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REDUCED BASIS FINITE ELEMENT METHODS FOR THE KORTEWEG-DE VRIES-BURGERS EQUATION

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Abstract. In this paper, the B-spline Galerkin finite element method and reduced order method for the Korteweg-de Vries-Burgers equation are considered. The semi-discrete and the fully discrete schemes are both provided. The reduced order model of the Korteweg-de Vries-Burgers equation by using proper orthogonal decomposition are provided. The stability and the error estimates of the corresponding schemes are then analyzed. Finally, numerical simulations are presented to show the efficiency of our proposed methods.

Key words. Korteweg-de Vries-Burgers, proper orthogonal decomposition, reduced order modeling, error analysis.

1. Introduction

In this paper, we propose numerical methods for solving the Korteweg-de Vries-Burgers (KdVB) equation: Given $\Omega = [-L, L]$, determine u such that

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0, \qquad x \in \Omega, 0 \le t \le T, \tag{1}$$

$$u(-L, t) = u(L, t) = 0, \qquad x \in \Omega, 0 \le t \le T, \tag{2}$$

$$u_x(-L, t) = u_x(L, t) = 0, \qquad x \in \Omega, 0 \le t \le T, \tag{3}$$

$$u\left(x, 0\right) = u_0\left(x\right), \qquad x \in \Omega, \tag{4}$$

where $\varepsilon, \nu, \mu \in \mathbb{R}$ are real positive parameters with $\varepsilon\nu\mu \neq 0$.

For the past several decades, many mathematicians and physicists have paid great attention to such kind of problems. The KdVB equation is one of the most important non-linear partial differential equations, which was developed by Su and Gardner [25] to describe the weak effects of dispersion, dissipation and nonlinearity of wave propagation in a liquid-filled elastic tube. For the parameter $\nu = 0$, (1)-(4) will be reduced to the KortewegDe Vries (KdV) equation which has been used to describe the dynamical effects, i.e., ion sound, plasma shock wave [10, 23, 24, 30]. For the parameter $\mu = 0$, (1)-(4) will be simplified to the Burgers equation that has a widely physical application in many fields, i.e., shock wave propagation, turbulence flow, etc. Some theoretical regularities such as the existence, uniqueness, stability of KdV-type equations have been studied in [1, 9, 14, 22], etc. The KdVB equation incorporates the properties of the KdV equation and Burgers equation which are of great interest to be studied, and has high research value in applied mathematics.

Many numerical methods have been already studied for the KdV-type equations, i.e., the finite element method [11, 28] and finite difference discontinuous Galerkin method [12], finite difference method [16, 27], etc. In this paper, we will numerically analyze and simulate the KdVB equation. Numerical simulations of nonlinear systems are relatively expensive with respect to both the storage and the computational complexity, where the iterative methods for the nonlinear system are usually required. To efficiently solve this kind of problems, many reduced-order modeling

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techniques are developed. One of the popular reduced order methods at least for the applications is the proper orthogonal decomposition (POD) analysis. The POD techniques combined with the Galerkin methods have been widely used to formulate the reduced order modelings for dynamic systems [2, 5, 6, 15, 17, 20, 21], which can provide precise approximation with reduced number of degrees of freedom. Moreover, the induced lower dimensional models alleviate the computational load and memory requirements [3]. In this paper, the approaches to efficiently handle the nonlinear terms and generate the snapshots are referred to the techniques for the reduced-order modeling for the Navier–Stokes equations [5, 6, 29]. Similarly to the fourth order equations [7], the third-order KdVB equation inherits higher regularity than that for the second order partial differential equation. In turn, the usual C^0 finite element basis with less regularity for the second order partial differential equation is usually not feasible for the KdVB equation.

In this paper, the quadratic B-spline basis with continuous first derivative is used The main contribution of paper is to perform theoretical analyses of the quadratic B-spline Galerkin finite element approximation for the KdVB equation and its related reduced order modeling based on the POD Galerkin finite element approximation.

