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ERROR ESTIMATES AND BLOW-UP ANALYSIS OF A FINITE-ELEMENT APPROXIMATION FOR THE PARABOLIC-ELLIPTIC KELLER-SEGEL SYSTEM

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Abstract. The Keller-Segel equations are widely used for describing chemotaxis in biology. Recently, a new fully discrete scheme for this model was proposed in [46], mass conservation, positivity and energy decay were proved for the proposed scheme, which are important properties of the original system. In this paper, we establish the error estimates of this scheme. Then, based on the error estimates, we derive the finite-time blowup of nonradial numerical solutions under some conditions on the mass and the moment of the initial data.

Key words. Parabolic-elliptic systems, finite element method, error estimates, finite-time blowup.

1. Introduction.

Keller and Segel first proposed a nonlinear model in the 1970s to describe the effect of cell aggregation in [27, 28]. A simplified Keller-Segel model in 2-D is given by

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \quad x \in \Omega,\ t > 0, \tag{1}$$

$$0 = \Delta v - v + \alpha u, \quad x \in \Omega,\ t > 0, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. The unknown $u = u(x,t)$ and $v = v(x,t)$ represent the concentration of the organism and chemoattractant respectively. The parameters χ, α are positive constants with χ being the sensitivity of chemotaxis. The model is supplemented with initial conditions

$$u(x, t=0) = u_0(x),\ v(x, t=0) = v_0(x),\ x \in \Omega,$$

and no flux boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{n}} - \chi u \frac{\partial v}{\partial \boldsymbol{n}} = 0,\ \frac{\partial v}{\partial \boldsymbol{n}} = 0, \quad x \in \partial\Omega,\ t > 0,$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega$, $\partial/\partial\boldsymbol{n}$ represents differentiation along $\boldsymbol{n}$ on $\partial\Omega$.

A different version of the Keller-Segel model consists in replacing (2) by

$$\frac{\partial v}{\partial t} = \Delta v - v + \alpha u, \quad x \in \Omega,\ t > 0. \tag{3}$$

The equation (1) describes the motion of the organism u. The term $F = -\nabla u + \chi u \nabla v$ is the flux, and the effect of diffusion $-\Delta u$ and that of chemotaxis $\chi\nabla \cdot (u\nabla v)$ are competing for u to vary. The equation (2) describes the change in concentration of the chemoattractant v, it is influenced by the diffusion and the decay of the chemoattractant as well as the growth of the organism. In general, the chemoattractant particles are much smaller than the organism particles, thus it diffuses faster, which means that the diffusion of the chemoattractant will reach the

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equilibrium state in a relatively short time. The model (1)-(2) is called parabolic-elliptic system. On the other hand, (1) with (3) is a parabolic-parabolic system.

The solution of the Keller-Segel model (1)-(2) has several well-known properties, particularly, it may blow up in finite time. Various aspects and results for the classical Keller-Segel model since 1970, along with some open questions, are summarized in [25]. Positivity, mass conservation and energy dissipation of Keller-Segel equations can be found in [35],[36],[47],[29] and [6], which plays an important role to study the Keller-Segel system. Blanchet, Dolbeault and Perthame presented in [3] a detail proof of the existence of weak solutions when the initial mass is below the critical mass, above which any solution to the parabolic-elliptic systems blows up in finite time in the whole Euclidean space. In [37], Nagai demonstrated the finite-time blowup of nonradial solutions under some assumptions on the mass and the moment of the initial data. As for the parabolic-parabolic systems, Blanchet proved in [2] the optimal critical mass of the solutions in $\mathbb{R}^d$ with $d \geq 3$. Wei proved that for every nonnegative initial data in $L^1(R^2)$, the 2-D Keller-Segel equation is globally well-posed if and only if the total mass $M \leq 8\pi$ in [49].

Although the large time behavior of the solution of the Keller-Segel model (1)-(2) has been well studied, there is still much to explore on the numerical side. Since the Keller-Segel equations possess three important properties: positivity, mass conservation and energy dissipation, it is preferable that numerical schemes can preserve these properties at the discrete level. In [26], the existence of weak solutions and upper bounds for the blow-up time for time-discrete (including the implicit Euler, BDF and Runge-Kutta methods) approximations of the parabolic-elliptic Keller-Segel models in the two-dimensional whole space are established. Liu, Li and Zhou proposed a numerical method in [34] which preserves both positivity and asymptotic limit, the proposed numerical method does not generate negative density if initialized properly under a less strict stability condition. Saito and Suzuki presented a finite difference scheme in [42] which satisfies the conservation of a discrete L^1 norm.

Some finite element methods are proposed in previous works. Saito presented a finite element scheme for parabolic-elliptic systems in [43] that satisfies both positivity and mass conservation properties. Under some assumptions on the regularity of solutions, the error estimates were established. Saito further constructed the finite element methods to the parabolic-parabolic systems in [44] and derived error analysis by using analytical semigroup theory. Gurusamy and Balachandran proposed a finite element method for parabolic-parabolic systems and established the existence of approximate solutions by using Schauder's fixed point theorem in [23]. Further the error estimates for the approximate solutions in H^1-norm were derived.

The discontinuous Galerkin methods can be also used to solve the Keller-Segel equations. Epshteyn and Kurganov developed a family of new interior penalty discontinuous Galerkin methods and proved error estimates for the proposed high-order discontinuous Galerkin methods in [15]. Epshteyn and Izmirlioglu further constructed a discontinuous Galerkin method for Keller-Segel model in [16] and obtained fully discrete error estimates for the proposed scheme. In 2017, Li, Shu and Yang applied the local discontinuous Galerkin (LDG) method to 2D Keller-Segel chemotaxis model in [30], they improved the results upon [15] and gave optimal rate of convergence under special finite element spaces before the blow-up occurs. In 2019, Guo, Li and Yang constructed a consistent numerical energy and prove the energy dissipation with the LDG discretization in [22].

Another important numerical methods for Keller-Segel models are finite volume methods since the positivity property can be naturally preserved. Filbet proposed in [18] a finite volume scheme for the parabolic-elliptic system, and by assuming the CFL condition $\chi\Delta t\mathcal{D}_{\mathcal{T},1} < 1$ and the initial datum $n^0 \geq a^0 > 0$, he proved existence and uniqueness of the numerical solution by using the Browder fixed point theorem, and showed that the numerical approximation converges to the exact solution under some assumptions. In 2016, Zhou and Saito proposed a finite volume scheme in [52], and established error estimates in L^p norm with a suitable $p > 2$ for the two dimensional case under some regularity assumptions of solutions and admissible mesh. By focusing on the radially symmetirc solution, they derived some *a prior* estimates to study the blow-up phenomenon of numerical solution.

There have been growing interests in positivity-preserving analysis for gradient flows with logarithmic energy potential. Some theoretical analysis of the positivity-preserving property and the energy stability have been explored for these numerical schemes for certain systems, such as Cahn-Hilliard systems in [7, 10, 11, 51, 12], the Poisson-Nernst-Planck-Cahn-Hilliard systems in [41], the Poisson-Nernst-Planck systems in [32], the thin film model without slope selection in [31] and a structure-preserving, operator splitting scheme for reaction-diffusion systems in [33]. The techniques of the higher order consistency analysis combined with rough error estimate and refined one have been presented in [32, 13, 14] which will be utilized in the following to obtain the convergence analysis.

Recently, a new approach for constructing positivity preserving schemes was proposed in [46]. The key for this approach is to write Δu as $\nabla \cdot (u\nabla \log u)$ in (1), and then use a convex splitting idea to construct mass conservative, bound preserving, and uniquely solvable schemes for (1)-(2) and for (1)-(3). The main purposes of this paper are to establish the convergence of the fully discrete scheme proposed in [46], and to show the finite-time blowup of numerical solutions under some conditions on the mass and moments of the initial data. More precisely, let u_h^k be an approximation of $u(\cdot, k\tau)$, where $\tau > 0$ is the time step and $k \in \mathbb{N}$. Let $\theta = (u_0, 1)$ be the initial mass, and $M^k = (u_h^k, \varphi)_h$ be the moment of u_h^k. Our first goal is to establish the error estimates for the fully discrete scheme proposed in [46] (cf. Theorem 6). Another important feature of the Keller-Segel system (1)-(2) is that the solution may blow up in finite time under certain conditions on the initial data. Our second goal is to show that the numerical solution will also blow up in finite time under similar conditions on the initial data (cf. Theorem 15). Many previous works (see [42, 43, 46]) show that the numerical solution seems to blow up under large initial data by several numerical experiments. However, there is still much to explore on the theoretical proof of blowup phenomenon besides the radial numerical solution in [52] mentioned before.

The rest of the paper is organized as follows. In Section 2, we recall some properties of the classical Keller-Segel equations, including its finite-time blowup behavior. In Section 3, we introduce the fully discrete scheme constructed in [46] and carry out a rigorous error analysis. In Section 4, we show that the numerical solution will blow up in finite time under suitable conditions on the initial data.

2. The Keller-Segel equations

In this section, we recall some properties for the Keller-Segel system (1)-(2) with no flux boundary conditions. In addition, we assume the initial value $u_0 \in W^{2,p}(\Omega)$, $1 < p < \infty$, and satisfies

$$u_0 \geq 0 \quad \text{and} \quad u_0 \not\equiv 0, \quad \forall x \in \Omega.$$

It was shown in [35] that there exist some $T > 0$ such that (1)-(2) is well posed in the time interval $[0, T]$. Moreover, it holds that

THEOREM 1. *The Keller-Segel system* (1)-(2) *satisfies the following properties:*

(i) Positivity preserving:

$$u(x,t) > 0, \quad v(x,t) > 0 \quad on \quad \Omega \times (0, T].$$

In fact, it is a consequence of the strong maximum principle [8] .

(ii) Mass conservation:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx, \quad \text{for all} \quad t > 0.$$

It is immediately follows from

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = \int_{\partial\Omega}\left(\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial v}{\partial \boldsymbol{n}}\right) dS = 0\ .$$

As a consequence of (i) and (ii), we obtain the conservation of L^1 norm, namely

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}.$$

(iii) Energy decay:

$$\frac{dF[u(t),v(t)]}{dt} = -\int_\Omega u|\nabla\cdot(\log u - \chi v)|^2\,dx \le 0,$$

where the free energy of (1)-(2) *is defined by*

$$F[u,v] = \int_\Omega \left(u(\log u - 1) - \chi uv + \frac{\chi}{2\alpha}|\nabla v|^2 + \frac{\chi}{2\alpha}v^2\right) dx\ .$$

The following result is shown in [37].

