INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 2-3, Pages 315–328

ON AN ADAPTIVE LDG FOR THE P-LAPLACE PROBLEM

DONGJIE LIU, LE ZHOU*, AND XIAOPING ZHANG

Abstract. In this paper we consider the adaptive local discontinuous Galerkin(LDG) method for the p-Laplace problem in polygonal regions in $\mathbb{R}^2$. We present new sharper a posteriori error estimate for the LDG approximation of the p-Laplacian in the new framework. Several examples are given to confirm the reliability of the estimate.

Key words. p-Laplace, local discontinuous Galerkin methods, quasi-norm, a posteriori error estimate.

1. Introduction

Let Ω be a bounded polyhedral domain in $\mathbb{R}^2$ with polygonal boundary Γ. We consider the classical p-Laplace problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ u = g_D = 0 & \text{on } \Gamma \end{cases}, \tag{1}$$

for $2 < p < \infty$ and given $f \in L^q(\Omega)$ (q conjugate of p). The p-Laplace problem (1) admits a unique weak solution satisfying [7, 17]

$$u = \arg\min E(v) \quad \text{for } v \in W_0^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega), v|_\Gamma = 0\}. \tag{2}$$

where

$$E(v) := \int_\Omega W(\nabla v)dx - \int_\Omega fv \, \mathrm{d}x. \tag{3}$$

The energy density function $W : \mathbb{R}^2 \to \mathbb{R}$ reads $W(\mathbf{a}) := |\mathbf{a}|^p/p$ with the derivative $\mathcal{A}(\mathbf{a}) := |\mathbf{a}|^{p-2}\mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^2$.

The Euler-Lagrange equation of (2) consists in finding $u \in W_0^{1,p}(\Omega)$ with

$$\int_\Omega \mathcal{A}(\nabla u) \cdot \nabla v \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x = 0 \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{4}$$

The embedding of $W_0^{1,p}(\Omega)$ into $W_0^{1,2}$ is continuous when $2 < p < \infty$ and Ω is bounded domain (see [5]).

The p-Laplacian occurs in many mathematical models of physical processes such as glaciology, nonlinear diffusion and filtration, power-law materials, and quasi-Newtonian flows. Furthermore it is viewed as one of the typical examples of a large class of difficult problems-degenerate nonlinear systems.

The numerical approximation for p-Laplace problem has been studied extensively in the literature. The previous analysis of finite element method (FEM) for this kind of problem was undertaken in [12], where the error estimates have been shown in the $W^{1,p}$-norm. The results were further improved in [1, 13, 28]. Recently, sharper error estimates were derived in [2, 4, 20, 23, 24, 26] by developing the quasi-norm techniques.

Received by the editors July 18, 2021 and, in revised form, August 31, 2021.
2000 *Mathematics Subject Classification.* 65N15, 65N30.
*Corresponding author.

Over the last two decades, there has been an increasing interest in discontinuous Galerkin (DG) methods for p-Laplace problem; see[6, 7, 17, 22]. Partically, Local discontinuous Galerkin (LDG) method [11, 14, 15] for p-structure problem was studied in [17, 22], where the quasi-norm interpolation estimates [18] were applied in the frame of broken spaces.

This paper aims at deriving a new explicit and reliable a posteriori error estimate for the LDG applied to the p-Laplacian. We generalize the Helmholtz decomposition of the gradient of the error [3, 9], derive the reliable a posteriori error estimate in the new defined distance, and the error of the energy can be presented at the same time in an easy way.

The remaining parts of this paper are organized as follows. In Section 2 we describe the LDG formulation and the equivalent minimization problem. In Section 3, we introduce the distance $\|F(\nabla u) - F(\nabla v)\|^2_{2,p,\Omega}$ to quantify the quality of approximations via $F(\mathbf{a}) := |\mathbf{a}|^{p/2-1}\mathbf{a}, \mathbf{a} \in \mathbb{R}^2$. The a posteriori error estimators based on new defined distance is presented in Section 4, Some numerical experiments conclude the paper in Section 5 with empirical evidence of the expected convergence.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$. Denote $\|\cdot\|_{L^p(\Omega)} := \|\cdot\|_{p,\Omega}$, $\|\cdot\|_{L^p(\Gamma)} := \|\cdot\|_{p,\Gamma}$. Denote the expression "$\lesssim$" abbreviates an inequality up to some multiplicative generic constant, i.e. $A \lesssim B$ means $A \leq C\ B$ with some generic constant $0 \leq C \leq \infty$, which depends on the interior angles of the triangles but not their sizes. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Discontinuous finite element approximation

2.1. Discontinuous finite element space and Local L^2-projection. In order to obtain LDG formulation of (1), we introduce the gradient $\boldsymbol{\theta} := \nabla u$ and the flux $\boldsymbol{\sigma} := \mathcal{A}(\boldsymbol{\theta}) = |\nabla u|^{p-2}\nabla u$, then (1) can be reformulated as the follow problem: Find $(u, \boldsymbol{\theta}, \boldsymbol{\sigma})$ in appropriate space such that

$$\left\{ \begin{array}{ll} \boldsymbol{\theta} = \nabla u, \boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\theta}), -\mathrm{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ u = g_D = 0 & \text{on } \Gamma \end{array} \right. . \tag{5}$$

Let $\mathcal{T}_h = \bigcup\{T\}$ be a shape-regular triangulation of $\bar{\Omega}$ such that $\bar{\Omega} = \bigcup\{T : T \in \mathcal{T}_h\}$, where straight triangle T has diameter h_T and unit outward normal to ∂T given by $\mathbf{n}_T$. $h := \max\{h_T : T \in \mathcal{T}_h\}$. We denote by $\Gamma_h = \bigcup\{E \subset \partial T : T \in \mathcal{T}_h\}$ the union of all edges of $\mathcal{T}_h$ and $\Gamma_I = \Gamma_h \backslash \Gamma$ an union of all interior edges of $\mathcal{T}_h$. The discontinuous finite element space of scalar function and vector function space are defined by

$$V_h = \{v \in L^p(\Omega) : v|_T \in \mathbf{P}_1(T) \quad \forall\ T \in \mathcal{T}_h\},$$
$$\Sigma_h = \{\boldsymbol{\theta} \in [L^p(\Omega)]^2 : \boldsymbol{\theta}|_T \in [\mathbf{P}_0(T)]^2 \quad \forall\ T \in \mathcal{T}_h\}.$$

$\mathbf{P}_k(T)$ denotes the polynomial of degree at most k on T. Similarly we have the piecewise smooth function space on $\mathcal{T}_h$

$$W^{1,p}(\mathcal{T}_h) = \{v \in L^p(\Omega) : v|_T \in W^{1,p}(T) \quad \forall\ T \in \mathcal{T}_h\}.$$

Let T_1 and T_2 be two adjacent elements with a common edge E. Denote $v_i := v|_{\partial T_i}$ the trace of function v restricted to E in element T_i with $\mathbf{n}_i := \mathbf{n}|_{\partial T_i}$ on E pointing exterior to T_i. Define jump and average of function v on E,

$$[\![v]\!] = v_1\mathbf{n}_1 + v_2\mathbf{n}_2, \quad \{v\} = \frac{1}{2}(v_1 + v_2), \quad E \in \Gamma_I.$$

For vector function $\boldsymbol{\tau} \in [W^{1,p}(\mathcal{T}_h)]^2$,

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}_1 \cdot \mathbf{n}_1 + \boldsymbol{\tau}_2 \cdot \mathbf{n}_2, \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2), \quad E \in \Gamma_I.$$

Particularly, if $E \subset \Gamma$,

$$[\![v]\!] = v\mathbf{n}, \quad \{v\} = v; \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau} \cdot \mathbf{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}.$$

We introduce some notation for convenience $(f,v) := \int_\Omega fv \ \mathrm{d}x$, $\langle f,g\rangle_E := \int_E fg \ \mathrm{d}s$ for $E \in \Gamma_h$, $V(h) := V_h + W^{1,p}(\Omega)$.