This paper is organized as follows. In following part of Section 1, we introduce the notation and preliminaries which are used throughout the paper. In Section 2, the Galerkin finite element methods are provided. The semidiscrete and fully discrete schemes are analyzed. In Section 3, we obtain the reduced dimension surrogate model of the KdVB equation by using proper orthogonal decomposition technique. We also indicate the error between the reduced model solution and its regular solution. Numerical simulations are presented in the final section.

1.1. Notation and Preliminaries. We use standard notation for the function spaces. For any integer $k \geq 0$, $H^k(\Omega)$ denotes the Sobolev space on Ω associated with inner product $(\cdot,\cdot)_{H^k}$ and norm $\|\cdot\|_k$. On the space $L^2(\Omega) := H^0(\Omega)$, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the L^2 inner product and norm, respectively. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The Sobolev space $H_0^k(\Omega)$ with $k = 2$ is then defined as

$$H_0^k(\Omega) = \left\{ u \in H^k(\Omega) : \partial_x^j u(x) = \partial_x^j u(x+L) = 0, j = 0, \ldots, k-1 \right\}, \tag{5}$$

where $\partial_x^j u(x)$ represents the jth derivative in the sense of distribution with respect to x of a function u. Let $C([0,T];H^k)$ be the space of all continuous functions $u : [0,T] \to H^k(\Omega)$ with $\|u\|_{C([0,T];H^k)} = \max_{0\leq t\leq T} \|u(t)\|_k < \infty$. Denote by $L^2([0,T];H^k)$ the space of square integrable functions $u : [0,T] \to H^k(\Omega)$ with $\|u\|^2_{L^2([0,T];H^k)} = \int_0^T \|u(t)\|_k^2 dt < \infty$. To be brief, we set $u_x := \partial_x u$ and $u_{xx} := \partial_x^2 u$. The variational form of (1)-(4) is derived by multiplying (1) with a function $v(x) \in H_0^2(\Omega)$ and integrating by parts on Ω. The weak formulation of (1)-(4) is then written as

$$(u_t, v) - \frac{\varepsilon}{2}\left(u^2, v_x\right) + \nu\left(u_x, v_x\right) - \mu\left(u_{xx}, v_x\right) = 0 \tag{6}$$

for almost every $t \in [0,T]$.

Let $M \in \mathbb{N}^+$ be a positive integer. Define the spatial mesh $\mathcal{T}_h$ with mesh size $h = 2L/M$. The grid points are denoted as $x_j = -L + jh, j = 0, 1, \ldots, M$ with subintervals $I_j = [x_j, x_{j+1}], j = 0, 1, \cdots, M-1$. Let $P_r(I)$ denote the space of polynomials on the interval I of degree no greater than $r \in \mathbb{N}^+$. We seek a discrete approximation u_h to the solution of (1)-(4) such that for all $t \in [0,T]$, $u_h(t)$ belongs to the space

$$S_h(\Omega) = \{v_h \in H_0^2(\Omega) : v_h|_{I_j} \in P_r(I_j), j = 0, 1, \dots, M-1\}. \tag{7}$$

$S_h(\Omega)$ possesses some approximation properties [8, 26]: For $u \in H^k(\Omega) \bigcap H_0^2(\Omega)$, then

$$\inf_{v_h \in S_h(\Omega)} \left\{ \|u - v_h\| + h \|u - v_h\|_1 \right\} \le C h^{r+1} \|u\|_{r+1}, \quad 0 \le r < k. \tag{8}$$

To do the error estimates, we introduce the Ritz projection P_h onto $S_h(\Omega)$, which is defined as an orthogonal projection with respect to the inner product (v_x, w_x) : For $v \in H^1(\Omega)$, so that

$$((P_h v)_x, \chi_x) = (v_x, \chi_x), \quad \forall \chi \in S_h(\Omega), \tag{9}$$

where P_h satisfies following Lemma 1.1.