LEMMA 2. [37] *Let $q \in \Omega$ and $0 < r_1 < r_2 < dist(q, \partial\Omega)$, where $dist(q, \partial\Omega)$ is the distance between q and $\partial\Omega$. Then there exist positive constants C_1, C_2 depending only on r_1, r_2 and $dist(q, \partial\Omega)$ such that for $t \in (0, T]$,*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u(x,t)\Phi(x)\,dx \\
&\le 4\int_\Omega u_0(x)\,dx - \frac{\alpha\chi}{2\pi}\left(\int_\Omega u_0(x)\,dx\right)^2 + C_1\left(\int_\Omega u_0(x)\,dx\right)\left(\int_\Omega u(x,t)\Phi(x)\,dx\right) \\
&\quad + C_2\left(\int_\Omega u_0(x)\,dx\right)^{3/2}\left(\int_\Omega u(x,t)\Phi(x)\,dx\right)^{1/2},
\end{aligned}$$

where $\Phi(x) = \phi(|x - q|)$ with

$$\phi(r) = \begin{cases} r^2 & if \quad 0 \le r \le r_1, \\ a_1 r^2 + a_2 r + a_3 & if \quad r_1 < r \le r_2, \\ r_1 r_2 & if \quad r > r_2, \end{cases}$$

where $a_1 = -\dfrac{r_1}{r_2 - r_1}, \quad a_2 = \dfrac{2r_1 r_2}{r_2 - r_1}, \quad a_3 = -\dfrac{r_1^2 r_2}{r_2 - r_1}.$

The finite-time blowup behavior is then proved using the above result.

THEOREM 3. [37] *Assume that $\int_\Omega u_0(x)\,dx > 8\pi/(\alpha\chi)$, if $\int_\Omega u_0(x)|x - q|^2\,dx$ is sufficiently small, then the solution (u, v) to* (1)-(2) *blows up in finite time.*

Moreover, the following pointwise estimates for v is established in [19]. An application of the Neumann semigroup leads to

$$\begin{aligned} v(x,t) &\geq \left(\int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t+\frac{(\mathrm{diam}\Omega)^2}{4t})} \mathrm{d}t \right) \int_\Omega u(x,t) \mathrm{d}x \\ &= \|u_0\|_{L^1(\Omega)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t+\frac{(\mathrm{diam}\Omega)^2}{4t})} \mathrm{d}t, \end{aligned}$$

for all $x \in \Omega,\ t \in (0,T]$, whenever (u,v) solves (1)-(2) in $\Omega \times (0,T]$ for some $T > 0$.

In this paper, we assume that

$$u(x,t) \geq \epsilon_0 \quad \text{for some } \epsilon_0 > 0, \quad (x,t) \in \overline{\Omega} \times (0,T].$$

3. The fully discrete scheme and error estimates

In this section, we describe the fully discrete scheme in [46] for (1)-(2), construct the error equations and establish the error estimates.

We now give a precise description of our finite element space X_h. Given a triangulation $\mathcal{T}$ for Ω, we let Z_h consists of all the vertices excluding those where Dirichlet boundary conditions are prescribed. We define X_h to be the finite element space spanned by the piecewise linear continuous functions based on $\mathcal{T}$. Let e be a triangle of the triangulation $\mathcal{T}$, and $P_{e,i}, i = 1,2,3$ be its vertices, we define the quadrature formula

$$Q_e(f) = \frac{1}{3}\mathrm{area}(e) \sum_{i=1}^{3} f(P_{e,i}) \approx \int_e f \mathrm{d}x.$$

We recall that [48]

$$|Q_e(f) - \int_e f \mathrm{d}x| \leq Ch^2 \sum_{|\alpha|=2} \|D^\alpha f\|_{L^1(e)}.$$

We then define the discrete inner product in X_h by

$$(u,v)_h = \sum_{e \in \mathcal{T}} Q_e(uv),$$

the corresponding norm is defined by $\|\cdot\|_{L^2_h}$. We have the following estimates in [48] for the quadrature error.

LEMMA 4. *Let $\epsilon_h(\cdot,\cdot) = (\cdot,\cdot)_h - (\cdot,\cdot)$ denote the quadrature error, then we have*

$$|\epsilon_h(u,v)| \leq Ch^2 \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)},\ \forall\ u,v \in X_h.$$

Applying Lemma 4, the norm $\|\cdot\|_{L^2_h}$ has the following property

$$\|\nabla u\|_{L^2_h(\Omega)} = \|\nabla u\|_{L^2(\Omega)},\ \forall\ u \in X_h. \tag{4}$$

Let $I_h : C(\Omega) \to X_h$ be the Lagrange interpolation operator, which has the approximation property [4] that for all $g \in H^2(\Omega) \cap H^1_0(\Omega)$,

$$\|I_h g - g\|_{L^2(\Omega)} \leq Ch^2 \|g\|_{H^2(\Omega)} \quad \text{and} \quad \|\nabla(I_h g - g)\|_{L^2(\Omega)} \leq Ch\|g\|_{H^2(\Omega)}. \tag{5}$$

The fully discrete scheme proposed in [46] for (1)-(2) is to find $(u_h^{k+1}, v_h^{k+1}) \in X_h \times X_h$ such that for all $(\varphi,\psi) \in X_h \times X_h$,

$$\left(\overline{\partial} u_h^k, \varphi\right)_h + \left(u_h^k \nabla \left(I_h \log u_h^{k+1}\right), \nabla\varphi\right)_h - \chi \left(u_h^k \nabla v_h^k, \nabla \varphi\right)_h = 0, \tag{6}$$

$$(\nabla v_h^{k+1}, \nabla\psi) + (v_h^{k+1}, \psi) - \alpha(u_h^{k+1}, \psi)_h = 0. \tag{7}$$

Here, the $(\cdot,\cdot)$ represents the usual L^2 inner product, the $(\cdot,\cdot)_h$ is the discrete inner product defined above, and $\overline{\partial} u_h^k$ is the forward Euler difference quotient approximating to $\partial_t u(t^k)$ defined by

$$\overline{\partial} u_h^k = \frac{u_h^{k+1} - u_h^k}{\tau}.$$

In this setting, the authors in [46] proved the following.

LEMMA 5. [46] *The numerical scheme* (6)-(7) *has the following properties:*

(i) Unique solvability:
The scheme (6)-(7) *has a unique solution* $(u_h^{k+1}, v_h^{k+1}) \in X_h \times X_h$.

(ii) Positivity preserving: If $u^k > 0$*, then* $u^{k+1} > 0$.

(iii) Mass conservation:

$$(u_h^{k+1}, 1)_h = (u_h^k, 1)_h = (u_h^0, 1)_h,$$

It is immediately derived by taking $\varphi = 1$ *in* (6).

(iv) Energy decay:

$$\tilde{F}^{k+1} - \tilde{F}^k \le -\tau \left(u_h^k \nabla (I_h(\log u_h^{k+1} - \chi v_h^k)), \nabla (I_h(\log u_h^{k+1} - \chi v_h^k))\right)_h \le 0,$$

where the discrete energy of (6)-(7) *is defined by*

$$\tilde{F}^k[u,v] = (u_h^k(\log u_h^k - 1), 1)_h - \chi(u_h^k, v_h^k)_h + \frac{\chi}{2\alpha}\|\nabla v_h^k\|^2 + \frac{\chi}{2\alpha}\|v_h^k\|^2.$$

We denote by (u,v) the exact solution pair to the original equations (1)-(3), and all the upper bounds for the exact solution are denoted as C. We set $u^k = u(t_k)$, $v^k = v(t_k)$, and denote

$$e_u^k := u^k - u_h^k, \quad e_v^k := v^k - v_h^k, \quad \forall k \in \mathbb{N}.$$

The following theorem is the main result of this section.

THEOREM 6. *Assume* $u_0 \in W^{2,p}(\Omega)(1 < p < \infty)$ *and the exact solution pair* (u,v) *is smooth enough for a fixed final time* $T > 0$*. Then, provided* τ *and* h *are sufficiently small and under the mild mesh-sizes requirement* $\tau \le Ch$*, we have the following error estimates*

$$\|e_u^m\|_{L_h^2(\Omega)} + \|e_v^m\|_{H_h^1(\Omega)} + (\tau \sum_{k=0}^{m-1} \|\nabla e_u^{k+1}\|_{L_h^2(\Omega)}^2)^{1/2} \le C(\tau + h), \quad \forall m \in \mathbb{N},$$

where $t_m = m\tau \le T$*,* $C > 0$ *is independent of* τ *and* h*.*

The proof for this theorem will be carried out with a sequence of procedures that we describe below.

REMARK 7. *The mesh-sizes requirement* $\tau \le Ch$ *in Theorem 6 is proposed to obtain a higher order consistency analysis via a perturbation argument, which is needed to get the separation property and the* $W^{1,\infty}$ *bound for the numerical solution.*

3.1. Higher order consistent approximation to (6)**-**(7)**.** In this subsection, we apply the perturbation argument method in [32] to the finite element scheme to construct f_1, f_2, f_3 such that

$$\hat{u} := u + hf_1 + h^2 f_2 + h^3 f_3,$$

is consistent with the given numerical scheme (6)-(7) at the order $O(h^4)$. The following lemma is used to construct f_1, f_2, f_3 and the proof is given in Appendix.

By applying a perturbation argument, a higher order $O(h^4)$ consistency is satisfied for $\hat{u}$, which is needed to obtain the separation property and a $W^{1,\infty}$ bound for the numerical solution.

LEMMA 8. *Suppose that $\tau \le Ch$ and u is smooth enough, then there exist bounded smooth functions f_1, f_2, f_3, such that $\hat{u} = u + hf_1 + h^2 f_2 + h^3 f_3$ satisfies*

$$(8) \qquad \left(\overline{\partial}\hat{u}^k, \varphi\right)_h + \left(\hat{u}^k \nabla I_h \log \hat{u}^{k+1}, \nabla\varphi\right)_h - \chi\left(\hat{u}^k \nabla A_h \hat{u}^k, \nabla\varphi\right)_h = \langle \mathcal{R}^k(\hat{u}), \varphi\rangle,$$

for all $\varphi \in X_h$, $k \in \mathbb{N}$, where $A_h = \alpha(-\Delta_h + I)^{-1}Q_h$ and $\langle\cdot,\cdot\rangle$ denotes the duality product satisfying

$$(9) \qquad |\langle \mathcal{R}^k(\hat{u}^k), \varphi\rangle| \le Ch^4 \|\varphi\|_{H^1},$$

where C depends on the regularity of the solution u.

