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Analysis of a Prey-predator Model with Disease in Prey*

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Abstract: In this paper, a system of reaction-diffusion equations arising in eco-epidemiological systems is investigated. The equations model a situation in which a predator species and a prey species inhabit the same bounded region and the predator only eats the prey with transmissible diseases. Local stability of the constant positive solution is considered. A number of existence and non-existence results about the non-constant steady states of a reaction diffusion system are given. It is proved that if the diffusion coefficient of the prey with disease is treated as a bifurcation parameter, non-constant positive steady-state solutions may bifurcate from the constant steady-state solution under some conditions.

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1 Introduction

Mathematical ecology and mathematical epidemiology are major fields of study. Since transmissible disease in ecological situation cannot be ignored, it is very important from both the ecological and the mathematical points of view to study ecological systems subject to epidemiological factors. A number of studies have been performed in this direction; see [1]–[9] and the references therein. Combining a typical SI model with an open system of variable size and a general predator-prey model, Bairagi et al. proposed a eco-epidemiological model in [10] as follows:

$$\begin{cases} u_t = ru\Big(1 - \dfrac{u+v}{k}\Big) - m_1uv, \\ v_t = m_1uv - \dfrac{m_2vw}{a+v} - m_3v, \\ w_t = \dfrac{m_4vw}{a+v} - m_5w, \\ u(0) > 0, \quad v(0) > 0, \quad w(0) > 0, \end{cases} \tag{1.1}$$

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where u, v and w are the densities of susceptible prey, infected prey and predator, respectively; r, k, m_i, $i = 1, 2, \cdots, 5$ are positive constants; m_1 is the rate of transmission; m_2 is the search rate; m_3 is the death rate of infected prey; m_4 represents the conversion factor; m_5 is the total death of predator population and a is the half saturation coefficient.

This model implies that the prey is divided into two disjoint classes, susceptible prey u and infected prey v. Only susceptible prey has capability of reproducing, but the infected prey still contributes with u to population growth towards the carrying capacity k. Disease transmission follows the simple law of mass action. The disease is spread among the prey population only. The infected population do not recover or become immune. It is assumed that predator consume only infected preys at the rates $m_2v/(a + v)$. For more detailed biological meaning the reader may consult [10].

As we know, most of the eco-epidemiological models are ODE systems. If we take into account the distribution of the species in spatial locations within a fixed bounded domain $\Omega \in \mathbf{R}^N$ with smooth boundary $\partial\Omega$ and both species diffuse, i.e., move from points of high to points of low population density, then (1.1) may be rewritten as

$$\begin{cases} u_t = d_1\Delta u + ru\Big(1 - \dfrac{u+v}{k}\Big) - m_1uv, & x \in \Omega, \quad t > 0, \\ v_t = d_2\Delta v + m_1uv - \dfrac{m_2vw}{a+v} - m_3v, & x \in \Omega, \quad t > 0, \\ w_t = d_3\Delta w + \dfrac{m_4vw}{a+v} - m_5w, & x \in \Omega, \quad t > 0, \\ \partial_n u = \partial_n v = \partial_n w = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) \geq 0, \quad v(x,0) \geq 0, \quad w(x,0) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where ∂_n is the outward directional derivative normal to $\partial\Omega$ and the positive constants d_1, d_2 and d_3 are the diffusion rates. The initial data $u(x,0)$, $v(x,0)$ and $w(x,0)$ are continuous functions on $\bar{\Omega}$. The homogeneous Neumann boundary condition means that (1.2) is self-contained and has no population flux across the boundary $\partial\Omega$.

The positive steady state solutions of (1.2) satisfy the following elliptic system:

$$\begin{cases} d_1\Delta u + ru\Big(1 - \dfrac{u+v}{k}\Big) - m_1uv = 0, & x \in \Omega, \\ d_2\Delta v + m_1uv - \dfrac{m_2vw}{a+v} - m_3v = 0, & x \in \Omega, \\ d_3\Delta w + \dfrac{m_4vw}{a+v} - m_5w = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

For the simplicity of notation, we denote

$$\Lambda = (r, k, m_1, m_2, m_3, m_4, m_5), \qquad U = (u, v, w).$$

We note that (1.2) has a unique nonnegative global solution U which can be proved by using the method of upper and lower solutions. In addition, if $u(x,0) \not\equiv 0$, $v(x,0) \not\equiv 0$, $w(x,0) \not\equiv 0$, then the solution is positive, i.e., $u > 0$, $v > 0$, $w > 0$ on $\bar{\Omega}$ for all $t > 0$. The equation (1.2) and hence (1.3) has a constant positive solution $U_0 = (u_0, v_0, w_0)$, where

$$u_0 = k - \frac{m_5 a(r + km_1)}{r(m_4 - m_5)}, \quad v_0 = \frac{m_5 a}{m_4 - m_5}, \quad w_0 = \frac{1}{m_2}(a + v_0)(m_1 u_0 - m_3), \tag{1.4}$$

provided

$$m_1 k > m_3, m_4 > m_5 + \frac{m_1 m_5 a(r + km_1)}{r(m_1 k - m_3)}. \tag{1.5}$$

Hereafter, we always suppose that (1.5) holds if there is no special demonstrativeness.

There are many works on the existence of positive steady states of ecological models. For predator-prey models, see [11]–[15] and the references therein. The general form of a weakly-coupled parabolic equation

$$\partial_t u_i = d_i \Delta u_i + u_i M_i(u_1, u_2, \cdots, u_n) \qquad \text{in} \ \ \Omega \times (0, T)$$

was discussed by Brown[16], where the asymptotic stability of critical points was given.

The main aim of this paper is to study the local stability of positive constant steady solution as well as the non-existence of non-constant positive steady state and bifurcation of non-constant solutions of (1.3).