Lemma 1.1 ([8, 26])**.** *Assuming that (8) holds, then there exists a constant $C > 0$ such that the following inequalities holds with $v, v_t \in H^k(\Omega)$, $k > r$ and $P_h v \in S_h(\Omega)$,*

$$\|P_h v - v\| + h \|(P_h v - v)_x\| \le C h^{r+1} \|v\|_{r+1}, \tag{10}$$

$$\|(P_h v - v)_t\| + h \|(P_h v - v)_{tx}\| \le C h^{r+1} \|v_t\|_{r+1}. \tag{11}$$

Lemma 1.2. *Let $\mathcal{T}_h$ be a quasi-uniform triangulation. Then the inverse inequality holds with $v_h \in S_h(\Omega)$,*

$$\|\partial_x(v_h)\| \le C h^{-1} \|v_h\|, \tag{12}$$

where C is independent of h.

Now, we are going to introduce the Gronwall inequality.

Lemma 1.3 ([13])**.** *Assuming that the continuous function $\phi(t)$ holds*

$$|\phi(t)| \le \beta + \alpha \int_a^t |\phi(\tau)| \, d\tau, \quad a \le t \le b, \tag{13}$$

where α, β are nonnegative constants. Then

$$|\phi(t)| \le \beta e^{\alpha(t-a)}, \quad a \le t \le b. \tag{14}$$

Lemma 1.4. *For integers $n, N \ge 0$, let κ_n, a_n, b_n, c_n and D be non-negative numbers such that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \Delta t \sum_{n=0}^{N} \kappa_n a_n + \Delta t \sum_{n=0}^{N} c_n + D, \quad \forall N \ge 0. \tag{15}$$

Suppose that for all n, $\Delta t \kappa_n < 1$, then it follows

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \exp\left(\sum_{n=0}^{N} \frac{\Delta t \kappa_n}{1 - \Delta t \kappa_n} \right) \left(\Delta t \sum_{n=0}^{N} c_n + D \right), \quad \forall N \ge 0. \tag{16}$$

2. Galerkin finite element for the KdVB equation

2.1. Semi-discrete Galerkin method. In this part, we consider stability and the error estimate for the semi-discrete Galerkin finite element approximation of the KdVB equation (1)-(4), which can be expressed as: Find $u_h = u_h(t) \in S_h(\Omega)$ such that for all $v \in S_h(\Omega)$

$$(u_{ht}, v) - \frac{\varepsilon}{2} \left(u_h^2, v_x \right) + \nu \left(u_{hx}, v_x \right) - \mu \left(u_{hxx}, v_x \right) = 0, \tag{17}$$

where the initial condition $u_h(x, 0)$ is defined as $P_h u_0$.

Lemma 2.1 (Boundedness)**.** *Let u be induced by (6) and u_h be a solution of (17). $\|u(t)\|$ and $\|u_h(t)\|$ are bounded for all $t \in [0,T]$, such that*

$$\|u(t)\| \leq \|u(0)\|, \tag{18}$$

$$\|u_h(t)\| \leq \|u_h(0)\|. \tag{19}$$

Proof. We consider the discrete weak formulation (17). Taking $v = u_h \in S_h(\Omega)$ in (17), we briefly have

$$(u_{ht}, u_h) - \frac{\varepsilon}{2}\left(u_h^2, u_{hx}\right) + \nu\left(u_{hx}, u_{hx}\right) - \mu\left(u_{hxx}, u_{hx}\right) = 0. \tag{20}$$

Since the terms (u_h^2, u_{hx}) and (u_{hxx}, u_{hx}) are vanishing, (20) can be directly rewritten as

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 = -\nu\|u_{hx}\|^2. \tag{21}$$

Integrating (21) from 0 to t, we obtain that

$$\|u_h(t)\|^2 = \|u_h(0)\|^2 - 2\nu\int_0^t \|u_{hx}\|^2 dt \leq \|u_h(0)\|^2. \tag{22}$$

The same assertion for the boundedness of $\|u(t)\|$ can be concluded by taking $v = u$ in (6). □

Given $t \in (0,T]$, we consider the following error decomposition

$$u_h(t) - u(t) = (u_h(t) - P_h u(t)) - (u(t) - P_h u(t)) = \eta - \xi. \tag{23}$$