REMARK 9. *Under the conditions that the exact solution $u \ge \epsilon_0$ for some $\epsilon_0 > 0$, and h is sufficiently small, we obtain that*

$$(10) \qquad \hat{u} \ge \frac{\epsilon_0}{2}.$$

Since the correction functions $f_j, j = 1, 2, 3$ only depend on the exact solution u, they are bounded in $W^{1,\infty}$ norm. Then, we can obtain the following $W^{1,\infty}$ bound for $\hat{u}$:

$$(11) \qquad \|\hat{u}^k\|_{W^{1,\infty}} \le C, \quad \forall k \ge 0.$$

3.2. A rough error estimate. In this subsection, we derive the strict separation property and a uniform $W^{1,\infty}$ bound for the numerical solution. Define an alternative error function:

$$\tilde{u}^k := I_h \hat{u}^k - u_h^k, \quad \forall k \in \mathbb{N}.$$

Subtracting the numerical scheme (6) from the consistency estimate (8) implies that

$$(12) \qquad (\overline{\partial}\tilde{u}^k, \varphi)_h = -(\tilde{u}^k \nabla \mathcal{V}_u^k + u_h^k \nabla \tilde{\mu}_u^k, \nabla\varphi)_h + \langle \mathcal{R}^k, \varphi\rangle,$$

where

$$\mathcal{V}_u^k := I_h \log \hat{u}^{k+1} - \chi A_h \hat{u}^k,$$
$$\tilde{\mu}_u^k := I_h \log \hat{u}^{k+1} - I_h \log u_h^{k+1} - \chi A_h \tilde{u}^k.$$

Since $\mathcal{V}_u^k$ only depends on the exact solution, we can assume

$$(13) \qquad \|\mathcal{V}_u^k\|_{W^{2,\infty}} \le C.$$

LEMMA 10. *The numerical solutions of the scheme* (6)-(7) *have the strict separation property and a uniform $W^{1,\infty}$ bound:*

$$u_h^k \ge \frac{\epsilon_0}{4}, \ \|u_h^k\|_{W^{1,\infty}} \le C^*,$$

for all $0 \le k \le T/\tau$, where ϵ_0 and C^* are positive constants.

Proof. We shall first make the following assumption at the previous time step:

$$(14) \qquad \|\tilde{u}^k\|_2 \le C(\tau^{15/4} + h^{15/4}).$$

Then, we will demonstrate that such an assumption will be recovered at the next time step.

Using the inverse inequality and $\tau \le Ch$, we obtain a $W^{1,\infty}$ bound for the numerical error functions:

$$(15) \qquad \|\tilde{u}^k\|_\infty \le \frac{C\|\tilde{u}^k\|_2}{h} \le \frac{C(\tau^{15/4} + h^{15/4})}{h} \le C(\tau^{11/4} + h^{11/4}) \le 1,$$

$$\|\nabla \tilde{u}^k\|_\infty \le \frac{C\|\tilde{u}^k\|_\infty}{h} \le \frac{C(\tau^{11/4}+h^{11/4})}{h} \le C(\tau^{7/4}+h^{7/4}) \le 1.$$

A combination of the above with (11), we get a $W^{1,\infty}$ bound for u_h^k at the previous time step:

$$\|u_h^k\|_\infty \le \|\hat{u}^k\|_\infty + \|\tilde{u}^k\|_\infty \le C+1 \le C^*,$$
$$\|\nabla u_h^k\|_\infty \le \|\nabla \hat{u}^k\|_\infty + \|\nabla \tilde{u}^k\|_\infty \le C+1 \le C^*.$$

Because of (15), taking τ and h sufficiently small, we have

$$\|\tilde{u}^k\|_\infty \le C(\tau^{11/4}+h^{11/4}) \le \frac{\epsilon_0}{4}.$$

Then the strict separation property is valid for u_h^k:

$$u_h^k \ge \hat{u}^k - \|\tilde{u}^k\|_\infty \ge \frac{\epsilon_0}{4}. \tag{16}$$

Taking $\varphi = \tau\tilde{\mu}_u^k$ in (12) leads to

$$(\tilde{u}^{k+1},\tilde{\mu}_u^k)_h + \tau(u_h^k\nabla\tilde{\mu}_u^k,\nabla\tilde{\mu}_u^k)_h = (\tilde{u}^k,\tilde{\mu}_u^k)_h - \tau(\tilde{u}^k\nabla\mathcal{V}_u^k,\nabla\tilde{\mu}_u^k)_h + \tau\langle\mathcal{R}^k,\tilde{\mu}_u^k\rangle. \tag{17}$$

Now we deal with the left hand side of (17). To proceed the first term on the left hand side of (17), using the Hölder inequality, we have

$$\begin{aligned}
(\tilde{u}^{k+1},\tilde{\mu}_u^k)_h =& (\tilde{u}^{k+1}, I_h\log\hat{u}^{k+1} - I_h\log u_h^{k+1})_h + (\tilde{u}^{k+1}, -\chi A_h\tilde{u}^k)_h\\
&\ge (\tilde{u}^{k+1}, \frac{1}{\zeta}\tilde{u}^{k+1})_h - C\|\tilde{u}^{k+1}\|_2\|\tilde{u}^k\|_2\\
&\ge \frac{1}{C^*}\|\tilde{u}^{k+1}\|_2^2 - \frac{1}{C^*}\|\tilde{u}^{k+1}\|_2^2 - C\|\tilde{u}^k\|_2^2\\
&= -C\|\tilde{u}^k\|_2^2,
\end{aligned} \tag{18}$$

where ζ lies between $\hat{u}^{k+1}$ and u_h^{k+1}. As for the second term on the left hand side of (17), using the strict separation property of the numerical solution (16), we have

$$(u_h^k\nabla\tilde{\mu}_u^k,\nabla\tilde{\mu}_u^k)_h \ge \frac{\epsilon_0}{4}\|\nabla\tilde{\mu}_u^k\|_2^2. \tag{19}$$

Next, we deal with the right hand side of (17). We apply the Hölder inequality and the Young inequality:

$$|(\tilde{u}^k,\tilde{\mu}_u^k)_h| \le \|\tilde{u}^k\|_2\|\tilde{\mu}_u^k\|_2 \le C\|\tilde{u}^k\|_2\|\nabla\tilde{\mu}_u^k\|_2 \le \frac{6C}{\epsilon_0\tau}\|\tilde{u}^k\|_2^2 + \frac{\epsilon_0\tau}{24}\|\nabla\tilde{\mu}_u^k\|_2^2. \tag{20}$$

An application of the Cauchy-Schwarz inequality and (13) leads to

$$\begin{aligned}
|-(\tilde{u}^k\nabla\mathcal{V}_u^k,\nabla\tilde{\mu}_u^k)_h| &\le \|\tilde{u}^k\nabla\mathcal{V}_u^k\|_2\|\nabla\tilde{\mu}_u^k\|_2 \le \|\nabla\mathcal{V}_u^k\|_\infty\|\tilde{u}^k\|_2\|\nabla\tilde{\mu}_u^k\|_2\\
&\le \frac{6C}{\epsilon_0}\|\tilde{u}^k\|_2^2 + \frac{\epsilon_0}{24}\|\nabla\tilde{\mu}_u^k\|_2^2.
\end{aligned}$$

Using the inequality (9), we have

$$|\langle\mathcal{R}^k,\tilde{\mu}_u^k\rangle| \le Ch^4\|\nabla\tilde{\mu}_u^k\|_2 \le \frac{6C}{\epsilon_0}h^8 + \frac{\epsilon_0}{24}\|\nabla\tilde{\mu}_u^k\|_2^2. \tag{21}$$

Substitution of (18)-(21) into (17) leads to

$$\frac{\epsilon_0}{4}\tau\|\nabla\tilde{\mu}_u^k\|_2^2 - C\|\tilde{u}^k\|_2^2 \le \frac{6C}{\epsilon_0\tau}\|\tilde{u}^k\|_2^2 + \frac{6C\tau}{\epsilon_0}\|\tilde{u}^k\|_2^2 + \frac{6C}{\epsilon_0}h^8\tau + \frac{\epsilon_0\tau}{8}\|\nabla\tilde{\mu}_u^k\|_2^2.$$

Then we have

$$\|\nabla\tilde{\mu}_u^k\|_2 \le C(\tau^{11/4}+h^{11/4}). \tag{22}$$

Again, taking $\varphi = \tilde{u}^{k+1} - \tilde{u}^k$ in the error equation (12) leads to

$$\|\tilde{u}^{k+1} - \tilde{u}^k\|_2 \le \tau \left(\|\nabla \cdot (\tilde{u}^k \nabla \mathcal{V}_u^k)\|_2 + \|\nabla \cdot (u_h^k \nabla \tilde{\mu}_u^k)\|_2 + \frac{1}{h}\|\mathcal{R}^k\|_{H^{-1}} \right). \tag{23}$$

Now we estimate the first term on the right hand side of (23). Using the Young inequality, we have

$$\begin{aligned}
\|\nabla \cdot (\tilde{u}^k \nabla \mathcal{V}_u^k)\|_2 &\le \|\tilde{u}^k \Delta \mathcal{V}_u^k\|_2 + \|\nabla \tilde{u}^k \nabla \mathcal{V}_u^k\|_2 \\
&\le \|\Delta \mathcal{V}_u^k\|_\infty \|\tilde{u}^k\|_2 + \|\nabla \mathcal{V}_u^k\|_\infty \|\nabla \tilde{u}^k\|_2 \\
&\le C(\|\tilde{u}^k\|_2 + \|\nabla \tilde{u}^k\|_2) \\
&\le C(\tau^{11/4} + h^{11/4}),
\end{aligned} \tag{24}$$

where (14) has been used in the last inequality. For the second term on the right hand side of (23), we have

$$\begin{aligned}
\|\nabla \cdot (u_h^k \nabla \tilde{\mu}_u^k)\|_2 &\le \|\nabla u_h^k \nabla \tilde{\mu}_u^k\|_2 + \|u_h^k \Delta \tilde{\mu}_u^k\|_2 \\
&\le \|\nabla u_h^k\|_\infty \|\nabla \tilde{\mu}_u^k\|_2 + \|u_h^k\|_\infty \|\Delta \tilde{\mu}_u^k\|_2 \\
&\le C^*(\|\nabla \tilde{\mu}_u^k\|_2 + \|\Delta \tilde{\mu}_u^k\|_2) \\
&\le C(\tau^{7/4} + h^{7/4}),
\end{aligned} \tag{25}$$

where (22) and the inverse inequality have been used in the last inequality. Substitution of (24)-(25) and (9) into (23) leads to

$$\|\tilde{u}^{k+1} - \tilde{u}^k\|_2 \le C(\tau^{11/4} + h^{11/4}).$$

Then, we can obtain a rough estimate for $\tilde{u}^{k+1}$:

$$\|\tilde{u}^{k+1}\|_2 \le \|\tilde{u}^{k+1} - \tilde{u}^k\|_2 + \|\tilde{u}^k\|_2 \le C(\tau^{11/4} + h^{11/4}).$$

An application of the inverse inequality and $\tau \le Ch$ implies that

$$\begin{aligned}
\|\tilde{u}^{k+1}\|_\infty &\le \frac{C\|\tilde{u}^{k+1}\|_2}{h} \le C(\tau^{7/4} + h^{7/4}) \le 1, \\
\|\nabla \tilde{u}^{k+1}\|_\infty &\le \frac{C\|\tilde{u}^{k+1}\|_\infty}{h} \le C(\tau^{3/4} + h^{3/4}) \le 1.
\end{aligned}$$

We take τ and h sufficiently small such that

$$\|\tilde{u}^{k+1}\|_\infty \le C(\tau^{7/4} + h^{7/4}) \le \frac{\epsilon_0}{4}.$$

A combination of above with (10) leads to the strict separation property:

$$u_h^{k+1} \ge \hat{u}^{k+1} - \|\tilde{u}^{k+1}\|_\infty \ge \frac{\epsilon_0}{4}.$$

In addition, we can obtain the following $W^{1,\infty}$ bound for the numerical solution u_h at time step t^{k+1}:

$$\begin{aligned}
\|u_h^{k+1}\|_\infty &\le \|\hat{u}^{k+1}\|_\infty + \|\tilde{u}^{k+1}\|_\infty \le C + 1 \le C^*, \\
\|\nabla u_h^{k+1}\|_\infty &\le \|\nabla \hat{u}^{k+1}\|_\infty + \|\nabla \tilde{u}^{k+1}\|_\infty \le C + 1 \le C^*,
\end{aligned}$$

which completes the proof. □

3.3. Proof of Theorem 6. In this subsection, we shall make use of the strict separation property and the uniform $W^{1,\infty}$ bound for the numerical solution derived in the above to prove Theorem 6.