The structure of the paper is as follows. In Section 2, dissipation of (1.2) is considered. Stability of the positive constant solution of ODE system (1.1) is studied in Section 3. In Section 4, we give a priori estimates for the positive solutions of (1.3) by using maximum principle and a Harnack-type inequality. In Section 5, non-existence of non-constant positive solutions is proved if the diffusion coefficient of w is large enough. Finally, in Section 6, bifurcation of non-constant solutions of (1.3) is studied with respect to the parameter d_2.

In the sequel, unless otherwise stated, all solutions considered will be classical solutions.

2 Dissipation

In this section, we study dissipation of (1.2). We first estimate the $L^1(\bar{\Omega})$ norm of the non-negative solutions of (1.2), and then transform L^1 estimates to L^p estimates for sufficiently large p. This approach has been used by many authors, as in [17] and [18]. First, we notice that

$$u(x,t) \le \max\{\max_{\bar{\Omega}} u(x,0), \ k\}$$

by the comparison principle.

Lemma 2.1 *Let*

$$b = m_2/m_4, \qquad c = \min\{m_3, \ m_5\}.$$

For any positive solution of (1.2), *there exists a positive M such that*

$$\begin{aligned} &\|u\|_{L^1} + \|v\|_{L^1} + b\|w\|_{L^1} \\ \le\ & \mathrm{e}^{-ct} \int_\Omega (u(x,0) + v(x,0) + bw(x,0))\mathrm{d}x + \frac{(r+c)^2 k}{4r} |\Omega|(1 - \mathrm{e}^{-ct}) \\ \le\ & M. \end{aligned} \tag{2.1}$$

Furthermore, for each $q > 1$, there exists a positive constant

$$C = C(q, \ d_1, \ d_2, \ d_3, \ \Omega)$$

such that

$$\begin{cases} \|u\|_{L^q} \le C(\|u\|_{L^1} + \|v\|_{L^1} + b\|w\|_{L^1}), \\ \|v\|_{L^q} \le C(\|u\|_{L^1} + \|v\|_{L^1} + b\|w\|_{L^1}), \\ \|w\|_{L^q} \le C(\|u\|_{L^1} + \|v\|_{L^1} + b\|w\|_{L^1}). \end{cases} \tag{2.2}$$

Proof. Multiplying b to the third equation of (1.2), and integrating over Ω , we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+v+bw) \le & \int_\Omega \Big(ru - \frac{r}{k}u^2 - m_3 v - bm_5 w\Big) \\ \le & -c\int_\Omega (u+v+bw) + \frac{k(r+c)^2}{4r}|\Omega|. \end{aligned}$$

Integrating the inequality, we have

$$\int_\Omega (u+v+bw) \le \mathrm{e}^{-ct}\int_\Omega (u(x,0)+v(x,0)+bw(x,0)) + \frac{k(r+c)^2}{4rc}|\Omega|(1-\mathrm{e}^{-ct}),$$

which implies that

$$\|u\|_{L^1} + \|v\|_{L^1} + b\|w\|_{L^1} \le M$$

for sufficiently large M.

Assuming that (2.2) holds for some $q \ge 1$ (it holds for $q = 1$ obviously from the above proof), we are to prove that it holds for exponent $2q$. Multiplying the first three equations of (1.2) by u^{2q-1}, v^{2q-1} and w^{2q-1} respectively, integrating over Ω, and then summing the above results, we have

$$\begin{aligned} \frac{1}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u^{2q}+v^{2q}+w^{2q}) \le & -\frac{2q-1}{q^2}\Big\{d_1\int_\Omega |\nabla u^q|^2 + d_2\int_\Omega |\nabla v^q|^2 + d_3\int_\Omega |\nabla w^q|^2\Big\} \\ & + \int_\Omega (ru^{2q} + m_1 v^{2q} u + m_4 w^{2q}), \end{aligned}$$

namely,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u^{2q}+v^{2q}+w^{2q}) \le & -\frac{d(4q-2)}{q}\Big(\int_\Omega |\nabla u^q|^2 + \int_\Omega |\nabla v^q|^2 + \int_\Omega |\nabla w^q|^2\Big) \\ & + 2qD\int_\Omega (u^{2p}+v^{2p}+w^{2p}), \end{aligned} \tag{2.3}$$

where $d = \min\{d_1,\ d_2,\ d_3\}$ and D is a positive constant independent of the initial value when t is large enough. Using (2.3) and the Nirenberg-Gagliardo inequality and Young's inequality, we see that (see [18] or [19])

$$\int_\Omega u^{2q} \le \epsilon\Big[\int_\Omega |\nabla u^q|^2 + \Big(\int_\Omega u^q\Big)^2\Big] + D(\epsilon)\Big(\int_\Omega u^q\Big)^p \tag{2.4}$$

for some positive constants p, where $D(\epsilon)$ is a constant depending on ϵ. Choosing

$$\epsilon = d(4q-2)/q(2qD+1),$$

from (2.3) and (2.4), we get that there are positive constants l_1 and l_2 such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u^{2q}+v^{2q}+w^{2q}) \le & -\Big(\int_\Omega u^{2q}+\int_\Omega v^{2q}+\int_\Omega w^{2q}\Big)\\ & + l_1\Big(\Big(\int_\Omega u^q\Big)^2+\Big(\int_\Omega v^q\Big)^2+\Big(\int_\Omega w^q\Big)^2\Big)\\ & + l_2\Big(\Big(\int_\Omega u^q\Big)^p+\Big(\int_\Omega v^q\Big)^p+\Big(\int_\Omega w^q\Big)^p\Big). \qquad (2.5)\end{aligned}$$

Integrating the above inequality, the asserted estimates now follow by applying the induction hypotheses.