Theorem 2.1. *Assume that $u \in H^{r+1}(\Omega) \cap H_0^2(\Omega)$ is a solution of (1)-(4). Let u_h be a solution of (17) in $S_h(\Omega)$. There has a constant C depending on u and T for so that*

$$\|u_h(t) - u(t)\| \leq Ch^{r-1}. \tag{24}$$

Proof. It is obvious that ξ in (23) is bounded with Lemma 1.1. Our goal is to estimate the error for $\eta \in S_h(\Omega)$. Subtracting the equation (6) from (17) with $v \in S_h(\Omega)$, the following error equation is then obtained

$$\begin{aligned}(\eta_t, v) - (\xi_t, v) - \frac{\varepsilon}{2}\left(u_h^2 - u^2, v_x\right) + \nu\left(\eta_x, v_x\right)\\ - \nu\left(\xi_x, v_x\right) - \mu\left(\eta_{xx}, v_x\right) + \mu\left(\xi_{xx}, v_x\right) = 0.\end{aligned} \tag{25}$$

Setting $v = \eta \in S_h(\Omega)$ in (25), we have

$$\begin{aligned}(\eta_t, \eta) - (\xi_t, \eta) - \frac{\varepsilon}{2}\left(u_h^2 - u^2, \eta_x\right) + \nu\left(\eta_x, \eta_x\right)\\ - \nu\left(\xi_x, \eta_x\right) - \mu\left(\eta_{xx}, \eta_x\right) + \mu\left(\xi_{xx}, \eta_x\right) = 0.\end{aligned} \tag{26}$$

With the Young's inequality and the boundedness of $\|u\|$ and $\|u_h\|$ from Lemma 2.1, taking $\sigma = \frac{2\varepsilon}{\nu}$ for the nonlinear term, we have

$$\begin{aligned}\frac{\varepsilon}{2}\left(u_h^2 - u^2, \eta_x\right) &\leq \frac{\varepsilon}{2}\sigma\left\|u_h^2 - u^2\right\|^2 + \frac{\varepsilon}{8\sigma}\left\|\eta_x\right\|^2\\ &\leq \frac{C\varepsilon^2}{\nu}\left(\|\eta\|^2 + \|\xi\|^2\right) + \frac{\nu}{16}\left\|\eta_x\right\|^2.\end{aligned} \tag{27}$$

For the term related to (η_{xx}, η_x), we have that

$$(\eta_{xx}, \eta_x) = \frac{1}{2}\int_\Omega \frac{d(\eta_x)^2}{dx}dx = (\eta_x(L))^2 - (\eta_x(-L))^2 = 0. \tag{28}$$

According to the inverse inequality and Young's inequality, (26) can be written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta\|^2+\nu\|\eta_x\|^2 &= (\xi_t,\eta)+\frac{\varepsilon}{2}\left(u_h^2-u^2,\eta_x\right)+\nu(\xi_x,\eta_x)-\mu(\xi_{xx},\eta_x)\\ &\le \frac{1}{2}\left(\|\xi_t\|^2+\|\eta\|^2\right)+\frac{C\varepsilon^2}{\nu}\left(\|\eta\|^2+\|\xi\|^2\right)+\frac{\nu}{16}\|\eta_x\|^2\\ &+\frac{\nu}{2}\left(\|\xi_x\|^2+\|\eta_x\|^2\right)+\frac{C\mu^2}{\nu h^2}\|\xi_x\|^2+\frac{\nu}{16}\|\eta_x\|^2.\end{aligned}\tag{29}$$

After simplification, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta\|^2+\frac{3\nu}{8}\|\eta_x\|^2 &\le \frac{1}{2}\left(\|\xi_t\|^2+\|\eta\|^2\right)+\frac{C\varepsilon^2}{\nu}\left(\|\eta\|^2+\|\xi\|^2\right)\\ &+\frac{\nu}{2}\|\xi_x\|^2+\frac{C\mu^2}{\nu h^2}\|\xi_x\|^2.\end{aligned}\tag{30}$$