We recall the following inverse estimate in [4, p.111, Lemma 4.5.3].

LEMMA 11. *Given a quasi-uniform triangulation $\mathcal{T}$ on domain $\Omega \subset \mathbb{R}^n$, and X_h be a finite-dimensional subspace of $W^{l,p}(K) \cap W^{m,q}(K)$, where $1 \le p \le \infty, 1 \le q \le \infty$ and $0 \le m \le l$. Then there exists a positive constant C such that for all $u \in X_h$, we have*

$$\|u\|_{W^{l,p}(K)} \le Ch^{m-l+n/p-n/q}\|u\|_{W^{m,q}(K)},$$

where C is independent of u.

We will also use the following discrete Gronwall inequality in [45, 50].

LEMMA 12. *Assume that $\tau > 0, B > 0, \{a_k\}, \{b_k\}, \{\gamma_k\}$ are non-negative sequences such that*

$$a_m + \tau \sum_{k=1}^{m} b_k \le \tau \sum_{k=1}^{m-1} \gamma_k a_k + B,\ \ m \ge 1.$$

Then

$$a_m + \tau \sum_{k=1}^{m} b_k \le B\exp\left(\tau \sum_{k=1}^{m-1} \gamma_k\right),\ \ m \ge 1.$$

A weak formulation of (1)-(2) is

$$(u_t, \varphi) + (\nabla u, \nabla \varphi) - \chi(u\nabla v, \nabla\varphi) = 0,\ \ \forall \varphi \in H_0^1(\Omega), \tag{26}$$

$$(\nabla v, \nabla \psi) + (v, \psi) = \alpha(u, \psi),\ \ \forall \psi \in H_0^1(\Omega). \tag{27}$$

Substituting $I_h u(t)$ into (26) at $t = t_{k+1}$, we have

$$\begin{aligned}
&\left(\overline{\partial} I_h u^k,\ \varphi\right)_h + \left(\nabla I_h u^{k+1},\ \nabla\varphi\right)_h - \chi\left(I_h u^k \nabla v^k,\ \nabla\varphi\right)_h \\
&= \left(\overline{\partial} u^k - u_t^{k+1},\ \varphi\right) + \left(\overline{\partial} I_h u^k - \overline{\partial} u^k,\ \varphi\right) + \left(\nabla(I_h u^{k+1} - u^{k+1}),\ \varphi\right) \\
&\quad + \chi\left((u^k - I_h u^k)\nabla v^k,\ \varphi\right) + \chi\left(u^{k+1}\nabla v^{k+1} - u^k \nabla v^k,\ \varphi\right) \\
&\quad + \left(\overline{\partial} I_h u^k, \varphi\right)_h - \left(\overline{\partial} I_h u^k, \varphi\right) + \left(\nabla I_h u^{k+1}, \nabla\varphi\right)_h - \left(\nabla I_h u^{k+1}, \nabla\varphi\right) \\
&\quad + \chi\left(I_h u^k \nabla v^k, \nabla\varphi\right) - \chi\left(I_h u^k \nabla v^k, \nabla\varphi\right)_h \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 := R_1,
\end{aligned} \tag{28}$$

where R_1 represents the truncation error. Similarly, substituting $I_h u(t), I_h v(t)$ into (27) at $t = t_k$ leads to

$$\begin{aligned}
(\nabla I_h v^k,\ \nabla\psi) + (I_h v^k,\ \psi) =& \alpha(I_h u^k,\ \psi) + (\nabla(I_h v(t_k) - v^k),\ \psi) \\
&+ (I_h v^k - v^k,\ \psi) + \alpha(u^k - I_h u^k,\ \psi).
\end{aligned} \tag{29}$$

We rewrite the numerical scheme (6) as

$$\begin{aligned}
&\left(\overline{\partial} u_h^k, \varphi\right)_h + \left(\nabla u_h^{k+1}, \nabla\varphi\right)_h - \chi\left(u_h^k \nabla v_h^k, \nabla\varphi\right)_h \\
&= \left(u_h^k \nabla(I - I_h)\log u_h^{k+1}, \nabla\varphi\right)_h + \left((u_h^{k+1} - u_h^k)\nabla \log u_h^{k+1}, \nabla\varphi\right)_h \\
&:= I_9 + I_{10} := R_2.
\end{aligned} \tag{30}$$

We split the error functions as

$$\begin{aligned}
e_u^k &= u^k - u_h^k = (u^k - I_h u^k) + (I_h u^k - u_h^k) := \rho_u^k + \sigma_u^k, \\
e_v^k &= v^k - v_h^k = (v^k - I_h v^k) + (I_h v^k - v_h^k) := \rho_v^k + \sigma_v^k.
\end{aligned}$$

Then using the property of the interpolation (5), we have

$$\begin{aligned}\|\rho_u^k\|_{L^2(\Omega)} + h\|\nabla\rho_u^k\|_{L^2(\Omega)} &\le Ch^2\|u\|_{H^2(\Omega)},\\ \|\rho_v^k\|_{L^2(\Omega)} + h\|\nabla\rho_v^k\|_{L^2(\Omega)} &\le Ch^2\|v\|_{H^2(\Omega)}.\end{aligned} \tag{31}$$

Subtracting the numerical scheme formulation (30) and (7) from the weak form (28) and (29), we obtain the following error equations:

$$\left(\overline{\partial}\sigma_u^k,\ \varphi\right)_h + \left(\nabla\sigma_u^{k+1},\ \nabla\varphi\right)_h = I_{11} + R_1 - R_2, \tag{32}$$

$$\begin{aligned}(\nabla\sigma_v^k,\ \nabla\psi) + (\sigma_v^k,\ \psi) = \alpha(\sigma_u^k,\ \psi) - \alpha\epsilon_h(u_h^k,\ \psi) + (\nabla(I_hv - v),\ \psi)\\ + (I_hv - v,\ \psi) + \alpha(u - I_hu,\ \psi),\end{aligned} \tag{33}$$

for all $\varphi, \psi \in X_h, k \ge 1$, where R_1, R_2 are defined before and I_{11} is defined as follows

$$I_{11} = \chi\left(I_hu(t_k)\nabla v^k - u_h^k\nabla v_h^k,\ \nabla\varphi\right)_h.$$

Taking $\varphi = \sigma_u^{k+1}$ in (32) leads to

$$\frac{1}{2\tau}\left(\|\sigma_u^{k+1}\|_{L_h^2}^2 - \|\sigma_u^k\|_{L_h^2}^2 + \|\sigma_u^{k+1} - \sigma_u^k\|_{L_h^2}^2\right) + \|\nabla\sigma_u^{k+1}\|_{L_h^2}^2 = I_{11} + R_1 - R_2. \tag{34}$$

Now we estimate the terms on the right-hand side of (35). For the first term I_{11}, applying the Cauchy-Schwarz inequality and the Young inequality yields

$$\begin{aligned}|I_{11}| &= |\chi\left(\sigma_u^k\nabla v^k + u_h^k\nabla e_v^k,\ \nabla\sigma_u^{k+1}\right)_h|\\ &\le \|\sigma_u^k\nabla v^k\|_{L_h^2(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L_h^2(\Omega)} + \|u_h^k\nabla e_v^k\|_{L_h^2(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L_h^2(\Omega)}\\ &\le C\|\sigma_u^k\|_{L_h^2(\Omega)}^2 + \frac{1}{20}\|\nabla\sigma_u^{k+1}\|_{L_h^2(\Omega)}^2 + C\|\nabla e_v^k\|_{L_h^2(\Omega)}^2.\end{aligned} \tag{35}$$

In order to estimate $\|\nabla e_v^k\|_{L_h^2(\Omega)}^2$ above, taking $\psi = \sigma_v^k$ in (33) and applying Lemma 4 leads to

$$\begin{aligned}\|\nabla\sigma_v^k\|_{L^2(\Omega)}^2 + \|\sigma_v^k\|_{L^2(\Omega)}^2 &\le \alpha\|\sigma_u^k\|_{L^2(\Omega)}\|\sigma_v^k\|_{L^2(\Omega)} + Ch^2\|\nabla u_h^k\|_{L^2(\Omega)}\|\nabla\sigma_v^k\|_{L^2(\Omega)}\\ &\quad + Ch\|v\|_{H^2(\Omega)}\|\sigma_v^k\|_{L^2(\Omega)} + Ch^2\|u\|_{H^2(\Omega)}\|\sigma_v^k\|_{L^2(\Omega)}\\ &\le C\|\sigma_u^k\|_{L^2(\Omega)}^2 + \|\sigma_v^k\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla\sigma_v^k\|_{L^2(\Omega)}^2 + Ch^2,\end{aligned}$$

where the property of the interpolation has been used in the first inequality. Thus we obtain the following estimate for $\|\nabla\sigma_v^k\|_{L^2(\Omega)}$:

$$\|\nabla\sigma_v^k\|_{L^2(\Omega)}^2 \le C\|\sigma_u^k\|_{L^2(\Omega)}^2 + Ch^2. \tag{36}$$

Applying Lemma 4 indicates that

$$\|\nabla\sigma_v^k\|_{L_h^2(\Omega)}^2 \le C\|\sigma_u^k\|_{L_h^2(\Omega)}^2 + Ch^2\|\nabla\sigma_u^k\|_{L^2(\Omega)}^2 + Ch^2.$$

A combination of the above estimates for $\|\nabla\sigma_v^k\|_{L_h^2(\Omega)}$ with (31) leads to

$$\|\nabla e_v^k\|_{L_h^2(\Omega)}^2 \le C\|\sigma_u^k\|_{L_h^2(\Omega)}^2 + Ch^2\|\nabla\sigma_u^k\|_{L^2(\Omega)}^2 + Ch^2. \tag{37}$$