Theorem 2.1 *For any positive solution of* (1.2)*, there exist positive constants* K_1 *and* K_2 *such that*

$$\limsup_{t\to\infty} u(x,t)\le k,\quad \limsup_{t\to\infty} v(x,t)\le K_1,\quad \limsup_{t\to\infty} w(x,t)\le K_2 \qquad \text{on } \bar\Omega. \qquad (2.6)$$

Proof. Since u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}-d_2u\le ru\Big(1-\dfrac{u}{k}\Big), & x\in\Omega,\ t>0,\\ \partial_n u=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)\ge 0, & x\in\Omega,\end{cases}$$

the first inequality in (2.6) follows by the comparison principle.

Let

$$D(A_1):=\{v(\,\cdot\,,t)\in C^1(\bar\Omega)\cap C^2(\Omega)|\ \partial_n v(\,\cdot\,,t)=0 \text{ on } \partial\Omega,\ t\ge 0\}.$$

For $v\in D(A_1)$ we define

$$A_1 v=-d_2\Delta v+m_3 v,$$

and for

$$w\in D(A_2):=\{w(\,\cdot\,,t)\in C^1(\bar\Omega)\cap C^2(\Omega)|\ \partial_n w(\,\cdot\,,t)=0 \text{ on } \partial\Omega,\ t\ge 0\}$$

define

$$A_2 w=-d_3\Delta w+m_5 w.$$

We can regard the problem (1.2) in the larger space $Y=L^p(\Omega)$, $p>N/2$ in the sense of [17]. Then $\mathrm{Re}\sigma(A_i)>\delta>0$ for some constant δ, $\mathrm{e}^{-A_i t}$ is an analytic semigroup on Y, and

$$\begin{cases} v(x,t)=\mathrm{e}^{-A_1t}v(x,0)+\displaystyle\int_0^t \mathrm{e}^{-(t-\tau)A_1}\Big(m_1u(x,\tau)v(x,\tau)-\frac{m_2v(x,\tau)w(x,\tau)}{a+v(x,\tau)}\Big)\mathrm{d}\tau,\\ w(x,t)=\mathrm{e}^{-A_2t}w(x,0)+\displaystyle\int_0^t \mathrm{e}^{-(t-\tau)A_2}\frac{m_4v(x,\tau)w(x,\tau)}{a+v(x,\tau)}\mathrm{d}\tau.\end{cases} \qquad (2.7)$$

The semigroup of operator e^{-A_it} maps Y into the space $Y^\alpha=D(A_i^\alpha)$ with the graph norm

$$\|v\|_{Y^\alpha}:=\|A_i^\alpha v\|_{L^p},$$

where A_i^α is the fractional power of A_i. We choose p so that $N/2p<\alpha<1$ and note the imbedding

$$Y^\alpha\hookrightarrow C^\nu,\qquad 0\le\nu<2\alpha-N/p \qquad (2.8)$$

(see [17] Th 1.6.1). Multiplying both sides of (2.7) by A_i^α, we have

$$\begin{aligned}\|v\|_{Y^\alpha} &= \|A_1^\alpha v\|_{L^p}\\ &\le \|A_1^\alpha \mathrm{e}^{-A_1 t} v(x,0)\|_{L^p} + \int_0^t \|A_1^\alpha \mathrm{e}^{-(t-\tau)A_1}\| \Big\| m_1 uv - \frac{m_2 vw}{a+v}\Big\|_{L^p} \mathrm{d}\tau\\ &\le C_\alpha t^{-\alpha}\mathrm{e}^{-\delta t}\|v(x,0)\|_{L^p} + C_\alpha G_1(\|u\|_{L_p} + \|v\|_{L_p} + \|w\|_{L_p}) \int_0^t (t-\tau)^{-\alpha}\mathrm{e}^{-\delta(t-\tau)}\mathrm{d}\tau,\\ \|w\|_{Y^\alpha} &= \|A_2^\alpha w\|_{L^p}\\ &\le \|A_2^\alpha \mathrm{e}^{-A_2 t} w(x,0)\|_{L^p} + \int_0^t \|A_2^\alpha \mathrm{e}^{-(t-\tau)A_2}\| \Big\| \frac{m_4 v(x,\tau) w(x,\tau)}{a+v(x,\tau)}\Big\|_{L^p} \mathrm{d}\tau\\ &\le C_\alpha t^{-\alpha}\mathrm{e}^{-\delta t}\|w(x,0)\|_{L^p} + C_\alpha G_2(\|u\|_{L_p} + \|v\|_{L_p} + \|w\|_{L_p}) \int_0^t (t-\tau)^{-\alpha}\mathrm{e}^{-\delta(t-\tau)}\mathrm{d}\tau.\end{aligned}$$

From (2.8) and Lemma 2.1, we complete the proof.

3 Stability of (u_0, v_0, w_0): ODE system (1.1)

In this section, we consider the stability of the positive steady-state $(u_0\ , v_0,\ w_0)$ of (1.1). Let $(u(t),\ v(t),\ w(t))$ be a positive solution of (1.1). It is easy to see that $u(t)$, $v(t)$ and $w(t)$ are bounded (see Lemma 1 in [10]).