Further, we briefly have

$$\frac{d}{dt}\|\eta\|^2\le C\left(\|\xi_t\|^2+\|\eta\|^2+\|\xi\|^2+\|\xi_x\|^2+h^{-2}\|\xi_x\|^2\right).\tag{31}$$

Integrating (31) from 0 to t yields

$$\begin{aligned}&\|\eta(t)\|^2\le\|\eta(0)\|^2+C\int_0^t\|\eta(\tau)\|^2\,d\tau\\ &+C\int_0^t\left(\|\xi(\tau)\|^2+\|\xi_t(\tau)\|^2+\|\xi_x(\tau)\|^2+h^{-2}\|\xi_x(\tau)\|^2\right)d\tau.\end{aligned}\tag{32}$$

Since $\eta(0)=u_h(0)-P_hu_0=0$, and $\|\xi\|$,$\|\xi_t\|$ and $\|\xi_x\|$ are also bounded as claimed in Lemma 1.1, with Lemma 1.3, it follows that

$$\begin{aligned}\|\eta\|^2 &\le Ch^{2r-2}\int_0^t\left(\left(1+h^2+h^4\right)\|u(\tau)\|_{r+1}^2+h^4\|u_t(\tau)\|_{r+1}^2\right)d\tau\\ &+C\int_0^t\|\eta(\tau)\|^2\,d\tau\\ &\le Ch^{2r-2}+C\int_0^t\|\eta(\tau)\|^2\,d\tau\\ &\le \exp(CT)h^{2r-2}.\end{aligned}\tag{33}$$

With the triangle inequality $\|u_h-u\|\le\|\eta\|+\|\xi\|$ and Lemma 1.1, we conclude the proof. □

Remark 2.1. *Since the inverse inequality in Lemma 1.2 is used to approximate the term (ξ_{xx},η_x), the obtained convergence rate of the KdVB equation is only sub-optimal.*

2.2. Fully discrete Galerkin method. In this part, we proceed to see some properties of the fully discrete scheme of (1)-(4) according to the backward Euler discretization in time. Using the variant of the Brouwer fixed point lemma, the existence and local uniqueness of the fully discrete scheme are proved. The error estimate of the scheme is then provided.

Let N be a positive integer. The time step is expressed as k such that $k=T/N$ with $t_n=nk, n=0,1,\cdots N$. For convenience, the following notations are used for $v(t)\in S_h(\Omega)$ on $[0,T]$,

$$v^n=v(t^n).\tag{34}$$

The fully discrete Galerkin finite element approximation of $u(t^n)$ is to find $u_h^n \in S_h(\Omega)$, $i = 1, 2, \cdots, N$ such that for all $v \in S_h(\Omega)$,

$$\left(\frac{u_h^n - u_h^{n-1}}{k}, v\right) - \frac{\varepsilon}{2}\left(\left(u_h^n\right)^2, v_x\right) + \nu\left(u_{hx}^n, v_x\right) - \mu\left(u_{hxx}^n, v_x\right) = 0 \tag{35}$$

with the initial condition $u_h^0 = P_h u(0) \in S_h(\Omega)$.

Lemma 2.1 (Stability). *Let u_h^n be a solution of (35). The following L^2 boundedness is hold*

$$\|u_h^n\| \le \left\|u_h^0\right\|, \quad n \ge 1. \tag{36}$$

Proof. Taking $v = u_h^n \in S_h(\Omega)$ in (35), we get that

$$\left(\frac{u_h^n - u_h^{n-1}}{k}, u_h^n\right) - \frac{\varepsilon}{2}\left(\left(u_h^n\right)^2, u_{hx}^n\right) + \nu\left(u_{hx}^n, u_{hx}^n\right) - \mu\left(u_{hxx}^n, u_{hx}^n\right) = 0. \tag{37}$$