Substitution of above into (35) leads to

$$|I_{11}| \le C\|\sigma_u^k\|_{L_h^2(\Omega)}^2 + \frac{1}{20}\|\nabla\sigma_u^{k+1}\|_{L_h^2(\Omega)}^2 + Ch^2\|\nabla\sigma_u^k\|_{L^2(\Omega)}^2 + Ch^2. \tag{38}$$

Next we estimate the second term $R_1 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8$. For the first term, we can derive [48]

$$
\begin{aligned}
|I_1| &= |\left(\overline{\partial} u^k - u_t^{k+1},\ \sigma_u^{k+1}\right)| \le C\tau^{1/2}\left(\int_{t_k}^{t_{k+1}} \|u_{tt}\|_{L^2(\Omega)}^2 \mathrm{d}s\right)^{1/2} \|\sigma_u^{k+1}\|_{L^2(\Omega)} \\
&\le C\tau \int_{t_k}^{t_{k+1}} \|u_{tt}\|_{L^2(\Omega)}^2 \mathrm{d}s + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{39}
$$

An application of the property of the interpolation and the Young inequality leads to

$$
\begin{aligned}
|I_2| =& |\left(\overline{\partial} I_h u^k - \overline{\partial} u^k,\ \sigma_u^{k+1}\right)| \\
\le& \frac{1}{\tau}\|I_h(u^{k+1} - u^k) - (u^{k+1} - u^k)\|_{L^2(\Omega)} \|\sigma_u^{k+1}\|_{L^2(\Omega)} \\
=& \frac{1}{\tau}\|(I_h u_t - u_t)\tau\|_{L^2(\Omega)} \|\sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch^2 \|u_t\|_{H^2(\Omega)} \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch^4 + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{40}
$$

Using the Cauchy-Schwarz inequality and the property of the interpolation, we have

$$
\begin{aligned}
|I_3| =& |\left(\nabla(I_h u^{k+1} - u^{k+1}),\ \nabla \sigma_u^{k+1}\right)| \\
\le& \|\nabla(I_h u^{k+1} - u^{k+1})\|_{L^2(\Omega)} \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch\|u^{k+1}\|_{H^2(\Omega)} \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch^2 + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{41}
$$

Similarly, we have

$$
\begin{aligned}
|I_4| =& |\chi\left((u^k - I_h u^k)\nabla v^k,\ \nabla \sigma_u^{k+1}\right)| \\
\le& \chi\|u^k - I_h u^k\|_{L^2(\Omega)} \|\nabla v^k\|_{L^\infty(\Omega)} \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch^2 \|u^k\|_{H^2(\Omega)} \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& Ch^4 + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{42}
$$

Now we apply the Cauchy-Schwarz inequality and the Young inequality:

$$
\begin{aligned}
|I_5| =& |\chi\left(\tau u_t \nabla v^{k+1} + \tau u^k \nabla v_t,\ \nabla \sigma_u^{k+1}\right)| \\
\le& C\left(\|u_t\|_{L^\infty(\Omega)} \|\nabla v\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} \|\nabla v_t\|_{L^\infty(\Omega)}\right)\tau \|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)} \\
\le& C\tau^2 + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{43}
$$

Notice that

$$
\begin{aligned}
I_6 &= \frac{1}{\tau}\left(I_h u^{k+1} - I_h u^k,\ \sigma_u^{k+1}\right) - \frac{1}{\tau}\left(I_h u^{k+1} - I_h u^k,\ \sigma_u^{k+1}\right)_h \\
&= -\epsilon_h\left(I_h u_t,\ \sigma_u^{k+1}\right),
\end{aligned}
$$

therefore, using Lemma 4 leads to the following estimate for I_6:

$$
|I_6| = |\epsilon_h(I_h u_t, \sigma_u^{k+1})| \le Ch^2 \|\nabla I_h u_t\| \|\nabla \sigma_u^{k+1}\| \le Ch^4 + \frac{1}{20}\|\nabla \sigma_u^{k+1}\|_{L^2(\Omega)}^2. \tag{44}
$$

We recall the quadrature formula defined before and Lemma 4, and arrive at

$$
I_7 = \left(I_h u^{k+1}, \nabla \sigma_u^{k+1}\right)_h - \left(I_h u^{k+1}, \nabla \sigma_u^{k+1}\right) = 0, \tag{45}
$$

An application of the Cauchy-Schwarz inequality and the property of the interpolation leads to

$$
\begin{aligned}
|I_8| &= |\chi\left(I_h u^k \nabla v^k, \nabla\sigma_u^{k+1}\right) - \chi\left(I_h u^k \nabla v^k, \nabla\sigma_u^{k+1}\right)_h| \\
&= |\chi\left(I_h u^k \nabla(I-I_h) v^k, \nabla\sigma_u^{k+1}\right) - \chi\left(I_h u^k \nabla(I-I_h) v^k, \nabla\sigma_u^{k+1}\right)_h| \\
&\leq Ch\|I_h u^k\|_{L^\infty}\|u^k\|_{L^2(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L^2(\Omega)} \\
&\leq Ch^2 + \frac{1}{20}\|\nabla\sigma_u^{k+1}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{46}
$$

Substituting estimates (39)-(46) into R_1 and applying the property (4), we obtain

$$
\begin{aligned}
|R_1| \leq & C\tau\int_{t_k}^{t_{k+1}}\|u_{tt}\|^2_{L^2(\Omega)}\mathrm{d}s + \frac{7}{20}\|\nabla\sigma_u^{k+1}\|^2_{L^2(\Omega)} + Ch^2 + C\tau^2 \\
= & C\tau\int_{t_k}^{t_{k+1}}\|u_{tt}\|^2_{L^2(\Omega)}\mathrm{d}s + \frac{7}{20}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)} + Ch^2 + C\tau^2.
\end{aligned} \tag{47}
$$

It remains to bound each term of $R_2 = I_9 + I_{10}$. Now we use the Cauchy-Schwarz inequality and the Young inequality:

$$
\begin{aligned}
|I_9| =& |\left(u_h^k \nabla(I-I_h)\log u_h^{k+1}, \nabla\sigma_u^{k+1}\right)_h| \\
&\leq \|u_h^k \nabla(I-I_h)\log u_h^{k+1}\|_{L^2_h(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)} \\
&\leq \|u_h^k\|_{L^\infty(\Omega)}\|\nabla(I-I_h)\log u_h^{k+1}\|_{L^2_h(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)} \\
&\leq Ch^2 + \frac{1}{20}\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)},
\end{aligned} \tag{48}
$$

where we have used the following inequality:

$$
\begin{aligned}
\|\nabla(I-I_h)\log u_h^{k+1}\|_{L^2_h(\Omega)} &\leq Ch\sum_e\sum_{|\alpha|=2}(\int_e |D^\alpha \log u_h^{k+1}|^2\mathrm{d}x)^{1/2} \\
&\leq Ch\sum_e(\int_e \frac{1}{(u_h^{k+1})^4}|\nabla u_h^{k+1}|^4\mathrm{d}x)^{1/2} \\
&\leq Ch\frac{1}{(\epsilon_0/4)^4}\sum_e(\int_e |\nabla u_h^{k+1}|^4\mathrm{d}x)^{1/2} \\
&\leq Ch.
\end{aligned}
$$

An application of the strict separation property and the $W^{1,\infty}$ bound of the numerical solution leads to

$$
\begin{aligned}
|I_{10}| =& |\left((u_h^{k+1}-u_h^k)\nabla\log u_h^{k+1}, \nabla\sigma_u^{k+1}\right)_h| \\
&\leq \|\frac{u_h^{k+1}-u_h^k}{u_h^{k+1}}\nabla u_h^{k+1}\|_{L^2_h(\Omega)}\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)} \\
&\leq \frac{\|\nabla u_h^{k+1}\|_{L^\infty(\Omega)}}{\epsilon_0/4}\left(\|\sigma_u^{k+1}-\sigma_u^k\|_{L^2_h(\Omega)} + C\tau\right)\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)} \\
&\leq C(\|\sigma_u^{k+1}-\sigma_u^k\|_{L^2_h(\Omega)} + C\tau)\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)} \\
&\leq C\|\sigma_u^{k+1}-\sigma_u^k\|^2_{L^2_h(\Omega)} + C\tau^2 + \frac{1}{20}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)}.
\end{aligned} \tag{49}
$$

Combining the estimates (48)-(49), we obtain

$$
|R_2| \leq C\|\sigma_u^{k+1}-\sigma_u^k\|^2_{L^2_h(\Omega)} + \frac{1}{10}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)} + Ch^2 + C\tau^2. \tag{50}
$$

Finally, combining (38),(47) and (50) in (34), we find

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\sigma_u^{k+1}\|^2_{L^2_h(\Omega)}-\|\sigma_u^{k}\|^2_{L^2_h(\Omega)}+\|\sigma_u^{k+1}-\sigma_u^k\|^2_{L^2_h(\Omega)}\right)+\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)}\\
&\le C\|\sigma_u^k\|^2_{L^2(\Omega)}+C\|\sigma_u^{k+1}-\sigma_u^k\|^2_{L^2_h(\Omega)}+\frac{1}{2}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)}+Ch^2+C\tau^2\\
&\quad+Ch^2\|\nabla\sigma_u^k\|^2_{L^2(\Omega)}+C\tau\int_{t_k}^{t_{k+1}}\|u_{tt}\|^2_{L^2(\Omega)}\mathrm{d}s.
\end{aligned}\tag{51}
$$

Multiplying by 2τ on both sides of (51) and summing up from $k=0$ to $m-1$, we get

$$
\begin{aligned}
&\|\sigma_u^{m}\|^2_{L^2_h(\Omega)}-\|\sigma_u^{0}\|^2_{L^2_h(\Omega)}+(1-2C\tau)\|\sigma_u^{k+1}-\sigma_u^k\|^2_{L^2_h(\Omega)}+\tau\sum_{k=0}^{m-1}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)}\\
\le{}& C\tau\sum_{k=0}^{m-1}\|\sigma_u^k\|^2_{L^2_h(\Omega)}+Ch^2+C\tau^2+C\tau h^2\sum_{k=0}^{m-1}\|\nabla\sigma_u^k\|^2_{L^2(\Omega)}+C\tau^2\int_0^T\|u_{tt}\|^2_{L^2}\mathrm{d}s.
\end{aligned}
$$

Assuming $1-2C\tau>0$, $1-Ch^2>1/2$ since τ and h is small enough, and applying the discrete Gronwall inequality (Lemma 12) to the above leads to

$$
\|\sigma_u^m\|_{L^2_h(\Omega)}+(\tau\sum_{k=0}^{m-1}\|\nabla\sigma_u^{k+1}\|^2_{L^2_h(\Omega)})^{1/2}\le C(h+\tau),\ \forall m\in\mathbb{N}.
$$

A combination of the above estimates for $\|\sigma_u^m\|_{L^2_h(\Omega)}$ and $\|\nabla\sigma_u^{k+1}\|_{L^2_h(\Omega)}$ with (31) leads to the desired error estimates for u. Finally, we obtain the error estimates for v from (36) and (37).