Recall the **local L^2-projection operator** $\Pi_k : L^2(\Omega) \to V_h$ [8]

$$(u - \Pi_k u, v_h)_T = 0 \quad \forall v_h \in \mathbf{P}_k(T), \ T \in \mathcal{T}_h,$$

it holds that

$$\|u - \Pi_k u\|_{p,T} \le C h_T \|\nabla u\|_{p,T} \quad 1 \le p \le \infty, \ \forall T \in \mathcal{T}_h, \tag{6}$$

$$\|u - \Pi_k u\|_{p,\Omega} \le C h \|\nabla u\|_{p,\Omega} \quad \forall u \in W^{1,p}(\Omega). \tag{7}$$

Recall the **trace inequality**

$$\|u\|_{p,\partial T} \le C h_T^{-\frac{1}{p}} \left(\|u\|_{p,T} + h_T \|\nabla u\|_{p,T}\right) \ \forall u \in W^{1,p}(T), \ 1 < p \le \infty, \tag{8}$$

C is some constants independent of the meshsize.

2.2. Numerical fluxes and LDG formulation. Multiplying the equations in $(5)_1$ by $\boldsymbol{\tau}_h \in \Sigma_h, \boldsymbol{\zeta}_h \in \Sigma_h$ and $v_h \in V_h$ shows that

$$\begin{aligned}
\int_T \boldsymbol{\theta} \cdot \boldsymbol{\tau}_h \ \mathrm{d}x &= -\int_T u \operatorname{div}_h \boldsymbol{\tau}_h \ \mathrm{d}x + \int_{\partial T} u\,(\boldsymbol{\tau}_h \cdot \mathbf{n})\,\mathrm{d}s, \\
\int_T \boldsymbol{\sigma} \cdot \boldsymbol{\zeta}_h \ \mathrm{d}x &= \int_T \mathcal{A}(\boldsymbol{\theta}) \cdot \boldsymbol{\zeta}_h \ \mathrm{d}x, \\
\int_T \boldsymbol{\sigma} \cdot \nabla_h v_h \ \mathrm{d}x &= \int_T f v_h \ \mathrm{d}x + \int_{\partial T} v_h (\boldsymbol{\sigma} \cdot \mathbf{n})\mathrm{d}s.
\end{aligned} \tag{9}$$

where $\nabla_h \cdot$ is piecewise gradient. Replacing $u, \boldsymbol{\theta}, \boldsymbol{\sigma}$ with $u_h, \boldsymbol{\theta}_h, \boldsymbol{\sigma}_h$ and defining numerical fluxes $\hat{u}_h := \hat{u}(u_h), \hat{\boldsymbol{\sigma}}_h := \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h)$ on the boundary show that

$$\begin{aligned}
\int_T \boldsymbol{\theta}_{\boldsymbol{h}} \cdot \boldsymbol{\tau}_h \ \mathrm{d}x &= -\int_T u_h \operatorname{div}_h \boldsymbol{\tau}_h \ \mathrm{d}x + \int_{\partial T} \hat{u}_h\,(\boldsymbol{\tau}_h \cdot \mathbf{n})\,\mathrm{d}s, \\
\int_T \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta}_h \ \mathrm{d}x &= \int_T \mathcal{A}(\boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta}_h \ \mathrm{d}x, \\
\int_T \boldsymbol{\sigma}_h \cdot \nabla_h v_h \ \mathrm{d}x &= \int_T f v_h \ \mathrm{d}x + \int_{\partial T} v_h (\hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n})\mathrm{d}s.
\end{aligned} \tag{10}$$

with

$$\hat{u}\,(u_h) := \begin{cases} \{u_h\} & \text{on } \Gamma_I \\ 0 & \text{on } \Gamma \end{cases}, \tag{11}$$

$$\hat{\boldsymbol{\sigma}}\,(u_h, \boldsymbol{\sigma}_h) := \begin{cases} \{\boldsymbol{\sigma}_h\} - \eta \mathcal{A}\left(h_E^{-1} [\![u_h]\!]\right) & \text{on } \Gamma_I \\ \boldsymbol{\sigma}_h - \eta \mathcal{A}\left(h_E^{-1} u_h \mathbf{n}\right) & \text{on } \Gamma \end{cases}, \tag{12}$$

the penalty coefficient $\eta > 0$ is some constant. The fluxes $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are consistent, since $\hat{u}(u) = u$, and $\hat{\boldsymbol{\sigma}}(u, \boldsymbol{\sigma}) = \boldsymbol{\sigma}$ for regular functions u and $\boldsymbol{\sigma}$ satisfying the boundary conditions. The fluxes are also conservative since they are single-valued.

Denote $S : V(h) \times \Sigma_h \to \mathbb{R}$ be the bilinear form defined by

$$S(v, X_h) := \int_{\Gamma_I} \{\boldsymbol{\tau}_h\} \cdot [\![v]\!] \ \mathrm{d}s + \int_\Gamma v \boldsymbol{\tau}_h \cdot \mathbf{n} \ \mathrm{d}s, \tag{13}$$

and $\mathbf{S} : V(h) \to \Sigma_h$ be the linear and bounded operator induced by the bilinear form S, that is, given $v \in V(h)$, $\mathbf{S}(v)$ is the unique element in Σ_h such that

$$\int_\Omega \mathbf{S}(v) \cdot \boldsymbol{\tau}_h \, \mathrm{d}x = S(v, \boldsymbol{\tau}_h) \quad \forall \, \boldsymbol{\tau}_h \in \Sigma_h. \tag{14}$$

Similarly, let $\nabla_{\mathrm{DG}} v$ be the unique element in Σ_h such that

$$(\nabla_{\mathrm{DG}} v, \boldsymbol{\tau}_h) = (\nabla_h v, \boldsymbol{\tau}_h) - (\mathbf{S}(v), \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \tag{15}$$

then the following pointwise identity holds for $v \in V(h)$

$$\nabla_{\mathrm{DG}} v = \nabla_h v - \mathbf{S}(v). \tag{16}$$

Particularly, $\nabla_{\mathrm{DG}} v = \nabla v$ when $v \in W_0^{1,p}(\Omega)$.