The terms $((u_h^n)^2, u_{hx}^n)$ and (u_{hxx}^n, u_{hx}^n) are vanishing. Obviously, with the identity $2(a, a-b) = (a-b, a-b) + (a, a) - (b, b)$, we can obtain that

$$\|u_h^n - u_h^{n-1}\|^2 + \|u_h^n\|^2 - \left\|u_h^{n-1}\right\|^2 = -2k\nu\|u_{hx}^n\|^2. \tag{38}$$

Then, summing equation (38) from 1 to n induces that

$$\|u_h^n\|^2 \le \left\|u_h^0\right\|^2 \tag{39}$$

which concludes the proof. □

The fully discrete scheme (35) induces a nonlinear system. The variant of the Brouwer fixed point lemma is further considered to illustrate existence of u_h^n in (35).

Lemma 2.2 ([4]). *Let H be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Furthermore, let $\mathcal{F}$ be a continuous mapping from H to H such that $(\mathcal{F}(w), w)_H \ge 0$ for all $w \in H$ with $\|w\|_H = \sigma > 0$. Then, there exists a $w^* \in H$ with $\|w^*\|_H \le \sigma$ satisfying $\mathcal{F}(w^*) = 0$.*

Theorem 2.2 (Existence and uniqueness). *Assume that a sequence of u_h^0, u_h^1, $\cdots$,$u_h^{n-1} \in S_h(\Omega)$ is given, then there exists $u_h^n \in S_h(\Omega)$, $n > 1$ satisfying (35). Further, when $\frac{2\varepsilon k\|u_h^0\|^2}{h^2} < 1$, scheme (35) is uniquely solvable.*

Proof. For the fully discrete scheme (35), we construct a continuous mapping $\mathcal{F}: S_h(\Omega) \to S_h(\Omega)$ as

$$\begin{aligned}(\mathcal{F}(w), v) = (w, v) - \left(u_h^{n-1}, v\right) - \frac{\varepsilon k}{2}\left(w^2, v_x\right) \\ + \nu k\left(w_x, v_x\right) - \mu k\left(w_{xx}, v_x\right), \quad \forall w, v \in S_h(\Omega).\end{aligned} \tag{40}$$

For $S_h(\Omega) \subset H^2(\Omega)$, it is trivial that $\mathcal{F}$ is continuous. Setting $v = w \in S_h(\Omega)$, we have

$$\begin{aligned}(\mathcal{F}(w), w) &\ge \|w\|^2 - \left(u_h^{n-1}, w\right) + \nu k(w_x, w_x) \\ &\ge \|w\|\left(\|w\| - \left\|u_h^{n-1}\right\|\right).\end{aligned}$$

For $\|w\| = \|u_h^0\|$, obviously $(\mathcal{F}(w), w) \ge 0$, which yields the existence of w^* with Lemma 2.2 such that $\mathcal{F}(w^*) = 0$. In fact, taking $u_h^n = w^*$, (40) is equivalent to (35). In this way, we proved the existence of the solution of (35).

For the uniqueness of (35), given $u_h^{n-1} \in S_h(\Omega)$, let u_h^n and v_h^n be two solutions of (35). Setting $w_h^n = u_h^n - v_h^n$, it follows that for $\chi \in S_h(\Omega)$

$$\begin{aligned}\left(\frac{w_h^n - w_h^{n-1}}{k}, \chi\right) - \frac{\varepsilon}{2}\left(\left(u_h^n\right)^2 - \left(v_h^n\right)^2, \chi_x\right) + \nu\left(w_{hx}^n, \chi_x\right) \\ -\mu\left(w_{hxx}^n, \chi_x\right) = 0.\end{aligned} \tag{41}$$

Further, taking $\chi = w_h^n \in S_h(\Omega)$ in (41) gives

$$\begin{aligned}\left(\frac{w_h^n - w_h^{n-1}}{k}, w_h^n\right) - \frac{\varepsilon}{2}\left(\left(u_h^n\right)^2 - \left(v_h^n\right)^2, w_{hx}^n\right) + \nu\left(w_{hx}^n, w_{hx}^n\right) \\ -\mu\left(w_{hxx}^n, w_{hx}^n\right) = 0.\end{aligned} \tag{42}$$