The proof of **Theorem 6** is complete.

4. Finite-time blowup

In this section, we discuss whether the solution of the fully discrete scheme (6)-(7) will blow up in finite time.

We first prove a discrete analog of Lemma 2. Taking $\varphi=I_h\Phi$ in (6), where Φ is defined as in Lemma 2, then from (5), we have the following error estimate

$$
\|\varphi-\Phi\|_{L^2(\Omega)}+h\|\varphi-\Phi\|_{L^2(\Omega)}\le Ch^2\|\Phi\|_{H^2(\Omega)}.\tag{52}
$$

LEMMA 13. *Assume that u is smooth enough for a fix time $T>0$, let $M^k=(u_h^k,\ \varphi)_h$, $\theta=(u,1)$. Under the mild mesh-sizes requirement $\tau\le Ch$, if $(u_h^0,\varphi)_h<\infty$, then $(u_h^k,\varphi)_h<\infty$ and the following inequality holds*

$$
\frac{M^{k+1}-M^k}{\tau}\le 4\theta-\frac{\alpha\chi}{2\pi}\theta^2+C_1\theta M^k+C_2\theta^{3/2}(M^k+Ch)^{1/2}+C_0h.
$$

Proof. We rewrite (6) as

$$
\begin{aligned}
\frac{M^{k+1}-M^k}{\tau}&=-(\nabla u^{k+1},\nabla\Phi)+\chi(u^k\nabla v^k,\nabla\Phi)+\sum_{i=1}^{8}J_i\\
&\le 4\theta-\frac{\alpha\chi}{2\pi}\theta^2+C_1\theta(u^k,\Phi)+C_2\theta^{3/2}(u^k,\Phi)^{1/2}+\sum_{i=1}^{8}J_i,
\end{aligned}\tag{53}
$$

where Lemma 2 has been used in the last inequality, and J_i $(i = 1, \cdots, 8)$ are defined as follows

$$\begin{aligned}
J_1 =&(\nabla u^{k+1},\ \nabla(\Phi - \varphi)),\\
J_2 =&(\nabla(u^{k+1} - u_h^{k+1}),\ \nabla\varphi),\\
J_3 =&\left(\nabla u_h^{k+1}, \nabla\varphi\right) - \left(\nabla u_h^{k+1}, \nabla\varphi\right)_h,\\
J_4 =&\left((u_h^{k+1} - u_h^k)\nabla \log u_h^{k+1}, \nabla\varphi\right)_h,\\
J_5 =&\chi\left(u^k \nabla v^k, \nabla(\varphi - \Phi)\right),\\
J_6 =&\chi(u_h^k \nabla v_h^k - u^k \nabla v^k, \nabla\varphi),\\
J_7 =&\chi(u_h^k \nabla v_h^k, \nabla\varphi)_h - \chi(u_h^k \nabla v_h^k, \nabla\varphi),\\
J_8 =&\left(u_h^k \nabla((I - I_h)\log u_h^{k+1}), \nabla\varphi\right)_h.
\end{aligned}$$

Thanks to (52), we obtain

$$\begin{aligned}
&(u^k, \Phi) = (u^k, \Phi - \varphi) + (u^k - u_h^k, \varphi) + (u_h^k, \varphi) - (u_h^k, \varphi)_h + (u_h^k, \varphi)_h\\
&\le \|u^k\|_{L^2(\Omega)}\|\Phi - \varphi\|_{L^2(\Omega)} + \|e_u^k\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + Ch^2\|\nabla u_h^k\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + M^k\\
&\le Ch + M^k,
\end{aligned}$$

where the error estimates and $\tau \le Ch$ have been used in the last inequality. Substitution of above into (53) leads to

$$\frac{M^{k+1} - M^k}{\tau} \le 4\theta - \frac{\alpha\chi}{2\pi}\theta^2 + C_1\theta M^k + C_2\theta^{3/2}(M^k + Ch)^{1/2} + \sum_{i=1}^{8} J_i. \tag{54}$$

Next, we estimate J_i $(i = 1, \cdots, 8)$ respectively. Utilizing the property of the interpolation operator in (52), we have

$$|J_1| \le \|\nabla u^{k+1}\|_{L^2(\Omega)}\|\nabla(\Phi - \varphi)\|_{L^2(\Omega)} \le Ch. \tag{55}$$

We derive from **Theorem 6** that

$$\begin{aligned}
|J_2| &= |(\nabla(u^{k+1} - u_h^{k+1}),\ \nabla(\varphi - \Phi)) + (\nabla(u^{k+1} - u_h^{k+1}),\ \nabla\Phi)|\\
&= |(\nabla(u^{k+1} - u_h^{k+1}),\ \nabla(\varphi - \Phi)) - ((u^{k+1} - u_h^{k+1}), \Delta\Phi)|\\
&\le Ch\|\nabla e_u^k\|_{L^2(\Omega)}\|\Phi\|_{H^2(\Omega)} + \|e_u^k\|_{L^2(\Omega)}\|\Delta\Phi\|_{L^2(\Omega)}\\
&\le Ch,
\end{aligned} \tag{56}$$

where the property of interpolation and $\tau \le Ch$ has been used. Noticing the definition of $(\cdot,\cdot)_h$, we have

$$J_3 = \left(\nabla u_h^{k+1}, \nabla\varphi\right) - \left(\nabla u_h^{k+1}, \nabla\varphi\right)_h = 0. \tag{57}$$

An application of the strict separation property, the $W^{1,\infty}$ bound for the numerical solution and $\tau \le Ch$ indicates that

$$
\begin{aligned}
|J_4| &= |(\frac{u_h^{k+1}-u_h^k}{u_h^{k+1}}\nabla u_h^{k+1}, \nabla\varphi)_h| \\
&\le \|\frac{u_h^{k+1}-u_h^k}{u_h^{k+1}}\nabla u_h^{k+1}\|_{L_h^2(\Omega)}\|\nabla\varphi\|_{L_h^2(\Omega)} \\
&\le \frac{\|\nabla u_h^{k+1}\|_{L^\infty(\Omega)}}{\epsilon_0/4}\|u_h^{k+1}-u_h^k\|_{L_h^2(\Omega)}\|\nabla\varphi\|_{L_h^2(\Omega)} \\
&\le \frac{C^*}{\epsilon_0/4}(\|e_u^{k+1}-e_u^k\|_{L_h^2(\Omega)} + C\tau)\|\nabla\varphi\|_{L_h^2(\Omega)} \\
&\le C(\|e_u^{k+1}\|_{L_h^2(\Omega)} + \|e_u^k\|_{L_h^2(\Omega)} + C\tau)\|\nabla\varphi\|_{L_h^2(\Omega)} \\
&\le Ch.
\end{aligned}
\tag{58}
$$

Utilizing the property of the interpolation operator in (52), we have

$$
\begin{aligned}
|J_5| &\le \chi\|u^k\nabla v^k\|_{L^2(\Omega)}\|\nabla(\varphi-\Phi)\|_{L^2(\Omega)} \\
&\le Ch\|u^k\|_{L^\infty(\Omega)}\|\nabla v^k\|_{L^2(\Omega)}\|\Phi\|_{H^2(\Omega)} \\
&\le Ch\|u^k\|_{L^\infty(\Omega)}\|u^k\|_{L^2(\Omega)}\|\Phi\|_{H^2(\Omega)} \\
&\le Ch.
\end{aligned}
\tag{59}
$$

An application of the Cauchy-Schwarz inequality and the error estimates leads to

$$
\begin{aligned}
|J_6| &= |-\chi(u_h^k\nabla e_v^k + e_u^k\nabla v^k, \nabla\varphi)| \\
&\le \chi(\|u_h^k\|_{L^\infty(\Omega)}\|\nabla e_v^k\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|e_u^k\|_{L^2(\Omega)}\|\nabla v^k\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega)}) \\
&\le Ch,
\end{aligned}
\tag{60}
$$

where $\tau \le Ch$ has been used in the last inequality. An application of Lemma 4 to J_4 leads to

$$
|J_7| = |\chi(u_h^k\nabla v_h^k, \nabla\varphi)_h - \chi(u_h^k\nabla v_h^k, \nabla\varphi)| = 0. \tag{61}
$$

Using the property of the interpolation operator I_h as in the proof of the Theorem 6, we have

$$
|J_8| = |\left(u_h^k\nabla((I-I_h)\log u_h^{k+1}), \nabla\varphi\right)_h| \le Ch. \tag{62}
$$

Finally, a substitution of (55)-(62) into (54) implies that

$$
\frac{M^{k+1}-M^k}{\tau} \le 4\theta - \frac{\alpha\chi}{2\pi}\theta^2 + C_1\theta M^k + C_2\theta^{3/2}(M^k+Ch)^{1/2} + C_0 h.
$$

The proof is complete. □

REMARK 14. *The positive constant C_0 in Lemma 13 depends on the regularity of the exact solutions.*

We can then derive the following discrete analog of **Theorem 3**.

THEOREM 15. *Assume that $\theta = (u,1) > 8\pi/(\alpha\chi)$. If $(u_h^0,\varphi)_h$ is sufficiently small, h and $\tau \le Ch$ are sufficiently small, then the solution (u_h^k, v_h^k) to the fully discrete scheme* (6)-(7) *will blow up in finite time, namely the maximal existence time $T_{\max} = k_{\max}\tau$ of the discrete solutions is finite.*

Proof. Obviously, we can derive the following inequality from Lemma 13

$$\frac{M^{k+1}-M^k}{\tau} \leq 4\theta - \frac{\alpha\chi}{2\pi}\theta^2 + C_1\theta M^k + C_2\theta^{3/2}(M^k)^{1/2} + C_3 h^{1/2}.$$

Denote $\beta = -(4\theta - \frac{\alpha\chi}{2\pi}\theta^2 + C_3 h^{1/2})$, we have the following inequality

$$\frac{M^{k+1}-M^k}{\tau} \leq -\beta + C_1\theta M^k + C_2\theta^{3/2}\sqrt{M^k}. \tag{63}$$

Under the condition that $\theta = (u, 1) > 8\pi/(\alpha\chi)$, we can choose sufficiently small h such that $\beta > 0$. Since $M^0 = (u_h^0, \varphi)_h$ is sufficiently small, we have

$$\beta_0 := \beta - C_1\theta M^0 - C_2\theta^{3/2}\sqrt{M^0} > 0.$$

We claim that the following inequality holds for all $k \in \mathbb{N}$

$$M^{k+1} \leq M^k - \tau\beta_0. \tag{64}$$

We prove the above inequality by induction. Using the inequality (63) for $k = 0$, we have

$$M^1 \leq M^0 - \tau(\beta - C_1\theta M^0 - C_2\theta^{3/2}\sqrt{M^0}) = M^0 - \tau\beta_0.$$