A straightforward computation shows that for all $(g, \boldsymbol{\tau}) \in V_h \times \Sigma_h$

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} g \boldsymbol{\tau} \cdot \mathbf{n} \mathrm{d}x = \langle \{g\}, [\![\boldsymbol{\tau}]\!] \rangle_{\Gamma_I} + \langle [\![g]\!], \{\boldsymbol{\tau}\} \rangle_{\Gamma_h}. \tag{17}$$

Inserting (11) and (12) into (10), summing the result over $T \in \mathcal{T}_h$ and using (17) lead to

$$\begin{aligned} (\boldsymbol{\theta}_h, \boldsymbol{\tau}_h) &= - (u_h, \mathrm{div}_h \, \boldsymbol{\tau}_h) + \langle \{u_h\}, [\![\boldsymbol{\tau}_h]\!] \rangle_{\Gamma_I} \\ (\boldsymbol{\sigma}_h, \boldsymbol{\zeta}_h) &= (\mathcal{A}(\boldsymbol{\theta}_h), \boldsymbol{\zeta}_h), \\ (\boldsymbol{\sigma}_h, \nabla_h v_h) &= (f, v_h) + \langle \{\boldsymbol{\sigma}_h\}, [\![v_h]\!] \rangle_{\Gamma_I} + \langle \boldsymbol{\sigma}_h, v_h \mathbf{n} \rangle_\Gamma \\ &\quad - \eta \langle \mathcal{A}(h_E^{-1} [\![u_h]\!]), [\![v_h]\!] \rangle_{\Gamma_I} - \eta \langle \mathcal{A}(h_E^{-1} u_h \mathbf{n}), v_h \mathbf{n} \rangle_\Gamma. \end{aligned} \tag{18}$$

The integration by part implies that

$$\begin{aligned} (\boldsymbol{\theta}_h, \boldsymbol{\tau}_h) &= (\nabla_h u_h, \boldsymbol{\tau}_h) - \langle [\![u_h]\!], \{\boldsymbol{\tau}_h\} \rangle_{\Gamma_I} - \langle u_h \mathbf{n}, \boldsymbol{\tau}_h \rangle_\Gamma, \\ (\boldsymbol{\sigma}_h, \boldsymbol{\zeta}_h) &= (\mathcal{A}(\boldsymbol{\theta}_h), \boldsymbol{\zeta}_h), \\ (\boldsymbol{\sigma}_h, \nabla_h v_h) &= (f, v_h) + \langle \{\boldsymbol{\sigma}_h\}, [\![v_h]\!] \rangle_{\Gamma_I} + \langle \boldsymbol{\sigma}_h, v_h \cdot \mathbf{n} \rangle_\Gamma \\ &\quad - \eta \langle \mathcal{A}(h_E^{-1} [\![u_h]\!]), [\![v_h]\!] \rangle_{\Gamma_I} - \eta \langle \mathcal{A}(h_E^{-1} u_h \mathbf{n}), v_h \mathbf{n} \rangle_\Gamma. \end{aligned} \tag{19}$$

The flux formulation of (1) reads: Given data $f \in L^q(\Omega)$, find $(u_h, \boldsymbol{\theta}_h, \boldsymbol{\sigma}_h) \in V_h \times \Sigma_h \times \Sigma_h$ such that for all $(v_h, \boldsymbol{\tau}_h, \boldsymbol{\zeta}_h) \in V_h \times \Sigma_h \times \Sigma_h$

$$\begin{aligned} (\boldsymbol{\theta}_h, \boldsymbol{\tau}_h) &= (\nabla_{\mathrm{DG}} u_h, \boldsymbol{\tau}_h), \\ (\boldsymbol{\sigma}_h, \boldsymbol{\zeta}_h) &= (\mathcal{A}(\boldsymbol{\theta}_h), \boldsymbol{\zeta}_h), \\ (\boldsymbol{\sigma}_h, \nabla_{\mathrm{DG}} v_h) &= (f, v_h) - \eta \langle \mathcal{A}(h_E^{-1} [\![u_h]\!], [\![v_h]\!] \rangle_{\Gamma_I} - \eta \langle \mathcal{A}(h_E^{-1} u_h \mathbf{n}), v_h \mathbf{n} \rangle_\Gamma. \end{aligned} \tag{20}$$

Note that (20) implies that

$$\boldsymbol{\theta}_h = \nabla_{\mathrm{DG}} u_h, \ \boldsymbol{\sigma}_h = \mathcal{A}(\nabla_{\mathrm{DG}} u_h). \tag{21}$$

The last equation of (20) and (21) lead to the system only in terms of u_h,

$$(\mathcal{A}(\nabla_{\mathrm{DG}} u_h), \nabla_{\mathrm{DG}} v_h) = (f, v_h) - \eta \langle \mathcal{A}(h_E^{-1} [\![u_h]\!]), [\![v_h]\!] \rangle_{\Gamma_I} - \eta \langle \mathcal{A}(h_E^{-1} u_h \mathbf{n}), v_h \mathbf{n} \rangle_\Gamma. \tag{22}$$

2.3. The equivalent discrete variational formulation. Define the discrete energy on the DG space

$$E_{\mathrm{DG}}(v) := \int_\Omega W(\nabla_{\mathrm{DG}} v) \, \mathrm{d}x + \sum_{E \in \Gamma_h} \frac{\eta}{h_E^{p-1}} \int_E W([\![v]\!]) \, \mathrm{d}s - \int_\Omega f v \, \mathrm{d}x \quad \forall \, v \in V_h. \tag{23}$$

DG approximation u_h to (2) minimizes the energy E in V_h , written

$$u_h \in \arg\min E_{\mathrm{DG}}(V_h). \tag{24}$$

The discrete minimizer u_h exists and unique [7].

The discrete Euler-Lagrange equations of (24) consists in finding $u_h \in V_h$ for all $v_h \in V_h$

(25)
$$\int_\Omega \mathcal{A}(\nabla_{\mathrm{DG}} u_h) \cdot \nabla_{\mathrm{DG}} v_h \; \mathrm{d}x + \sum_{E \in \Gamma_h} \eta \int_E \mathcal{A}(h_E^{-1} [\![u_h]\!]) \cdot [\![v_h]\!] \; \mathrm{d}s = \int_\Omega f v_h \; \mathrm{d}x.$$

3. Distance

In this section, we generalize the definition quasi-norm [4, 19] to adaptive LDG method on the p-Laplacian. Define

$$F(\mathbf{a}) := |\mathbf{a}|^{p/2-1}\mathbf{a} \quad \forall \mathbf{a} \in \mathbb{R}^2, \tag{26}$$

let $\boldsymbol{\alpha} := \mathcal{A}(\mathbf{a})$, $\boldsymbol{\beta} := \mathcal{A}(\mathbf{b})$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, then it satisfies that [16]

$$|F(\mathbf{a}) - F(\mathbf{b})|^2 \sim \frac{|\boldsymbol{\alpha} - \boldsymbol{\beta}|^2}{(|\mathbf{a}| + |\mathbf{b}|)^{p-2}} \sim \frac{|\boldsymbol{\alpha} - \boldsymbol{\beta}|^2}{(|\mathbf{a}|^{p-2} + |\mathbf{b}|^{p-2})} \tag{27}$$

Define the distance

$$\int_\Omega \frac{|\boldsymbol{\alpha} - \boldsymbol{\beta}|^2}{(|\mathbf{a}| + |\mathbf{b}|)^{p-2}} dx := \|F(\mathbf{a}) - F(\mathbf{b})\|_{2,p,\Omega}^2 \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^2. \tag{28}$$

Lemma 3.1[24]. Given $2 \le p < \infty$ and the conjugate q, there exists positive constants $c_1(p)$ and $c_2(p)$ such that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\boldsymbol{\alpha} := \mathcal{A}(\mathbf{a})$, $\boldsymbol{\beta} := \mathcal{A}(\mathbf{b})$

(29)
$$\begin{aligned} c_1(p) \int_\Omega (W(\mathbf{b}) - W(\mathbf{a}) - \boldsymbol{\alpha} \cdot (\mathbf{b} - \mathbf{a})) \; \mathrm{d}x &\le \|F(\mathbf{a}) - F(\mathbf{b})\|_{2,p,\Omega}^2 \\ &\le c_2(p) \int_\Omega (W(\mathbf{b}) - W(\mathbf{a}) - \boldsymbol{\alpha} \cdot (\mathbf{b} - \mathbf{a})) \; \mathrm{d}x. \end{aligned}$$

The need for such an error concept comes from the nonlinear behavior of the residual, which is not reflected by standard Sobolev norms. The new defined distance concept permits us to prove a best approximation property for Galerkin solutions as well as optimal interpolation estimates, which are crucial ingredients in a posteriori finite element analysis.