The linearized problem of (1.1) at $(u_0,\ v_0,\ w_0)$ takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u\\ v\\ w\end{pmatrix} = \mathfrak{B}\begin{pmatrix} u\\ v\\ w\end{pmatrix} + O\begin{pmatrix} z\\ z\\ z\end{pmatrix},$$

where

$$z = (u-u_0)^2 + (v-v_0)^2 + (w-w_0)^2$$

and

$$\mathfrak{B} = \begin{pmatrix} -\dfrac{ru_0}{k} & -\left(\dfrac{r}{k} + m_1\right)u_0 & 0\\ m_1 v_0 & \dfrac{m_2 w_0 v_0}{(a+v_0)^2} & -\dfrac{m_2 m_5}{m_4}\\ 0 & \dfrac{m_4 a w_0}{(a+v_0)^2} & 0\end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & a_{23}\\ 0 & a_{32} & 0\end{pmatrix}.$$

Theorem 3.1 *If*

$$\max\left\{\frac{m_1 m_5 k}{r},\ m_5 + \frac{m_1 m_5 a(r+km_1)}{r(m_1 k - m_3)}\right\} < m_4 \le \frac{rm_1(k+m_5^2) - m_3 m_5 (r+m_5)}{r^2 + rm_1 m_5 - m_3 m_5},$$

then the constant positive steady state solution U_0 of (1.1) *is locally asymptotically stable.*

Proof. The characteristic polynomial of $\mathfrak{B}$ can be written as

$$\rho(\lambda) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3$$

with

$$\begin{cases} B_1 = -a_{11} - a_{22} = \Big(\dfrac{r}{k} - \dfrac{m_1 m_5}{m_4}\Big) u_0 + \dfrac{m_1 m_5}{m_4}, \\ B_2 = a_{11}a_{22} - a_{23}a_{32} - a_{12}a_{21}, \\ B_3 = a_{11}a_{23}a_{32} > 0. \end{cases}$$

Under the conditions of the theorem, we have

$$B_1 > 0, \qquad B_3 > 0.$$

Through a series of calculation, we have

$$B_1 B_2 - B_3 = -(a_{11} + a_{22})(a_{11}a_{22} - a_{23}a_{32} - a_{12}a_{21}) - a_{11}a_{23}a_{32} =: M.$$

We prove $M > 0$ under the conditions of the theorem. Notice that

$$\begin{aligned} M =& -\frac{r^2 m_2 u_0^2 v_0 w_0}{k^2(a+v_0)^2} + \frac{r m_2^2 u_0 v_0^2 w_0^2}{k(a+v_0)^4} - \frac{m_2^2 m_5 a v_0 w_0^2}{(a+v_0)^4} \\ &+ \frac{r}{k}\Big(\frac{r m_1}{k} + m_1^2\Big) u_0^2 v_0 - \frac{m_2 u_0 v_0^2 w_0}{(a+v_0)^2}\Big(\frac{r m_1}{k} + m_1^2\Big) \\ =& -\frac{r^2 m_5}{k^2 m_4}(m_1 u_0 - m_3) u_0^2 + \frac{r m_5^2}{k m_4^2}(m_1 u_0 - m_3)^2 u_0 \\ &- \frac{m_5^3 a}{m_4^2 v_0}(m_1 u_0 - m_3)^2 + \frac{r}{k}\Big(\frac{r m_1}{k} + m_1^2\Big) u_0^2 v_0 \\ &- \frac{m_1 m_5}{m_4}\Big(\frac{m_1 r}{k} + m_1^2\Big) u_0^2 v_0 + \frac{m_3 m_5}{m_4}\Big(\frac{r m_1}{k} + m_1^2\Big) u_0 v_0 \\ =&\Big(\frac{r}{k} - \frac{m_1 m_5}{m_4}\Big)\Big(\frac{r m_1}{k} + m_1^2\Big) u_0^2 v_0 + \frac{m_5^2 a}{m_4 v_0}\Big(\frac{r}{k} - \frac{m_1 m_5}{m_4}\Big)(m_1 u_0 - m_3) u_0 \\ &+ \frac{m_3 m_5}{m_4}\Big(\frac{r m_1}{k} + m_1^2\Big) u_0 v_0 \\ &+ \Big\{\frac{r m_5}{k m_4}\Big(\frac{m_1 m_5}{m_4} - \frac{r}{k}\Big) u_0^2 - \Big(\frac{r m_3 m_5^2}{k m_4^2} + \frac{r m_1 m_5^2 (m_4 - m_5)}{k m_4^2}\Big) u_0 \\ &\quad + \frac{m_3 m_5^2 (m_4 - m_5)}{m_4^2}\Big\}(m_1 u_0 - m_3). \end{aligned}$$

The first three terms in the right hand side of the above expression are positive if

$$\frac{r}{k} - \frac{m_1 m_5}{m_4} > 0.$$

M is positive if the expression within the curly bracket is positive. This expression can be put in the form $Au_0^2 + Bu_0 + C$, where

$$\begin{aligned} A &= \frac{r m_5}{k m_4}\Big(\frac{m_1 m_5}{m_4} - \frac{r}{k}\Big), \\ B &= -\Big(\frac{r m_3 m_5^2}{k m_4^2} + \frac{r m_1 m_5^2 (m_4 - m_5)}{k m_4^2}\Big), \\ C &= \frac{m_3 m_5^2 (m_4 - m_5)}{m_4^2}. \end{aligned}$$

We note that $A < 0$, $B < 0$, $C > 0$, and thus, a sufficient condition for the positivity of this expression is $Ak^2 + Bk + C \geqq 0$ since $u_0 < k$, which implies

$$m_4 \le \frac{r m_1 (k + m_5^2) - m_3 m_5 (r + m_5)}{r^2 + r m_1 m_5 - m_3 m_5}.$$

Thus, $M > 0$ when

$$m_4 \leq \frac{rm_1(k+m_5^2) - m_3m_5(r+m_5)}{r^2 + rm_1m_5 - m_3m_5}.$$

It follows from the Routh-Hurwitz criterion that the three roots λ_1, λ_2, λ_3 of $\rho(\lambda) = 0$ all have negative real parts. This implies that U_0 is locally asymptotically stable.

