$T^*_{\omega_i} B_i$ are equivalent if and only if there is a linear map $z : B \to B^*$ such that for all $a, b, c \in B$*

$$\begin{aligned}&\omega_1(a, b, c) - \omega_2(a, b, c)\\ &= (-1)^{(a+b)c} L^*(a, b)z(c) + (-1)^{ab} P^*(a, c)z(b) + R^*(b, c)z(a) - z([abc]).\end{aligned} \tag{4.2}$$

If this is the case, then the supersymmetric part z_s of z which is defined by

$$z_s(b)(d) := \frac{1}{2}(z(b)(d) + (-1)^{bd} z(d)(b)), \qquad b, d \in B$$

induces a symmetric invariant bilinear form on B, i.e.,

$$z_s(a)([dcb]) = (-1)^{ab+bc} z_s(d)([abc]), \qquad a, b, c, d \in B.$$

(ii) *$T^*_{\omega_i} B_i$ are isometrically equivalent if and only if there is a linear map $z : B \to B^*$ such that (4.2) holds for all $a, b, c \in B$ and, in addition, the supersymmetric part z_s of z vanishes.*

Proof. (i) The equivalence between $T^*_{\omega_1} B_1$ and $T^*_{\omega_2} B_2$ holds if and only if there is a homomorphism of LSTS

$$\Phi : T^*_{\omega_1} B_1 \to T^*_{\omega_2} B_2$$

satisfying

$$\Phi(b + g) = b + z(b) + g,\ b \in B,\ g \in B^*,$$

where z is the component of Φ that maps B to B^*. Indeed, by the definition, Φ must be the identity on B^* and we must have

$$b = p(b) = p(\Phi(b)) = z_1(b),$$

where $z_1(b)$ is the component of Φ that maps B to B. Clearly, Φ is a linear isomorphism for arbitrary z. Then for all $a, b, c \in B$ and $f, g, h \in B^*$, we have

$$\begin{aligned}&\Phi([a + f,\ b + g,\ c + h])\\ &= \Phi([abc] + \omega_1(a, b, c) + (-1)^{(a+b)c} L^*(a, b)h + (-1)^{ab} P^*(a, c)g + R^*(b, c)f)\\ &= [abc] + z([abc]) + \omega_1(a, b, c) + (-1)^{(a+b)c} L^*(a, b)h + (-1)^{ab} P^*(a, c)g + R^*(b, c)f,\end{aligned}$$

where the multiplication is formed in $T^*_{\omega_1}B_1$. On the other hand,

$$\begin{aligned}
&[\Phi(a+f)\Phi(b+g)\Phi(c+h)]\\
=\ &[a+z(a)+f,\ b+z(b)+g,\ c+z(c)+h]\\
=\ &[abc]+\omega_2(a,b,c)+(-1)^{(a+b)c}L^*(a,b)h+(-1)^{(a+b)c}L^*(a,b)z(c)\\
&+(-1)^{ab}P^*(a,c)g+(-1)^{ab}P^*(a,c)z(b)+R^*(b,c)f+R^*(b,c)z(a),
\end{aligned}$$

where the multiplication is formed in $T^*_{\omega_2}B_2$. Hence Φ is a homomorphism of LSTS if and only if (4.2) holds. Now split z into its anti-supersymmetric part z_a defined by

$$z_a(b)(d) := \frac{1}{2}(z(b)(d)-(-1)^{bd}z(d)(b)), \qquad b,d\in B,$$

and its supersymmetric part z_s defined above. Then $z = z_s + z_a$. We see that the right hand side of (4.2) evaluated on $d \in B$ has the following form:

$$\begin{aligned}
&(-1)^{ac+bc+ab}z_a(c)([bad])+(-1)^{a(b+c+d)}z_a(b)([cda])+(-1)^{bc+bd+cd}z_a(a)([dcb])\\
&+(-1)^{d(b+c+a)}z_a(d)([abc])+(-1)^{ac+bc+ab}z_s(c)([bad])+(-1)^{a(b+c+d)}z_s(b)([cda])\\
&+(-1)^{bc+bd+cd}z_s(a)([dcb])-(-1)^{d(b+c+a)}z_s(d)([abc]).
\end{aligned}$$

Writing the above summation as $s(abcd)$ and considering

$$s(abcd)-(-1)^{a(b+c+d)+b(c+d)+cd}s(dcba),$$

by Lemma 3.4(4), we get

$$z_s(a)([dcb]) = (-1)^{ab+bc}z_s(d)([abc]),$$

which proves the invariance of the supersymmetric bilinear form induced by z_s.

(ii) Let the isomorphism Φ be defined as in (i). Then, we have for all $b,d\in B$ and $f,g\in B^*$

$$\begin{aligned}
q_B(\Phi(b+f),\ \Phi(d+g)) &= q_B(b+z(b)+f,\ d+z(d)+g)\\
&= z(b)(d)+z(d)(b)+f(d)+g(b)\\
&= z(b)(d)+z(d)(b)+q_B(b+f,\ d+g),
\end{aligned}$$

from which it is clear that ϕ is an isometry if and only if $z_s = 0$.

References

[1] Okubo S, Kamiya N. Quasiclassical Lie superalgebras and Lie supertriple systems. *Comm. Algebra*, 2002, **30**: 3825–3850.

[2] Okubo S, Kamiya N. Jordan-Lie superalgebra and Jordan-Lie triple system. *J. Algebra*, 1997, **198**: 388–411.

[3] Okubo S. Parastatistic as Lie supertriple systems. *J. Math. Phys.*, 1994, **35**: 2785–2803.

[4] Zhang Z X, Shi Y Q, Zhao L N. Invariant symmetric bilinear forms on Lie triple system. *Comm. Algebra*, 2002, **30**(11): 5563–5573.

[5] Zhang Z X, Li H J, Dong L. Invariant bilinear forms on anti-Lie triple systems. *Chinese J. Contemp. Math.*, 2004, **25**: 237–244.

[6] Feng J Q. Metrisable Lie triple systems. *Acta Math. Sinica*, 2008, **51**(3): 457–468.

[7] Bordemann B. Nondegenerate invariant bilinear forms on nonassociative algebras. *Acta. Math. Univ. Comenian.*, 1997, **LXVI**(2): 151–201.