4. A posteriori error estimate of LDG

Let $\mathcal{N}_h : V(h) \to V(h)'$ be the pure nonlinear operator

$$\begin{aligned} [\mathcal{N}_h(w), v] &:= \int_\Omega \mathcal{A}\left(\nabla_{\mathrm{DG}} w\right) \cdot \left(\nabla_{\mathrm{DG}} v\right) \; \mathrm{d}x \\ &= \int_\Omega \mathcal{A}\left(\nabla_h w - \mathbf{S}(w)\right) \cdot \left(\nabla_h v - \mathbf{S}(v)\right) \; \mathrm{d}x \quad \forall w, v \in V(h). \end{aligned} \tag{30}$$

Since $\mathcal{A}$ satisfies the assumption (H.4) in [8] and the operator $\mathbf{S}$ is bounded, we deduce that $\mathcal{N}_h$ is Gâteaux differentiable at each $z \in V(h)$, and its derivative can be interpreted as the bounded bilinear form $D\mathcal{N}_h(z) : V(h) \times V(h) \to \mathbb{R}$,

$$D\mathcal{N}_h(z)(w, v) := \int_\Omega D\mathcal{A}(\tilde{\zeta}) \left(\nabla_h v - \mathbf{S}(v)\right) \cdot \left(\nabla_h w - \mathbf{S}(w)\right) dx \quad \forall w, v \in V(h) \tag{31}$$

where $\tilde{\zeta} := \nabla_h z - \mathbf{S}(z)$ is the Jacobian matrix of $\mathcal{A}$ at $\tilde{\zeta}$.

The $D\mathcal{A}$ is symmetric for all $z \in V(h)$ and the $D\mathcal{N}_h$ is hemicontinuous, mean value theorem leads to the existence of $\tilde{u} \in V(h)$,

$$D\mathcal{N}_h(\tilde{u})\left(u - u_h, v\right) = \left[\mathcal{N}_h(u) - \mathcal{N}_h\left(u_h\right), v\right] \quad \forall v \in V(h). \tag{32}$$

In this section, we will propose a posteriori error estimate under new distance frame. The key technique are the convexity of energy density function W and the Helmholtz decomposition for the term $\nabla_{\mathrm{DG}}(u-u_h)$.

Denote curl $\chi := (-\dfrac{\partial\chi}{\partial y}, \dfrac{\partial\chi}{\partial x})$ for any $\chi \in W^{1,p}(\Omega)$, then the following lemma holds.

Lemma 4.1(Helmholtz decomposition) There exists $\psi \in W_0^{1,p}(\Omega)$ and $\chi \in W^{1,p}(\Omega)$ such that

$$\nabla_{\mathrm{DG}}(u-u_h) = \nabla\psi + (D\mathcal{A}(\tilde{\boldsymbol{\theta}}))^{-1}\mathrm{curl}\ \chi. \tag{33}$$

Here, $D\mathcal{A}(\tilde{\boldsymbol{\theta}})$ is a symmetric positive definite matrix. Furthermore, there exists a constant $C > 0$ independent of h, satisfies

$$\|\nabla\psi\|_{p,\Omega} + \|\mathrm{curl}\chi\|_{p,\Omega} \le C\|\nabla_{\mathrm{DG}}(u-u_h)\|_{p,\Omega}. \tag{34}$$

Proof. Let $\psi \in W_0^{1,p}(\Omega)$ be the unique weak solution of the boundary value problem:

$$\begin{cases} -\operatorname{div}\left(D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla\psi\right) = -\operatorname{div}\left(D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla_{\mathrm{DG}}(u-u_h)\right) & \text{in } \Omega \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{35}$$

Since div $\left(D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla_{\mathrm{DG}}(u-u_h) - D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla\psi\right) = 0$ in the sense of distributions, and Ω is simply connected, Theorem 3.1 in Chapter I of [21] shows that

$$D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla_{\mathrm{DG}}(u-u_h) = D\mathcal{A}(\tilde{\boldsymbol{\theta}})\nabla\psi + \mathrm{curl}\ \chi. \tag{36}$$

A simple transformation leads to

$$\nabla_{\mathrm{DG}}(u-u_h) = \nabla\psi + (D\mathcal{A}(\tilde{\boldsymbol{\theta}}))^{-1}\mathrm{curl}\ \chi. \tag{37}$$

According to the Theorem 1.4 in [27], stability estimate can be obtained,

$$\|\nabla\psi\|_{p,\Omega} + \|(D\mathcal{A}(\tilde{\boldsymbol{\theta}}))^{-1}\mathrm{curl}\chi\|_{p,\Omega} \le C\|\nabla_{\mathrm{DG}}(u-u_h)\|_{p,\Omega}. \tag{38}$$

The coefficients matrix $D\mathcal{A}(\tilde{\boldsymbol{\theta}})$ is pointwise symmetric uniformly positive definite, there exists $0 < \mu < M < \infty$ such that for all $y \in R^2$

$$\mu|y|^2 \le y \cdot D\mathcal{A}(\tilde{\boldsymbol{\theta}})y \le M|y|^2. \tag{39}$$

and

$$\begin{aligned} \|\mathrm{curl}\chi\|_{p,\Omega}^p &= \int_\Omega |\mathrm{curl}\chi|^{2\cdot\frac{p}{2}}\mathrm{d}x \\ &\lesssim \int_\Omega \left(\mathrm{curl}\chi\cdot(D\mathcal{A}(\tilde{\boldsymbol{\theta}}))^{-2}\mathrm{curl}\chi\right)^{\frac{p}{2}}\mathrm{d}x \\ &= \|(D\mathcal{A}(\tilde{\boldsymbol{\theta}}))^{-1}\mathrm{curl}\chi\|_{p,\Omega}^p. \end{aligned} \tag{40}$$

The combination of (38) and (40) leads to the (34). □

Lemma 4.2 Let $\bar{u}_h \in W^{1,p}(\Omega)\cap C(\bar{\Omega})$ be an auxiliary function of the u_h and it satisfies,

(1) $\bar{u}_h|_T \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_h$.

(2) $\bar{u}_h(\bar{\mathbf{x}}) := u(\bar{\mathbf{x}}) = 0$ for each $\bar{\mathbf{x}} \in \Gamma$.

(3) for each node $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ not lying on the boundary Γ, $\bar{u}_h(\bar{\mathbf{x}})$ is the average of the values of $u_h(\bar{\mathbf{x}})$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs, i.e., $\bar{u}_h(\bar{\mathbf{x}}) = \dfrac{1}{n}\sum_{\bar{\mathbf{x}}\in T} u_h|_T(\bar{\mathbf{x}})$, where n is the number of elements T adjacent to $\bar{\mathbf{x}}$.