4 A Priori Estimates

In this section, the main purpose is to give a priori upper bound and Harnack Inequality for positive solutions of (1.3). To this aim, we first recall the following well known results:

Lemma 4.1 (Maximum principle) *If $u \in C^2(\bar{\Omega})$ satisfies $\partial_n u = 0$ on $\partial\Omega$ and $x_0 \in \bar{\Omega}$ is a point where u achieves its maximum, then $-\Delta u(x_0) \geq 0$.*

Lemma 4.2 (Harnack Inequality) *Let $c \in C(\bar{\Omega})$ and $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive classical solution to $\Delta w(x) + c(x)w(x) = 0$ in Ω subject to the zero flux boundary condition. Then there exists a positive constant $C = C(\Omega, \|c\|_\infty)$ such that $\max\limits_{\bar{\Omega}} w \leq C \min\limits_{\bar{\Omega}} w$.*

The first lemma can be found in [20] and the second is due to Lin *et al.*[21]. Our main result in this section is the following.

Theorem 4.1 *For any positive solution U of (1.3), we have*

$$\sup_{\bar{\Omega}} u(x) \leq k, \quad \sup_{\bar{\Omega}} v(x) \leq \frac{d_1 k}{d_2} + k, \quad \sup_{\bar{\Omega}} w(x) \leq \frac{m_4 k(d_1+d_2)}{d_3 m_2} + \frac{m_1 m_4 k^2 (d_1+d_2)}{d_2 m_2 m_5}. \tag{4.1}$$

Proof. Let $x_0 \in \bar{\Omega}$ be such that

$$u(x_0) = \max_{\bar{\Omega}} u(x).$$

Then by Lemma 4.1 and the positivity of u, v and w, one can obtain

$$ru(x_0)\Big(1 - \frac{u(x_0)+v(x_0)}{k}\Big) - m_1 v(x_0)u(x_0) \geq 0,$$

which in turn implies

$$u(x_0) = \max_{\bar{\Omega}} u(x) \leq k.$$

Let

$$y = d_1 u + d_2 v.$$

Then we can deduce that

$$\begin{cases} -\Delta y = ru\Big(1 - \dfrac{u+v}{k}\Big) - \dfrac{m_2 vw}{a+v} - m_3 v, & x \in \Omega, \\ \partial_n y = 0, & x \in \partial\Omega. \end{cases}$$

Let $x_1 \in \bar{\Omega}$ be a point such that

$$y(x_1) = \max_{\bar{\Omega}} y(x).$$

By the application of the maximum principle, it yields $v(x_1) \le k$. Consequently,

$$\begin{aligned} d_2 \max_{\bar\Omega} v(x) &\le \max_{\bar\Omega} y(x) \\ &= y(x_1) \\ &= d_1 u(x_1) + d_2 v(x_1) \\ &\le k(d_1 + d_2), \end{aligned}$$

and hence

$$\sup_{\bar\Omega} v(x) \le \frac{d_1 k}{d_2} + k.$$

Let

$$z = d_2 m_4 v + d_3 m_2 w.$$

We can prove the last inequality of the theorem similarly.

Theorem 4.2 *Suppose that Λ is fixed. Let $0 < d \le \min\{d_1,\ d_2,\ d_3\}$ be a constant. Then there exist positive constants $C_1(d)$, $C_2(d,\ d_1)$ and $C_3(d,\ d_1)$ depending on Λ and Ω, such that for any positive solution U of* (1.3) *the following Harnack-type inequalities holds*:

$$\frac{\max_{\bar\Omega} u(x)}{\min_{\bar\Omega} u(x)} \le C_1(d), \quad \frac{\max_{\bar\Omega} v(x)}{\min_{\bar\Omega} v(x)} \le C_2(d,\ d_1), \quad \frac{\max_{\bar\Omega} w(x)}{\min_{\bar\Omega} w(x)} \le C_3(d). \tag{4.2}$$

Proof. Note that u satisfies

$$\Delta u + u\frac{rk - ru - rv - m_1 kv}{d_1 k} \qquad \text{for } x \in \Omega$$

and

$$\frac{\partial u}{\partial n} = 0 \qquad \text{for } x \in \partial\Omega.$$

We see from Theorem 4.1 that

$$\Big\|\frac{rk - ru - rv - m_1 kv}{d_1 k}\Big\|_\infty \le \frac{1}{d_1}\Big[2r + \frac{d_1 r}{d_2} + r + \frac{m_1 d_1 k}{d_2} + m_1 k\Big] \le C(d).$$

Hence it follows from Lemma 4.2 that there exists a positive constant $C_1(d)$ such that

$$\max_{\bar\Omega} u(x) \le C_1(d) \min_{\bar\Omega} u(x).$$

The inequalities for v and w can be proved similarly.

5 Non-existence of Non-constant Positive Steady States

In this section, we give some sufficient conditions for the non-existence of non-constant positive solutions to (1.3). Let U be a positive solution to (1.3) and $\tilde U$ be the average of U in Ω. Multiplying the first three equations of (1.3) by $A(u-\tilde u)/u$ and $(v-\tilde v)/v$ and $(w-\tilde w)/w$ respectively, where $A = m_1 k/(r + m_1 k)$, and integrating the results over Ω, one can obtain

$$\begin{aligned} \int_\Omega A d_1 \frac{\tilde u}{u^2}|\nabla u|^2 &= A\int_\Omega (u - \tilde u)\Big(r - \frac{ru + rv}{k} - m_1 v\Big) \\ &= A\int_\Omega (u - \tilde u)\Big(\frac{r\tilde u + r\tilde v}{k} + m_1\tilde v - \frac{ru + rv}{k} - m_1 v\Big) \end{aligned}$$