Now assume that (64) holds for $k-1$, we have

$$M^k \leq M^{k-1} - \tau\beta_0.$$

Notice that M^k is decreasing about k, we have

$$\begin{aligned} M^{k+1} &\leq M^k - \tau(\beta - C_1\theta M^k - C_2\theta^{3/2}\sqrt{M^k}) \\ &\leq M^k - \tau(\beta - C_1\theta M^0 - C_2\theta^{3/2}\sqrt{M^0}) \\ &= M^k - \tau\beta_0. \end{aligned}$$

Next summing (64) over k shows that

$$M^{k+1} \leq M^0 - (k+1)\tau\beta_0.$$

Hence, if the solution (u_h^k, v_h^k) exists for all $k \geq 0$, then M^{k+1} becomes negative provided that $T > M^0/\beta_0$. This is a contradiction to the positivity of M^k. Thus, the proof is complete. □

Remark 16. *Note that in the classical Keller-Segel system, the solution may blow up in finite time. Based on the error estimates, we prove that the numerical solution can also blow up under large initial value. There are several numerical examples in* [46] *to validate the blowup behavior of the numerical solution to the fully discrete scheme* (6)-(7)*. The analysis of Theorem 15 depends on the regularity of the solution, it is very interesting whether we can still have similar results under weak regularity, we will continue to conduct on this issue in the future.*

5. Conclusion

In this paper, we established error estimates for a fully discrete scheme proposed in [46] for the classical parabolic-elliptic Keller-Segel system, and showed that the numerical solution will blow up in finite time under some assumptions, similar to the situation for the exact solution of the classical parabolic-elliptic Keller-Segel system.

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A. Appendix

LEMMA 17. *Denote* $A = \alpha(-\Delta + I)^{-1}, A_h = \alpha(-\Delta_h + I)^{-1}Q_h$, *where* Q_h *is defined as* $(Q_h u, \psi_h) = (u, \psi_h)_h$, *then the following estimate holds for all* $u \in H^2(\Omega)$,

$$\|A_h u - Au\|_{L^2(\Omega)} + h\|A_h u - Au\|_{H^1(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}. \tag{A.1}$$

Proof. Let $w = Au$ and $w_h = A_h u$, we have the following equations:

$$(\nabla w, \nabla\psi) + (w, \psi) = \alpha(u, \psi),\ \forall\psi \in H^1,$$
$$(\nabla w_h, \nabla\psi) + (w_h, \psi) = \alpha(u, \psi)_h,\ \forall\psi \in X_h.$$

Then we have

$$(\nabla(w_h - w), \nabla\psi) + (w_h - w, \psi) = \alpha((u, \psi)_h - (u, \psi)).$$

Taking $\psi = w_h - \psi_h$ and denoting $w_h - w$ by $w_h - \psi_h - (w - \psi_h)$ in the above equation leads to

$$\|w_h - \psi_h\|_{H^1}^2 \leq \|w - \psi_h\|_{H^1}\|w_h - \psi_h\|_{H^1} + \alpha((u, w_h - \psi_h)_h - (u, w_h - \psi_h)).$$

From [48] and Lemma 4, we have the following estimate

$$\begin{aligned}
&|(u, w_h - \psi_h) - (u, w_h - \psi_h)_h| \\
=&|(u - I_h u, w_h - \psi_h) - (u - I_h, w_h - \psi_h)_h + (I_h u, w_h - \psi_h) - (I_h u, w_h - \psi_h)_h| \\
\leq& Ch|u|_{H^1}\|w_h - \psi_h\|_{L^2} + Ch^2\|\nabla u\|_{L^2}\|\nabla(w_h - \psi_h)\|_{L^2} \\
\leq& Ch|u|_{H^1}\|w_h - \psi_h\|_{H^1}.
\end{aligned}$$

Combing the above estimates with the elliptic regularity estimate leads to

$$\begin{aligned}
\|w - w_h\|_{H^1} &\leq C \inf_{\forall\psi_h \in X_h} \|w - \psi_h\|_{H^1} + Ch|u|_{H^1} \\
&\leq Ch|w|_{H^2} + Ch|u|_{H^1} \leq Ch\|u\|_{H^1}.
\end{aligned}$$

The L^2 error estimate can be obtained by using the duality argument

$$\|w - w_h\|_{L^2} \leq Ch\|w - w_h\|_{H^1} + Ch^2|u|_{H^2} \leq Ch^2\|u\|_{H^2}.$$

Combing above estimates with the definitions of A and A_h shows that

$$\begin{aligned}
&\|A_h u - Au\|_{L^2(\Omega)} + h\|A_h u - Au\|_{H^1(\Omega)} \\
=&\|w_h - w\|_{L^2(\Omega)} + h\|w_h - w\|_{H^1(\Omega)} \\
\leq& Ch^2\|u\|_{H^2(\Omega)},
\end{aligned}$$

which completes the proof. □

In order to obtain Lemma 8, we proceed in several steps. Firstly, we deal with $\hat{u}_1 := u + hf_1$ and construct f_1 as follows.

LEMMA 18. *Assume that $\tau \leq Ch$ and u is smooth enough, then there exists a bounded smooth function f_1, such that $\hat{u}_1 := u + hf_1$ satisfies*

$$\left(\overline{\partial}\hat{u}_1^k, \varphi\right)_h + \left(\hat{u}_1^k \nabla I_h \log \hat{u}_1^{k+1}, \nabla\varphi\right)_h - \chi\left(\hat{u}_1^k \nabla A_h \hat{u}_1^k, \nabla\varphi\right)_h = \langle \mathcal{R}_1^k, \varphi\rangle, \tag{A.2}$$

for all $\varphi \in X_h$, $k \in \mathbb{N}$, where A_h is defined as in Lemma 17 and

$$|\langle \mathcal{R}_1^k, \varphi\rangle| \leq Ch^2 \|\varphi\|_{H^1}.$$

Proof. From (A.2), we have the following equality:

$$\begin{aligned}
&(\frac{\partial \hat{u}_1}{\partial t}, \varphi) + (\nabla \hat{u}_1, \nabla\varphi) - \chi(\hat{u}_1 \nabla A \hat{u}_1, \nabla\varphi) \\
=&(\frac{\partial \hat{u}_1}{\partial t}, \varphi) + (\nabla \hat{u}_1, \nabla\varphi) - \chi(\hat{u}_1 \nabla A \hat{u}_1, \nabla\varphi) + \langle \mathcal{R}_1^k, \varphi\rangle \\
&- \left(\overline{\partial}\hat{u}_1^k, \varphi\right)_h - \left(\hat{u}_1^k \nabla I_h \log \hat{u}_1^{k+1}, \nabla\varphi\right)_h + \chi\left(\hat{u}_1^k \nabla A_h \hat{u}_1^k, \nabla\varphi\right)_h,
\end{aligned} \tag{A.3}$$

where the operators A and A_h are defined as in Lemma 17. Moreover, the left hand of (A.3) can be rewritten as

$$\begin{aligned}
&(\frac{\partial \hat{u}_1}{\partial t}, \varphi) + (\nabla \hat{u}_1, \nabla\varphi) - \chi(\hat{u}_1 A \hat{u}_1, \nabla\varphi) \\
=&h((\frac{\partial f_1}{\partial t}, \varphi) + (\nabla f_1, \nabla\varphi) - \chi(uAf_1, \nabla\varphi) - \chi(f_1 Au, \nabla\varphi)) \\
&- h^2(f_1 A f_1, \nabla\varphi),
\end{aligned}$$

where equation (26) has been used.

Step 1: Construction for f_1. For any $\varphi \in H^1$, define $\langle \mathcal{R}_0^k(u), \varphi\rangle$ as

$$\begin{aligned}
\langle \mathcal{R}_0^k(u), \varphi\rangle :=&(\frac{\partial u}{\partial t} - \overline{\partial}u^k, \varphi) + (\nabla(u - u^{k+1}), \nabla\varphi) - (u^k \nabla(I_h - I)(\log u^{k+1}), \nabla\varphi)_h \\
&- ((u^k - u^{k+1})\nabla \log u^{k+1}, \nabla\varphi)_h + (\nabla u^{k+1}, \nabla\varphi) - (\nabla u^{k+1}, \nabla\varphi)_h \\
&+ \chi(u^k \nabla A_h u^k - u\nabla A_h u, \nabla\varphi)_h + \chi(u\nabla(A_h - A)u, \nabla\varphi) \\
:=&\langle \mathcal{N}(u), \varphi\rangle + \chi(u^k \nabla A_h u^k - u\nabla A_h u, \nabla\varphi)_h + \chi(u\nabla(A_h - A)u, \nabla\varphi),
\end{aligned}$$

we can show that $\frac{1}{h}\mathcal{R}_0^k(u) \in H^{-1}$ is well defined. Using the Cauchy-Schwarz inequality and the property of the interpolation, we obtain

$$\begin{aligned}
|(\nabla u^{k+1}, \nabla\varphi) - (\nabla u^{k+1}, \nabla\varphi)_h| =&(\nabla(I - I_h)u^{k+1}, \nabla\varphi) - (\nabla(I - I_h)u^{k+1}, \nabla\varphi)_h \\
\leq& Ch\|u^{k+1}\|_{H^2}\|\nabla\varphi\|_{L^2},
\end{aligned}$$

then the following estimate holds for $\langle \mathcal{N}(u), \varphi\rangle$:

$$\langle \mathcal{N}(u), \varphi\rangle \leq C_4 h\|\varphi\|_{H^1}, \tag{A.4}$$

and the positive constant

$$C_4 = C_4(\|u\|_{W^{2,\infty}(0,T;H^2)} + \|u\|_{W^{1,\infty}(0,T;L^\infty)}\|\log u\|_{L^\infty(0,T;H^2)}).$$

An application of Lemma 17 leads to

$$\begin{aligned}
|(u\nabla(A_h - A)u, \nabla\varphi)| \leq& \|u\|_{L^\infty}\|\nabla(A_h u - Au)\|_{L^2}\|\nabla\varphi\|_{L^2} \\
\leq& Ch\|u\|_{L^\infty}\|u\|_{H^2}\|\nabla\varphi\|_{L^2}.
\end{aligned}$$

Combining (A.4) with few direct calculations shows that the following estimate holds for $\langle \mathcal{R}_0^k(u), \varphi\rangle$:

$$|\langle \mathcal{R}_0^k(u), \varphi\rangle| \leq C_5 h\|\varphi\|_{H^1},$$

where

$C_5 = C_5(\|u\|_{W^{2,\infty}(0,T;H^2)} + \|u\|_{W^{1,\infty}(0,T;L^\infty)}\|\log u\|_{L^\infty(0,T;H^2)} + \|u\|^2_{W^{1,\infty}(0,T;L^\infty)})$,

then $\frac{1}{h}\langle \mathcal{R}_0^k(u),\varphi\rangle$ is well-defined. Combining $\langle \mathcal{R}_0^k(u),\varphi\rangle$ with (A.3) leads to the following linear partial differential equation for f_1:

$$(\frac{\partial f_1}{\partial t},\varphi) + (\nabla f_1,\nabla\varphi) - \chi(u\nabla A f_1,\nabla\varphi) - \chi(f_1\nabla A u,\nabla\varphi) = \frac{1}{h}\langle \mathcal{R}_0^k(u),\varphi\rangle, \tag{A.5}$$

for all $t\in(t_k,t_{k+1}]$. From [17, Chapter7.1, Theorem 3], there exists a weak solution f_1 of (A.5). In addition, from [17, Chapter7.1, Theorem 7], suppose that u is smooth enough such that $\mathcal{R}_0^k(u)$ is smooth enough in $\overline{\Omega}\times[t_k,t_{k+1}]$, and the m^{th}-order compatibility conditions hold for $m=0,1,\cdots$, then the problem (A.5) has a smooth enough solution f_1 in $\overline{\Omega}\times[t_k,t_{k+1}]$.