Then it holds

$$\sum_{T\in\mathcal{T}_h}\int_T(\nabla u-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x=\int_\Omega(\nabla_h\bar{u}_h-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x. \tag{41}$$

Proof. Integral by parts and $\operatorname{div}(\operatorname{curl}\chi)=0$ lead to

$$\begin{aligned}
&\int_\Omega(\nabla u-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x\\
=&\sum_{T\in\mathcal{T}_h}\int_T\nabla(u-\bar{u}_h+\bar{u}_h-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x\\
=&\int_\Omega(\nabla_h\bar{u}_h-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x+\sum_{T\in\mathcal{T}_h}\int_{\partial T}(u-\bar{u}_h)\operatorname{curl}\chi\cdot\mathbf{n}_T\ \mathrm{d}s\\
=&\int_\Omega(\nabla_h\bar{u}_h-\nabla_{\mathrm{DG}}u_h)\cdot\operatorname{curl}\chi\ \mathrm{d}x+\int_\Gamma(u-\bar{u}_h)\operatorname{curl}\chi\cdot\mathbf{n}_T\ \mathrm{d}s.
\end{aligned}$$

$\bar{u}_h|_\Gamma=u|_\Gamma=0$ concludes the proof. □

Lemma 4.3[24] For any $u\in W_0^{1,p}(\Omega)$

$$||\nabla u||_{p,\Omega}\le(pC_F\|f\|_{q,\Omega})^{\frac{1}{p-1}}, \tag{42}$$

where $C_F\le\dfrac{\mathrm{width}(\Omega)}{\pi}$, $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Theorem 4.4 Let u_h be a approximated solution of problem (24), it holds that

$$\|F(\nabla u)-F(\nabla_{\mathrm{DG}}u_h)\|_{2,p,\Omega}^2\lesssim\sum_{T\in\mathcal{T}_h}\eta_T, \tag{43}$$

$$\begin{aligned}
\eta_T:=&\,h_T^q\|\left(f+\operatorname{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\|_{q,T}^q\\
&+h_T\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T-\mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma}^q+\|\nabla\bar{u}_h-\boldsymbol{\theta}_h\|_{q,T}^q.
\end{aligned} \tag{44}$$

Proof. The choice $\mathbf{a}:=\nabla u,\mathbf{b}:=\nabla_{\mathrm{DG}}u_h$, and $\alpha:=\mathcal{A}(\nabla u)$ in Lemma 3.1 and Euler-Lagrange equations (4) and (25) lead to

$$\begin{aligned}
&\|F(\nabla u)-F(\nabla_{\mathrm{DG}}u_h)\|_{2,p,\Omega}^2+c_2(p)\left(E(u)-E_{\mathrm{DG}}(u_h)\right)\\
\le&c_2(p)\left[\int_\Omega fu_h\ \mathrm{d}x-\int_\Omega fu\ \mathrm{d}x-\sum_{E\in\Gamma_h}\frac{\eta}{h_E^{p-1}}\int_E W([\![u_h]\!])\ \mathrm{d}s\right.\\
&\left.-\int_\Omega\mathcal{A}(\nabla u)\cdot(\nabla_{\mathrm{DG}}u_h-\nabla u)\ \mathrm{d}x\right]\\
\le&c_2(p)\left[\int_\Omega\left(\mathcal{A}(\nabla_{\mathrm{DG}}u_h)-\mathcal{A}(\nabla u)\right)\cdot\nabla_{\mathrm{DG}}u_h\ \mathrm{d}x\right.\\
&\left.+\sum_{E\in\Gamma_h}\frac{\eta}{h_E^{p-1}}\int_E\mathcal{A}([\![u_h]\!])\cdot[\![u_h]\!]-W([\![u_h]\!])\ \mathrm{d}s\right].
\end{aligned} \tag{45}$$

The choice $\mathbf{a} := \nabla_{\mathrm{DG}} u_h, \mathbf{b} := \nabla u$, and $\alpha := \mathcal{A}(\nabla_{\mathrm{DG}} u_h)$ lead to

$$\begin{aligned}
&\|F(\nabla u) - F(\nabla_{\mathrm{DG}} u_h)\|_{2,p,\Omega}^2 + c_2(p)\,(E_{\mathrm{DG}}(u_h) - E(u)) \\
\leq& c_2(p)\left[\int_\Omega fu \,\mathrm{d}x - \int_\Omega fu_h \,\mathrm{d}x + \sum_{E\in\Gamma_h} \frac{\eta}{h_E^{p-1}} \int_E W([\![u_h]\!]) \,\mathrm{d}s\right. \\
&\left. - \int_\Omega \mathcal{A}(\nabla_{\mathrm{DG}} u_h)\cdot(\nabla u - \nabla_{\mathrm{DG}} u_h) \,\mathrm{d}x\right] \\
\leq& c_2(p)\left[\int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot \nabla u \,\mathrm{d}x\right. \\
&\left. - \sum_{E\in\Gamma_h} \frac{\eta}{h_E^{p-1}} \int_E \mathcal{A}([\![u_h]\!])\cdot[\![u_h]\!] - W([\![u_h]\!]) \,\mathrm{d}s\right].
\end{aligned} \tag{46}$$

The (45)-(46) shows that

$$\|F(\nabla u) - F(\nabla_{\mathrm{DG}} u_h)\|_{2,p,\Omega}^2 \lesssim \int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot(\nabla u - \nabla_{\mathrm{DG}} u_h) \,\mathrm{d}x. \tag{47}$$

Helmholtz decomposition from Lemma 4.1 shows that

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot(\nabla u - \nabla_{\mathrm{DG}} u_h) \,\mathrm{d}x \\
=& \int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot(\nabla\psi + D\mathcal{A}(\tilde{\boldsymbol{\theta}})^{-1}\mathrm{curl}\,\chi) \,\mathrm{d}x \\
=& \int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot\nabla\psi \,\mathrm{d}x \\
&+ \int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot D\mathcal{A}(\tilde{\boldsymbol{\theta}})^{-1}\mathrm{curl}\,\chi \,\mathrm{d}x,
\end{aligned} \tag{48}$$

where $\tilde{\boldsymbol{\theta}} := \nabla_{\mathrm{DG}}\tilde{u} = \nabla_h\tilde{u} - \mathbf{S}(\tilde{u})$.

Let Π_0 be a modified piecewise constant projection from $W^{1,p}(\Omega)$ onto $L^2(\Omega)$, so that for all $z \in W^{1,p}(\Omega)$, $(\Pi_0 z)|_T := \frac{1}{|T|}\int_T z \,\mathrm{d}x$ for each T with $\partial T \cap \Gamma = \emptyset$, and $(\Pi_0 z)|_T := 0$ on each $T \in \mathcal{T}_h$ with an edge on Γ. We can obtain

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot\nabla\psi \,\mathrm{d}x \\
=& \sum_{T\in\mathcal{T}_h} \int_T (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot\nabla_h(\psi - \Pi_0\psi) \,\mathrm{d}x \\
=& \sum_{T\in\mathcal{T}_h} \left\{ -\int_T \mathrm{div}\,(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot(\psi - \Pi_0\psi) \,\mathrm{d}x \right. \\
&\left. + \int_{\partial T} (\psi - \Pi_0\psi)\,(\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}} u_h))\cdot\mathbf{n}_T \,\mathrm{d}s \right\} \\
=& \sum_{T\in\mathcal{T}_h} \left\{ \int_T (f + \mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}} u_h))\,(\psi - \Pi_0\psi) \,\mathrm{d}x \right. \\
&\left. + \int_{\partial T\setminus\Gamma} (\psi - \Pi_0\psi)\,(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T - \mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T) \,\mathrm{d}s \right\}.
\end{aligned} \tag{49}$$