$$= A\int_\Omega \Big\{-\frac{r}{k}(u-\tilde u)^2 - \Big(\frac{r}{k}+m_1\Big)(u-\tilde u)(v-\tilde v)\Big\},$$

$$\begin{aligned}\int_\Omega d_2\frac{\tilde v}{v^2}|\nabla v|^2 &= \int_\Omega (v-\tilde v)\Big(m_1u-\frac{m_2w}{a+v}-m_3\Big)\\ &= \int_\Omega (v-\tilde v)\Big(m_1u-\frac{m_2w}{a+v}-m_1\tilde u+\frac{m_2\tilde w}{a+\tilde v}\Big)\\ &= \int_\Omega \Big\{m_1(u-\tilde u)(v-\tilde v)-\frac{m_2}{a+v}(v-\tilde v)(w-\tilde w)+\frac{m_2\tilde w(v-\tilde v)^2}{(a+v)(a+\tilde v)}\Big\}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega d_3\frac{\tilde w}{w^2}|\nabla w|^2 &= \int_\Omega (w-\tilde w)\Big(\frac{m_4v}{a+v}-m_5\Big)\\ &= \int_\Omega (w-\tilde w)\Big(\frac{m_4v}{a+v}-\frac{m_4\tilde v}{a+\tilde v}\Big)\\ &= \int_\Omega \frac{m_4a}{(a+v)(a+\tilde v)}(v-\tilde v)(w-\tilde w).\end{aligned}$$

So we have

$$\begin{aligned}&A\int_\Omega d_1\frac{\tilde u}{u^2}|\nabla u|^2+\int_\Omega d_2\frac{\tilde v}{v^2}|\nabla v|^2+\int_\Omega d_3\frac{\tilde w}{w^2}|\nabla w|^2\\ =&\int_\Omega\Big\{-\frac{m_1r}{r+m_1k}(u-\tilde u)^2+\Big(\frac{m_4a}{(a+v)(a+\tilde v)}-\frac{m_2}{a+v}\Big)(v-\tilde v)(w-\tilde w)\\ &+\frac{m_2\tilde w}{(a+v)(a+\tilde v)}(v-\tilde v)^2\Big\}.\end{aligned}$$

Furthermore, it follows from Theorem 4.1 and 4.2 that

$$\frac{\tilde v}{v^2}\ge\left(\frac{\min_{\bar\Omega} v(x)}{\max_{\bar\Omega} v(x)}\right)^2\frac{1}{\tilde v}\ge C_2(d,d_1),\qquad \frac{\tilde w}{w^2}\ge\left(\frac{\min_{\bar\Omega} w(x)}{\max_{\bar\Omega} w(x)}\right)^2\frac{1}{\tilde w}\ge C_3(d,d_1,d_2). \tag{5.1}$$

Using Poincaré's inequality

$$\int_\Omega (f-\tilde f)^2\le\frac{1}{\lambda}\int_\Omega |\nabla f|^2$$

and Young's inequality, we can obtain

$$\begin{aligned}&\int_\Omega Ad_1\lambda C_1(d)(u-\tilde u)^2+\int_\Omega d_2\lambda C_2(d,d_1)(v-\tilde v)^2+\int_\Omega d_3\lambda C_3(d,d_1,d_2)(w-\tilde w)^2\\ \le&\int_\Omega\Big\{-\frac{m_1r}{r+m_1k}(u-\tilde u)^2+\varepsilon(v-\tilde v)^2+C(\varepsilon)(w-\tilde w)^2\Big\}.\end{aligned} \tag{5.2}$$

Taking ε small enough, we have $U=\tilde U$ when d_3 large enough by (5.2). So we have the following theorem:

Theorem 5.1 *Let d be a constant such that $0<d\le\min\{d_1,\ d_2,\ d_3\}$. There exist positive constant $C_{d_3}=C_{d_3}(\Lambda,\ d,\ d_1,\ d_2)$ such that the system* (1.3) *has no non-constant solution provided that $d_3\ge C_{d_3}$.*

6 Bifurcation

In this section, we discuss the bifurcation of non-constant positive solutions of (1.3). Let the parameters $\Lambda,\ d_1,\ d_3>0$ be fixed. And consider $d_2>0$ as the bifurcation parameter.

We say that $(\hat{d}_2, U_0) \in (0,\infty)\times X$ is a bifurcation point of (1.3) if for any $\delta \in (0, \hat{d}_2)$, there exists $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$ such that (1.3) has a non-constant positive solution. Otherwise, we say that $(d_2,\ U_0)$ is a regular point.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and set

$$X = \{(u,v,w) \in [C^1(\bar{\Omega})]^3 \mid \partial_n u = \partial_n v = \partial_n w = 0 \ \text{ on } \partial\Omega\},$$
$$B_\delta = \{U \in X \mid \|U\|_X < \delta \ \text{ with } \ \delta > 0\},$$
$$E(\mu) = \{\varphi \mid -\Delta\varphi = \mu\varphi \text{ in } \Omega,\ \partial_n\varphi = 0 \text{ on } \partial\Omega \ \text{ with } \mu \in \mathbf{R}^1\}.$$

Let $\{\varphi_{ij}\}_{j=1}^{\dim\{E(\mu_i)\}}$ be an orthonormal basis of $E(\mu_i)$ and

$$X_{ij} = \{C\varphi_{ij} \mid C \in \mathbf{R}^3\}.$$

Then

$$X = \bigoplus_{i=0}^{\infty} X_i,$$

where

$$X_i = \bigoplus_{j=1}^{\dim\{E(\mu_i)\}} X_{ij}.$$

Let

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \qquad F(U) = \begin{pmatrix} ru\Big(1 - \dfrac{u+v}{k}\Big) - m_1uv \\ v\Big(m_1u - \dfrac{m_2w}{a+v} - m_3\Big) \\ w\Big(\dfrac{m_4v}{a+v} - m_5\Big) \end{pmatrix}.$$