Step 2: Construction for $\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle$. Let $\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle$ be

$$\begin{aligned}\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle :=& h\langle \mathcal{N}(f_1),\varphi\rangle + \chi h((u\nabla(A_h-A)f_1 + f_1\nabla(A_h-A)u,\nabla\varphi))\\ &+\chi h((u^k\nabla A_h f_1^k - u\nabla A_h f_1,\nabla\varphi)_h + (f_1^k\nabla A_h u^k - f_1\nabla A_h u,\nabla\varphi)_h)\\ &-(u^k\nabla I_h(\log(u^{k+1}+hf_1^{k+1}) - \log u^{k+1} - \frac{hf_1^{k+1}}{u^{k+1}}),\nabla\varphi)_h\\ &-h(u^k\nabla(I_h-I)\frac{f_1^{k+1}}{u^{k+1}},\nabla\varphi)_h - h((u^k-u^{k+1})\nabla\frac{f_1^{k+1}}{u^{k+1}},\nabla\varphi)_h\\ &+\left(\overline{\partial}u^k,\varphi\right) - \left(\overline{\partial}u^k,\varphi\right)_h + h^2(f_1^k\nabla\frac{f_1^{k+1}}{u^{k+1}},\nabla\varphi) + \chi h^2(f_1\nabla A f_1,\nabla\varphi)\\ &+\chi((u\nabla A_h u,\nabla\varphi)_h - (u\nabla A_h u,\nabla\varphi)).\end{aligned}$$

Similarly, the following estimate holds for $\langle \mathcal{N}(f_1),\varphi\rangle$ as discussed in (A.4):

$$\langle \mathcal{N}(f_1),\varphi\rangle \le C_6 h\|\varphi\|_{H^1}, \tag{A.6}$$

where the positive constant

$$C_6 = C_6(\|f_1\|_{W^{2,\infty}(0,T;H^2)} + \|f_1\|_{W^{1,\infty}(0,T;L^\infty)}\|\log u\|_{L^\infty(0,T;H^2)}).$$

Combining (A.6) with few calculations yields the following estimate for $\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle$:

$$|\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle| \le C_7 h^2\|\varphi\|_{H^1},$$

where the positive constant

$$\begin{aligned}C_7 = C_7(&\|u\|_{W^{1,\infty}(0,T;H^2)} + (\|u\|_{W^{1,\infty}(0,T;L^\infty)} + \|f_1\|_{L^\infty(0,T;L^\infty)})\|\frac{f_1}{u}\|_{L^\infty(0,T;H^2)}\\ &+\|u\|_{W^{1,\infty}(0,T;L^\infty)}\|f_1\|_{W^{1,\infty}(0,T;L^\infty)} + \|u\|^2_{L^\infty(0,T;H^2)} + \|f_1\|^2_{L^\infty(0,T;L^\infty)}) + C_6,\end{aligned}$$

then the $O(h^2)$ consistency for $\hat{u}_1 = u + hf_1$ is obtained, which leads to Lemma 18. □

After repeated application of the perturbation argument as illustrated in Lemma 18, Lemma 8 can be proved.

Proof of Lemma 8. The duality product $\frac{1}{h^2}\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle$ is well-defined since the fact that $\frac{1}{h^2}\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle$ is uniformly bounded as $h\to 0,\tau\to 0$ and $\tau\le Ch$. We can construct f_2 by solving the following linear partial differential equation:

$$(\frac{\partial f_2}{\partial t},\varphi) + (\nabla f_2,\nabla\varphi) - \chi(u\nabla A f_2,\nabla\varphi) - \chi(f_2\nabla A u,\nabla\varphi) = \frac{1}{h^2}\langle \mathcal{R}_1^k(\hat{u}_1),\varphi\rangle, \tag{A.7}$$

for all $t\in(t_k,t_{k+1}]$. As discussed in Step 1 above, the problem (A.7) has a smooth enough solution f_2 in $\overline{\Omega}\times[t_k,t_{k+1}]$.

By repeated application of the methods in Step 2 above, we can construct $\langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle$ by $\hat{u}_2 := \hat{u}_1 + h^2 f_2$ such that the $O(h^3)$ consistency for $\hat{u}_2$ is arrived:

$$\left(\overline{\partial} \hat{u}_2^k, \varphi\right)_h + \left(\hat{u}_2^k \nabla I_h \log \hat{u}_2^{k+1}, \nabla\varphi\right)_h - \chi\left(\hat{u}_2^k \nabla A_h \hat{u}_2^k, \nabla\varphi\right)_h = \langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle, \tag{A.8}$$

for all $\varphi \in X_h$, $k \in \mathbb{N}$, where

$$|\langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle| \le C h^3 \|\varphi\|_{H^1},$$

where C is a positive constant depending on the derivatives of $\hat{u}_2$, such that $\frac{1}{h^3}\langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle$ is well-defined.

Again, by using Step 1 in Lemma 18, the correction function f_3 can be constructed by the following linear partial differential equation:

$$\left(\frac{\partial f_3}{\partial t}, \varphi\right) + (\nabla f_3, \nabla\varphi) - \chi(u \nabla A f_3, \nabla\varphi) - \chi(f_3 \nabla A u, \nabla\varphi) = \frac{1}{h^3}\langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle, \tag{A.9}$$

for all $t \in (t_k, t_{k+1}]$, and the problem (A.9) has a smooth enough solution f_3 in $\overline{\Omega} \times [t_k, t_{k+1}]$.

Combing equations (A.5),(A.7) and (A.9) leads to

$$\begin{aligned}
&\left(\frac{\partial \hat{u}}{\partial t}, \varphi\right) + (\nabla \hat{u}, \nabla\varphi) - \chi(\hat{u} A \hat{u}, \nabla\varphi) \\
=&\langle \mathcal{R}_0^k(u), \varphi \rangle + \langle \mathcal{R}_1^k(\hat{u}_1), \varphi \rangle + \langle \mathcal{R}_2^k(\hat{u}_2), \varphi \rangle - h^4 \chi((f_2 \nabla A f_2, \nabla\varphi) \\
&+ (f_1 \nabla A f_3 + f_3 \nabla A f_1, \nabla\varphi)) - h^5 \chi(f_2 \nabla A f_3 + f_3 \nabla A f_2, \nabla\varphi) \\
&- h^6 \chi(f_3 \nabla A f_3, \nabla\varphi).
\end{aligned} \tag{A.10}$$

Denote $\langle \mathcal{R}^k(\hat{u}), \varphi \rangle$ as follows

$$\begin{aligned}
\langle \mathcal{R}^k(\hat{u}), \varphi \rangle := {}& h^3((\overline{\partial} f_3^k - \frac{\partial f_3}{\partial t}, \varphi) + (\overline{\partial} f_3^k, \varphi)_h - (\overline{\partial} f_3^k, \varphi) \\
&+ (f_3^k \nabla (I_h - I) \log \hat{u}_2^{k+1}, \varphi)_h - ((u^{k+1} - u^k) \nabla \frac{f_3^{k+1}}{\hat{u}_2^{k+1}}, \nabla\varphi)_h \\
&+ (\nabla f_3^{k+1}, \nabla\varphi)_h - (\nabla f_3^{k+1}, \nabla\varphi) + ((f_3^k - f_3^{k+1}) \nabla \log \hat{u}_2^{k+1}, \nabla\varphi)_h) \\
&+ h^6 (f_3^k \nabla \frac{f_3^{k+1}}{\hat{u}_2^{k+1}}, \nabla\varphi)_h + (\hat{u}^k \nabla(\log \hat{u}^{k+1} - \log \hat{u}_2^{k+1} - \frac{h^3 f_3^{k+1}}{\hat{u}_2^{k+1}}), \nabla\varphi)_h \\
&+ (\hat{u}_2^k \nabla(\log \hat{u}^{k+1} - \log \hat{u}_1^{k+1} - \frac{h^2 f_2^{k+1}}{\hat{u}_1^{k+1}}), \nabla\varphi)_h + h^4 (f_2^k \nabla \frac{f_2^{k+1}}{\hat{u}_1^{k+1}}, \nabla\varphi)_h \\
&+ h^2((\overline{\partial} f_2^k, \varphi)_h - (\overline{\partial} f_2^k, \varphi)) + h^3 \chi((f_3^k \nabla A_h \hat{u}_2^k + \hat{u}_2^k \nabla A_h f_3^k, \nabla\varphi)_h \\
&- (f_3 \nabla A_h \hat{u}_2 + \hat{u}_2 \nabla A_h f_3, \nabla\varphi)) + h^6 \chi((f_3^k \nabla A_h f_3^k, \nabla\varphi)_h \\
&- (f_3 \nabla A_h f_3, \nabla\varphi)) + h^4 \chi((f_2^k \nabla A_h f_2^k, \nabla\varphi)_h - (f_2 \nabla A_h f_2, \nabla\varphi)) \\
&+ h^2 \chi((f_2 \nabla A_h \hat{u}_1 + \hat{u}_1 \nabla A_h f_2, \nabla\varphi)_h - (f_2 \nabla A_h \hat{u}_1 + \hat{u}_1 \nabla A_h f_2, \nabla\varphi)).
\end{aligned}$$

Combining above with few direct calculations shows the following estimate for $\langle \mathcal{R}^k(\hat{u}), \varphi \rangle$

$$|\langle \mathcal{R}^k(\hat{u}), \varphi \rangle| \le C h^4 \|\varphi\|_{H^1},$$

where C depending on the derivatives of u, then the $\frac{1}{h^4}\langle \mathcal{R}^k(\hat{u}), \varphi \rangle$ is well defined and the $O(h^4)$ consistency holds for $\hat{u} = u + h f_1 + h^2 f_2 + h^3 f_3$, which leads to Lemma 8. □

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