The property of Π_0

$$\|\psi - \Pi_0\psi\|_{p,T} \lesssim h_T\|\nabla\psi\|_{p,T}. \tag{50}$$

and the Hölder inequality show that

$$\begin{aligned} &\int_T (f + \mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h))\,(\psi - \Pi_0\psi)\ \mathrm{d}x \\ \le&\|\,(f + \mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h))\,\|_{q,T}\|\psi - \Pi_0\psi\|_{p,T} \\ \lesssim& h_T\|\,(f + \mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h))\,\|_{q,T}\|\nabla\psi\|_{p,T}. \end{aligned} \tag{51}$$

The Hölder inequality and the trace inequality (8) imply that

$$\begin{aligned} &\int_{\partial T\backslash\Gamma} (\psi - \Pi_0\psi)\,(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T - \mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T)\ \mathrm{d}s \\ \le&\|\psi - \Pi_0\psi\|_{p,\partial T\backslash\Gamma}\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T - \mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma} \\ \lesssim& h_T^{1-\frac{1}{p}}\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T - \mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma}\|\nabla\psi\|_{p,T}. \end{aligned} \tag{52}$$

Now, it comes to the estimate of the second term of (48), the Lemma 4.2 shows that

$$\begin{aligned} &\int_\Omega (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla_{\mathrm{DG}}u_h))\cdot D\mathcal{A}(\tilde{\boldsymbol{\theta}})^{-1}\mathrm{curl}\ \chi\ \mathrm{d}x \\ =&\int_\Omega D\mathcal{A}(\tilde{\boldsymbol{\theta}})\,(\nabla u - \nabla_{\mathrm{DG}}u_h)\cdot D\mathcal{A}(\tilde{\boldsymbol{\theta}})^{-1}\mathrm{curl}\ \chi\ \mathrm{d}x \\ =&\int_\Omega (\nabla\bar{u}_h - \nabla_{\mathrm{DG}}u_h)\cdot\mathrm{curl}\,\chi\ \mathrm{d}x. \end{aligned} \tag{53}$$

The Hölder inequality leads to

$$\int_\Omega (\nabla\bar{u}_h - \nabla_{\mathrm{DG}}u_h)\cdot\mathrm{curl}\,\chi\ \mathrm{d}x \le \|\nabla\bar{u}_h - \boldsymbol{\theta}_h\|_{q,\Omega}\|\,\mathrm{curl}\,\chi\|_{p,\Omega}. \tag{54}$$

The equivalent property of quasi-norm shows that

$$\begin{aligned} \|\nabla u - \nabla_{\mathrm{DG}}u_h\|_{p,\Omega}^p =& \int_\Omega |\nabla_{\mathrm{DG}}(u - u_h)|^p\ \mathrm{d}x \\ \le& \int_\Omega |\nabla_{\mathrm{DG}}(u - u_h)|^2\,(|\nabla u| + |\nabla_{\mathrm{DG}}(u - u_h)|)^{p-2}\ \mathrm{d}x \\ =&|\nabla_{\mathrm{DG}}(u - u_h)|_{(w,p)} \sim \|F(\nabla u) - F(\nabla_{\mathrm{DG}}u_h)\|_{2,p,\Omega}^2, \end{aligned} \tag{55}$$

where $|\cdot|_{(w,p)}$ is the quasi-norm in [25]. More details can be refered to Proposition 3.1 in [25].

The combination of above estimates and Lemma 4.1 yields

$$\begin{aligned} &\|\nabla u - \nabla_{\mathrm{DG}}u_h\|_{p,\Omega}^{p-1} \\ \lesssim& \sum_{T\in\mathcal{T}_h} \Big\{ h_T\|\,(f + \mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h))\,\|_{q,T} + h_T^{1-\frac{1}{p}}\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T \\ &-\mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma} + \|\nabla\bar{u}_h - \boldsymbol{\theta}_h\|_{q,T}\Big\} \end{aligned} \tag{56}$$

and

$$\begin{aligned}
&\|\nabla u-\nabla_{\mathrm{DG}} u_h\|_{p,\Omega}^{p} \\
\lesssim &\sum_{T\in\mathcal{T}_h}\Big\{h_T\|\left(f+\mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\|_{q,T}+h_T^{1-\frac{1}{p}}\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T \\
&-\mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma}+\|\nabla\bar{u}_h-\boldsymbol{\theta}_h\|_{q,T}\Big\}^{\frac{p}{p-1}} \\
\lesssim &\sum_{T\in\mathcal{T}_h}\Big\{h_T^{q}\|\left(f+\mathrm{div}\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\|_{q,T}^{q}+h_T\|\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}_T \\
&-\mathcal{A}(\boldsymbol{\theta}_h)\cdot\mathbf{n}_T\|_{q,\partial T\backslash\Gamma}^{q}+\|\nabla\bar{u}_h-\boldsymbol{\theta}_h\|_{q,T}^{q}\Big\}.
\end{aligned}$$

This concludes the proof. □

Theorem 4.5 Let u_h be a approximated solution of problem (24), it holds that

$$\|F(\nabla u)-F(\nabla_{\mathrm{DG}}u_h)\|_{2,p,\Omega}^{2}+|E(u)-E_{\mathrm{DG}}(u_h)|\lesssim\sum_{T\in\mathcal{T}_h}\tilde{\eta}_T, \tag{57}$$

$$\begin{aligned}
\tilde{\eta}_T:=\ &\eta_T+h_T^{1-\frac{1}{p}}\|\left(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!])-\{\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\}\right)\cdot\mathbf{n}_T\|_{q,\partial T} \\
&+h_T\|f-\Pi_0 f\|_{q,T}+\left|\frac{\eta}{h_{\partial T}^{p-1}}\int_{\partial T}\mathcal{A}([\![u_h]\!])\cdot[\![u_h]\!]-W([\![u_h]\!])\ \mathrm{d}s\right|.
\end{aligned}\tag{58}$$

Proof. The (45)-(46) shows that

$$\begin{aligned}
&\|F(\nabla u)-F(\nabla_{\mathrm{DG}}u_h)\|_{2,p,\Omega}^{2}+c_2(p)|E(u)-E_{\mathrm{DG}}(u_h)| \\
\le &c_2(p)\left[\left|\int_\Omega\left(\mathcal{A}(\nabla u)-\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\cdot(\nabla u-\nabla_{\mathrm{DG}}u_h)\ \mathrm{d}x\right|\right. \\
&+\left|\int_\Omega\left(\mathcal{A}(\nabla u)-\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\cdot\nabla u\ \mathrm{d}x\right| \\
&\left.+\left|\sum_{E\in\Gamma_h}\frac{\eta}{h_E^{p-1}}\int_E\mathcal{A}([\![u_h]\!])\cdot[\![u_h]\!]-W([\![u_h]\!])\ \mathrm{d}s\right|\right] \\
:= &c_2(p)\left(|I_1|+|I_2|+|I_3|\right).
\end{aligned}\tag{59}$$