Then we can rewrite (1.3) as

$$\begin{cases} -\Delta U = D^{-1}F(U), & x \in \Omega, \\ \partial_n U = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

or equivalently,

$$G(d_2, U) = U - (I-\Delta)^{-1}\{D^{-1}F(U) + U\} = 0 \qquad \text{on } \ X, \tag{6.2}$$

where $(I-\Delta)^{-1}$ is the inverse of $I-\Delta$ with the homogeneous Neumann boundary condition. By direct computation, we have

$$G_U(d_2, U_0) = I - (I-\Delta)^{-1}\{D^{-1}F_U(U_0) + I\}. \tag{6.3}$$

Obviously, for each $i \in \{0,\ 1,\ 2,\ 3,\ \cdots\}$, X_i is invariant under the operator $G_U(d_2,\ U_0)$, and λ is an eigenvalue of $G_U(d_2,\ U_0)$ on X_i if and only if λ is an eigenvalue of the matrix

$$A_i = I - \frac{1}{1+\mu_i}\{D^{-1}F_U(U_0) + I\} = \frac{1}{1+\mu_i}\{\mu_i I - D^{-1}F_U(U_0)\}.$$

Define

$$\mathcal{N}(d_2) = \{\mu > 0 \mid \det H(d_2;\mu) = 0 \ \text{ for } \ d_2 > 0\},$$

where

$$H(d_2;\mu) = \{\mu I - D^{-1}F_U(U_0)\}.$$

A direct calculation gives

$$\det H(d_2;\mu)=\mu^3+\mu^2\left(\frac{ru_0}{d_1k}-\frac{m_2w_0v_0}{d_2(a+v_0)^2}\right)+\mu\left(\frac{m_2m_5aw_0}{d_2d_3(a+v_0)^2}+\frac{m_1u_0v_0}{d_1d_2}\left(\frac{r}{k}+m_1\right)\right.$$
$$\left.-\frac{rm_2u_0v_0w_0}{d_1d_2k(a+v_0)^2}\right)+\frac{rm_2m_5au_0v_0w_0}{d_1d_2d_3k(a+v_0)^2}. \tag{6.4}$$

Using S_p to denote the positive spectrum of $-\Delta$ on Ω with the homogeneous Neumann boundary condition, that is, $S_p=\{\mu_1,\ \mu_2,\ \mu_3,\ \cdots\}$, we have the following local bifurcation theorem:

Theorem 6.1 *Suppose that Λ satisfies* (1.5). *Let $\hat{d}_2>0$ and consider the point $(\hat{d}_2,\ U_0)$.*

(i) *If $S_p\cap\mathcal{N}(\hat{d}_2)=\emptyset$, then $(\hat{d}_2,\ U_0)$ is a regular point of* (1.3);

(ii) *Suppose that $S_p\cap\mathcal{N}(\hat{d}_2)\neq\emptyset$ and the positive root of $\det H(\hat{d}_2;\ \mu)=0$ is simple. If $\sum\limits_{\mu_j\in\mathcal{N}(\hat{d}_2)}\dim E(\mu_j)$ is odd, then $(\hat{d}_2,\ U_0)$ is a bifurcation point of* (1.3).

Proof. Let $V(x)=U(x)-U_0$. Then Problem (1.3) is equivalent to

$$\begin{cases}-\Delta V=D^{-1}F(U_0+V), & x\in\Omega,\\ \partial_n V=0, & x\in\partial\Omega,\end{cases}$$

which, in turn, is equivalent to

$$G(d_2,V)=V-(I-\Delta)^{-1}\{D^{-1}F(U_0+V)+V\}=0 \qquad \text{on } X.$$

By direct computation, one can obtain

$$G_V(d_2,0)=I-(I-\Delta)^{-1}\{D^{-1}F_U(U_0)+I\}.$$

(i) If $S_p\cap\mathcal{N}(\hat{d}_2)=\emptyset$, then $\det H(\hat{d}_2,\ \mu_i)\neq 0$ for all $i\in\{0,\ 1,\ 2,\ 3,\ \cdots\}$ by (5.4). Hence, 0 is not an eigenvalue of $G_V(\hat{d}_2,\ 0)$. This implies that $G_V(\hat{d}_2,\ 0)$ is a homeomorphism from X to itself. The implicit function theorem shows that for all d_2 close to $\hat{d}_2$, $V=0$ is the only solution to $G(d_2,\ V)=0$ in a small neighborhood of the origin, that is, $(\hat{d}_2,\ U_0)$ is a regular point of (1.3).

(ii) Suppose $S_p\cap\mathcal{N}(\hat{d}_2)\neq\emptyset$. By a direct computation, the eigenvalue λ of $H(\hat{d}_2;\ \mu_i)$ is given by

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \tag{6.5}$$

with

$$\begin{cases}A_1=-\left(3\mu_i-\dfrac{m_2v_0w_0}{\hat{d}_2(a+v_0)^2}+\dfrac{ru_0}{kd_1}\right),\\[2ex] A_2=3\mu_i^2+2\left(\dfrac{ru_0}{kd_1}-\dfrac{m_2w_0v_0}{\hat{d}_2(a+v_0)^2}\right)\mu_i\\[2ex] \qquad -\dfrac{rm_2w_0v_0u_0}{kd_1\hat{d}_2(a+v_0)^2}+\dfrac{m_2m_5aw_0}{\hat{d}_2d_3(a+v_0)^2}+\dfrac{m_1u_0v_0}{d_1\hat{d}_2}\left(\dfrac{r}{k}+m_1\right),\\[2ex] A_3=-\mu_i^3-\left(\dfrac{ru_0}{kd_1}-\dfrac{m_2w_0v_0}{\hat{d}_2(a+v_0)^2}\right)\mu_i^2\\[2ex] \qquad -\left(\dfrac{m_2m_5aw_0}{\hat{d}_2d_3(a+v_0)^2}+\dfrac{m_1u_0v_0}{d_1\hat{d}_2}\left(\dfrac{r}{k}+m_1\right)-\dfrac{rm_2w_0v_0u_0}{kd_1\hat{d}_2(a+v_0)^2}\right)\mu_i-\dfrac{rm_2m_5au_0w_0}{kd_1\hat{d}_2d_3(a+v_0)^2}.\end{cases}$$