We already have an estimate of I_1 from Theorem 4.4. Now we deal with term I_2, $(\nabla_{\mathrm{DG}}u_h)|_T\in[\mathbf{P}_0(T)]^2$ implies that

$$\begin{aligned}
I_2=&\int_\Omega\left(\mathcal{A}(\nabla_{\mathrm{DG}}u)-\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\right)\cdot\nabla u\ \mathrm{d}x \\
=&\int_\Omega fu\ \mathrm{d}x-\int_\Omega\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\cdot\nabla_h(\Pi_1 u)\ \mathrm{d}x \\
=&\int_\Omega f(u-\Pi_1 u)\ \mathrm{d}x-\sum_{E\in\Gamma_h}\int_E\{\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\}\cdot[\![\Pi_1 u]\!]\ \mathrm{d}s \\
&+\sum_{E\in\Gamma_h}\left(\eta\int_E\mathcal{A}(h_E^{-1}[\![u_h]\!])\ [\![\Pi_1 u]\!]\ \mathrm{d}s\right) \\
=&\int_\Omega f(u-\Pi_1 u)\ \mathrm{d}x+\sum_{T\in\mathcal{T}_h}\left(\int_{\partial T}\left(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!])-\{\mathcal{A}(\nabla_{\mathrm{DG}}u_h)\}\right)\cdot\mathbf{n}_T\ \Pi_1 u\ \mathrm{d}s\right).
\end{aligned}\tag{60}$$

Applying Lemma 4.3 and local L^2 projection property we introduced in Section 2, we obtain

$$\int_\Omega f(u - \Pi_1 u)\, \mathrm{d}x \lesssim \sum_{T\in\mathcal{T}_h} h_T \|f - \Pi_0 f\|_{q,T}. \tag{61}$$

On the other hand, The Lemma 4.1 and the Lemma 4.3 imply that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \left(\int_{\partial T} \big(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!]) - \{\mathcal{A}(\nabla_{\mathrm{DG}} u_h)\}\big) \cdot \mathbf{n}_T\ \Pi_1 u\ \mathrm{d}s \right) \\
&= \sum_{T\in\mathcal{T}_h} \left(\int_{\partial T} \big(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!]) - \{\mathcal{A}(\nabla_{\mathrm{DG}} u_h)\}\big) \cdot \mathbf{n}_T\ (u - \Pi_1 u)\ \mathrm{d}s \right) \\
&\le \sum_{T\in\mathcal{T}_h} \| \big(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!]) - \{\mathcal{A}(\nabla_{\mathrm{DG}} u_h)\}\big) \cdot \mathbf{n}_T\|_{q,\partial T}\ \|u - \Pi_1 u\|_{p,\partial T} \qquad (62)\\
&\lesssim \sum_{T\in\mathcal{T}_h} \| \big(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!]) - \{\mathcal{A}(\nabla_{\mathrm{DG}} u_h)\}\big) \cdot \mathbf{n}_T\|_{q,\partial T}\ h_T^{1-\frac{1}{p}} \|\nabla u\|_{p,T} \\
&\lesssim h_T^{1-\frac{1}{p}} \sum_{T\in\mathcal{T}_h} \| \big(\eta\mathcal{A}(h_{\partial T}^{-1}[\![u_h]\!]) - \{\mathcal{A}(\nabla_{\mathrm{DG}} u_h)\}\big) \cdot \mathbf{n}_T\|_{q,\partial T}.
\end{aligned}$$

I_3 is computable. The combination of above estimates verifies the assertion of the theorem. □

5. Numerical experiments

This section is devoted to the numerical investigation of the lowest-order schemes of LDG for the p-Laplace problem on square domain. The numerical experiments concern the practical application of Theorem 4.5. Denote the left-hand side of the estimates by $LHS = \|F(\nabla u) - F(\nabla_{\mathrm{DG}} u_h)\|_{2,p,\Omega}^2 + |E(u) - E_{\mathrm{DG}}(u_h)|$. The global upper bounds read $GUB = \sum_{T\in\mathcal{T}_h} \tilde{\eta}_T$.

The triangulations are either uniform refinement or with an adaptive mesh-refinement algorithm with initial mesh $\mathcal{T}_0$ and then, for any triangle T of a triangulation $\mathcal{T}_l$ at level $l = 0, 1, 2, 3...$. Given all those contributions, mark some set $T' \in \mathcal{M}_\ell$ of triangles in $\mathcal{T}_\ell$ of minimal cardinality with the bulk criterion

$$1/2 \sum_{T\in\mathcal{T}_\ell} \tilde{\eta}^2(T) \le \sum_{T'\in\mathcal{M}_\ell} \tilde{\eta}^2(T').$$

The refinement of all triangles in $\mathcal{M}_\ell$ plus minimal further refinements to avoid hanging nodes lead to the triangulation $T_{\ell+1}$ within the newest-vertex bisection. The choice of the refinement-indicator $\tilde{\eta}(T)$ is motivated by the convergence theory of adaptive mesh-refining algorithms e.g. in the review article [10] with further details on the mesh-refinement. The convergence history plots displays the left-hand sides and the upper bounds as function of the number of degrees of freedom (ndof) in a log-log scale.

5.1. Example 1. Consider the p-Laplace problem on the square domain $\Omega := (0,1)^2$ with the exact solution $u = \sin(\pi x)\sin(\pi y)$ and the right-hand side term $f = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ for $p = 3$.

Figure 1 displays the triangulations generated by ALDG for Theorem 4.5. Figure 2 displays the global upper bounds and the corresponding error terms of the estimates for uniform and adaptive mesh-refinement.

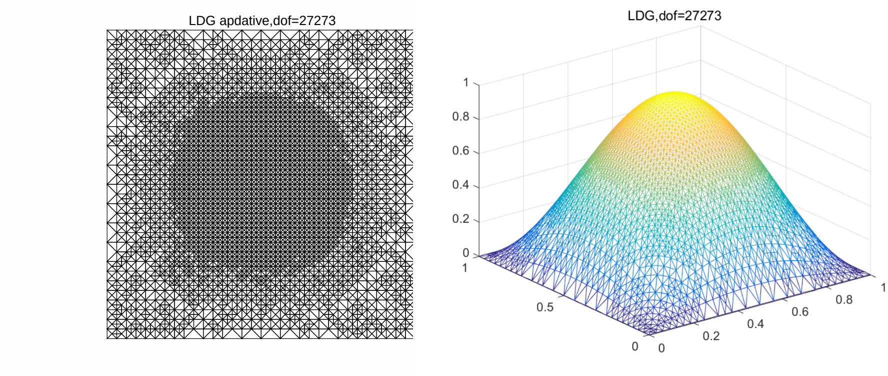

FIGURE 1. Adpative mesh of LDG method for $p = 3$.

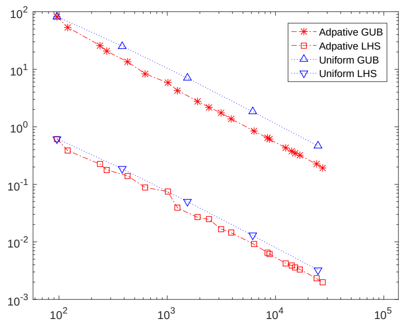

FIGURE 2. Convergence history of LDG method for $p = 3$.

5.2. Example 2. Consider the p-Laplace problem on the square domain $\Omega := (0,1)^2$ with the exact solution

$$u = 0.0005x^2(x-1)^2y^2(y-1)^2e^{10x^2+10y},$$

and the right-hand side term $f = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ for $p = 3$.

Figure 3 displays the triangulations generated by ALDG for Theorem 4.5. Figure 4 displays the global upper bounds and the corresponding error terms of the estimates for uniform and adaptive mesh-refinement.

5.3. Conclusions. We consider the adaptive LDG method based on the new defined distance. The main features of our technique is that the a posteriori error estimates provide reliability upper bounds with discretization error, and the error of the energy can be presented at the same time. The numerical examples show that the convergence results are consistent with the theoretical analysis. Furthermore, the associated adaptive method is shown to be much more efficient than a uniform refinement to compute the discrete solutions. In particular, the experiments illustrate the ability of the adaptive algorithm to localize the singularities of each problem.

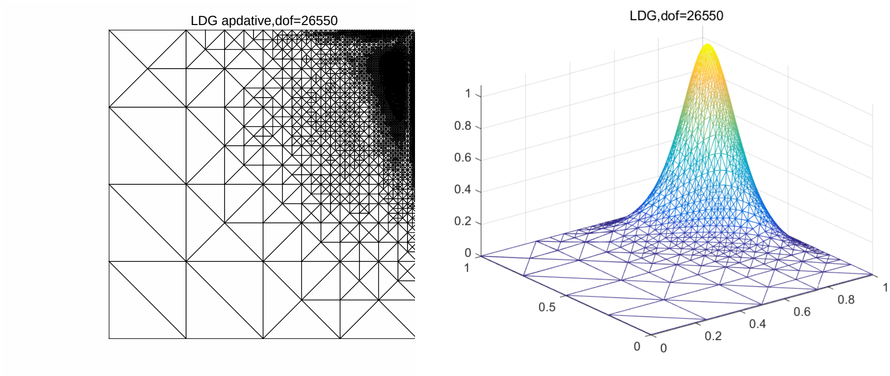

FIGURE 3. Adpative mesh of LDG method for $p = 3$.

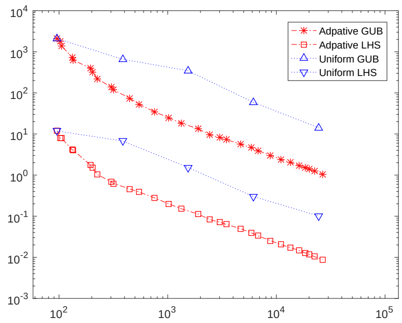

FIGURE 4. Convergence history of LDG method for $p = 3$.

Acknowledgments

The author would like to thank the referees for the helpful suggestions. This research was supported by National Natural Science Foundation of China (No. 11571226, 11671313, 11771364).

References

[1] M. Ainsworth and D. Kay, Approximation theory for the hp-version finite element method and application to the non-linear Laplacian, Appl. Numer. Math., 34 (2000), 329-344.
[2] J.W. Barrett and W.B. Liu, Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow, Numer. Math., 68 (1994), 437-456.
[3] R. Becker, P. Hansbo and M. G. Larson, Energy norm a posteriori error estimation for discontinuous Galerkin methods, Comput. Meth. Appl. Mech. Eng., 192 (2003), 723-733.
[4] L. Belenki and L. Diening, Optimality of an adaptive finite element method for the p-Laplacian equation, IMA J. Numer. Anal., 32 (2012), 484-510.
[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential, Springer, New York, 2010.
[6] A. Buffa and C. Ortner, Compact embeddings of broken Sobolev spaces and applications, IMA J. Numer. Anal., 29 (2009), 827-855.
[7] E. Burman and A. Ern, Discontinuous Galerkin approximations with discrete variational principle for the nonlinear Laplacian, C. R. Math. Acad. Sci. Paris., 346 (2008), 1013-1016.
[8] R. Bustinza and G.N. Gatica, A local discontinuous Galerkin method for nonlinear diffusion problems with mixed boundary conditions, SIAM J. Sci. Comput., 26 (2004), 152-177.

[9] R. Bustinza, G.N. Gatica, and B. Cockburn, An A Posteriori Error Estimate for the Local Discontinuous Galerkin Method Applied to Linear and Nonlinear Diffusion Problems, J. Sci. Comput., 22-23(1-3) (2005),147-185.
[10] C. Carstensen, M. Feischl, M. Page, D. Praetorius, Axioms of adaptivity, Comput. Meth. Appl. Mat., 67 (2014), 1195-1253.
[11] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM J. Numer. Anal., 38 (2000), 1676-1706.
[12] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, 1978.
[13] E. Creuse, M. Farhloul, L. Paquet, A posteriori error estimation for the dual mixed finite element method for the p-Laplacian in a polygonal domain, Comput. Method. Appl. M., 196 (2007), 2570-2582.
[14] B. Cockburn, G. Kanschat, I. Perugia, and D. Schötzau, Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids, SIAM J. Numer. Anal., 39 (2001), 264-285.
[15] B. Cockburn and C.W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, SIAM J. Numer. Anal., 35 (1998), 2440-2463.
[16] L. Diening and F. Ettwein, Fraction estimates for non-differentiable elliptic systems with general growth, Forum. Math., 20 (3) (2008), 523-556.
[17] L. Diening, D. Kröner, M. Růžička, and I. Toulopoulos, A local discontinuous Galerkin approximation for systems with p-structure, IMA J. Numer. Anal., 34 (2014), 1447-1488.
[18] L. Diening and M. Růžička, Interpolation operators in Orliz-Sobolev spaces, Numer. Math., 107 (2007), 107-129.
[19] L. Diening and M. Růžička, Non-Newtonian fluids and function spaces, Nonlinear Anal. Funct. Spaces Appl., 8 (2007), 95-143.
[20] C. Ebmeyer and W. B. Liu, Quasi-norm interpolation error estimates for finite element approximations of problems with p-structure, Numer. Math., 100(2005), 233-258.
[21] V. Girault, and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Ser. Comput. Math., 1986.
[22] D. Kröner, M. Růžička, and I. Toulopoulos, Numerical solutions of systems with (p,δ)-structure using local discontinuous Galerkin finite element methods, Int J. Numer. Meth. Fl., 76(11) (2014), 855-874.
[23] D.J. Liu, A.Q. Li and Z.R. Chen, Nonconforming FEMs for the P-Laplace Problem, Adv. Appl. Math. Mech., 10(6)(2018), 1365-1383.
[24] D.J. Liu and Z.R. Chen, The adaptive finite element method for the P-Laplace problem, Appl. Numer. Math., 152(2020), 323-337.
[25] W.B. Liu and N.N. Yan, Quasi-norm a priori and a posteriori error estimates for the nonconforming approximation of p-Laplacian, Numer. Math., 89 (2001), 341-378.
[26] W.B. Liu and N.N. Yan, On quasi-norm interpolation error estimation and a posteriori error estimates for p-Laplacian, SIAM J. Numer. Anal., 40 (2002), 1870-1895.
[27] C.G. Simader and H. Sohr, A new approach to the helmholtz decomposition and the neumann problem in L^q-spaces for bounded and exterior domains, Ser. Adv. Math. Appl. Sci., 1992(8):11.
[28] A. Veeser, Convergent adaptive finite elements for nonlinear Laplacian, Numer. Math., 92 (2002), 743-770.

Department of Mathematics, College of Sciences, Shanghai University, Shanghai 200444, PR China
E-mail: `liudj@shu.edu.cn and zuole@shu.edu.cn`

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, PR China
E-mail: `xpzhang.math@whu.edu.cn`