It is easy to show that 0 is a simple eigenvalue of $H(\hat{d}_2,\ \mu_j)$ for any j satisfying $\mu_j \in S_p \cap \mathcal{N}(\hat{d}_2)$ by (6.5). Now, suppose on the contrary that the assertion of the theorem were false. Then there would exist a $\hat{d}_2 > 0$ such that the following are true:

(a) $S_p \cap \mathcal{N}(\hat{d}_2) \neq \emptyset$ and $\displaystyle\sum_{\mu_j \in \mathcal{N}(\hat{d}_2)} \dim E(\mu_j)$ is odd, where $\mu_j \in S_p \cap \mathcal{N}(\hat{d}_2)$;

(b) There exists a $\delta \in (0,\ \hat{d}_2)$ such that for every $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$, $V = 0$ is the only solution to $G(d_2,\ V(x)) = 0$ in a neighborhood B_δ of the origin.

Since $G(d_2;\ \cdot\ )$ is a compact perturbation of the identity function, in view of (b), the Leray–Schauder degree $\deg(G(d_2;\ \cdot\ ),\ B_\delta,\ 0)$ is well defined and does not depend on $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$. In addition, for those $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$, where $G_V(d_2,\ 0)$ is invertible, $\deg(G(d_2;\ \cdot\ ),\ B_\delta,\ 0) = (-1)^{\nu(d_2)}$, where $\nu(d_2)$ is the total number of negative eigenvalues of $G_V(d_2,\ 0)$.

Let $\tilde{H}(d_2;\ \mu) = d_1 d_2 d_3 \det H(d_2;\ \mu)$. For $\mu_j \in S_p \cap \mathcal{N}(\hat{d}_2)$, we have

$$\tilde{H}(\hat{d}_2;\ \mu_j) = 0.$$

A direct computation yields

$$\frac{\partial}{\partial d_2}\tilde{H}(\hat{d}_2;\ \mu_j) = d_1 d_3 \mu_j^3 + \frac{d_3 r u_0}{k}\mu_j^2 > 0.$$

We may choose δ small enough so that

$$\frac{\partial}{\partial d_2}\tilde{H}(d_2;\ \mu_j) > 0, \qquad \mu_j \in S_p \cap \mathcal{N}(\hat{d}_2),\ d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta].$$

Therefore

$$\tilde{H}(\hat{d}_2 - \delta;\ \mu_j)\tilde{H}(\hat{d}_2 + \delta;\ \mu_j) < 0,$$

and in turn,

$$H(\hat{d}_2 - \delta;\ \mu_j)H(\hat{d}_2 + \delta;\ \mu_j) < 0, \qquad \mu_j \in S_p \cap \mathcal{N}(\hat{d}_2). \tag{6.6}$$

Since S_p does not have any accumulation point, by taking δ sufficiently small, we may assume that

$$S_p \cap N(d_2) = \emptyset, \qquad \forall\ d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2) \cup (\hat{d}_2,\ \hat{d}_2 + \delta].$$

Therefore, $G_V(d_2, 0)$ is invertible for all $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2) \cup (\hat{d}_2,\ \hat{d}_2 + \delta]$.

Note that $X = \displaystyle\bigoplus_{i=1}^{\infty} X_i$, where X_i is the eigenspace corresponding to μ_i. Also, for each $i \in \{0,\ 1,\ 2,\ 3,\ \cdots\}$ and $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$, X_i is invariant for the operator $G_V(d_2,\ 0)$ and the number of negative eigenvalues of $G_V(d_2,\ 0)$ on X_i is the same as that of the matrix A_i or $H(d_2;\ \mu_i)$. For every i, if $\mu_i \notin \mathcal{N}(\hat{d}_2)$, then the number of eigenvalues with negative real parts of $G_V(d_2,\ 0)$ on X_i is independent of $d_2 \in [\hat{d}_2 - \delta,\ \hat{d}_2 + \delta]$; whereas if $\mu_j \in \mathcal{N}(\hat{d}_2)$ then the difference between the number of eigenvalues with negative real parts of $G_V(d_2,\ 0)$ on X_j for $d_2 = \hat{d}_2 - \delta$ and $d_2 = \hat{d}_2 + \delta$ is 1 by (6.6). From the simpleness of the positive root of $H(\hat{d}_2;\ \mu) = 0$, we have

$$|\nu(\hat{d}_2 - \delta) - \nu(\hat{d}_2 + \delta)| = \sum_{\mu_j \in \mathcal{N}(\hat{d}_2)} \dim E(\mu_j),$$

which is odd. Therefore,

$$\deg(G(\hat{d}_2 - \delta;\ \cdot\ ),\ B_\delta,\ 0) \neq \deg(G(\hat{d}_2 + \delta;\ \cdot\ ),\ B_\delta,\ 0),$$

and we have a contradiction. This contradiction shows that $(\hat{d}_2,\ U_0)$ is a bifurcation point of (1.3).

Remark 6.1 From Theorems 3.1 and 6.1, we can conclude that diffusion-driven instability occurs when the diffusion coefficients is in a suitable range.

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