Using the boundedness of $\|u_h^n\|$ and $\|v_h^n\|$, Cauchy-Schwarz inequality and inverse inequality, we obtain that

$$\begin{aligned}\|w_h^n - w_h^{n-1}\|^2 + \|w_h^n\|^2 - \left\|w_h^{n-1}\right\|^2 &\leq \varepsilon k\left(\left(u_h^n\right)^2 - \left(v_h^n\right)^2, w_{hx}^n\right) \\ &\leq \varepsilon k\|(u_h^n)^2 - (v_h^n)^2\|\|w_{hx}^n\| \\ &\leq 2\varepsilon k\|u_h^0\|\|u_h^n - v_h^n\|\|w_{hx}^n\| \\ &\leq \frac{2\varepsilon k}{h^2}\|u_h^0\|\|w_h^n\|^2.\end{aligned} \tag{43}$$

Then we get

$$\left(1 - \frac{2\varepsilon k\|u_h^0\|}{h^2}\right)\|w_h^n\|^2 \leq \left\|w_h^{n-1}\right\|^2. \tag{44}$$

With the mathematical induction, for $\|w_h^{n-1}\| = 0$ and $\frac{2\varepsilon k\|u_h^0\|^2}{h^2} < 1$, we have $\|w_h^n\| = 0$ which means that $u_h^n = v_h^n$. In this way, we get the proof of the uniqueness of the solution of (35). ◻

We estimate the error between $u(t)$ of (1)-(4) and u_h^n of (35). Setting $u^n = u(t_n) \in H_0^2(\Omega)$, we decompose the error e^n at time instant t_n as

$$e^n = u_h^n - u^n = (u_h^n - P_h u^n) - (u^n - P_h u^n) = \eta^n - \xi^n. \tag{45}$$

Theorem 2.3. *Assume that $u(t) \in H^{r+1}(\Omega) \cap H_0^2(\Omega)$ of (6) is a regular solution with $u_{tt}, u_{ttt} \in L^2([0,T]; L^2) \cap C(0,T; H^{r+1})$ and $u_h^0 = P_h u_0$. Then, there exists a constant $C > 0$ which is independent of h and k, so that*

$$\|u_h^N - u(t_N)\|^2 + \nu k\sum_{n=1}^{N}\|u_h^N - u(t_N)\|^2 \leq C(h^{2r-2} + k^2). \tag{46}$$

Proof. Taking $v \in S_h(\Omega)$ in the continuous variational formulation (6) gives

$$(u_t^n, v) - \frac{\varepsilon}{2}\left(\left(u^n\right)^2, v_x\right) + \nu\left(u_x^n, v_x\right) - \mu\left(u_{xx}^n, v_x\right) = 0. \tag{47}$$

The Galerkin finite element approximation u_h^n of $u(t^n)$ is written as, with $v \in S_h(\Omega)$

$$\left(\frac{u_h^n - u_h^{n-1}}{k}, v\right) - \frac{\varepsilon}{2}\left(\left(u_h^n\right)^2, v_x\right) + \nu\left(u_{hx}^n, v_x\right) - \mu\left(u_{hxx}^n, v_x\right) = 0. \tag{48}$$

Subtracting the equation (47) from (48), we have the error equation as

$$\begin{aligned}\left(\frac{\eta^n - \eta^{n-1}}{k}, v\right) + \nu\left(\eta_x^n, v_x\right) - \mu\left(\eta_{xx}^n, v_x\right) = \left(\frac{\xi^n - \xi^{n-1}}{k}, v\right) + (\rho^n, v) \\ -\frac{\varepsilon}{2}\left(\left(u^n\right)^2 - \left(u_h^n\right)^2, v_x\right) + \nu\left(\xi_x^n, v_x\right) - \mu\left(\xi_{xx}^n, v_x\right)\end{aligned} \tag{49}$$

with $\rho^n = u_t(t_n) - \frac{u^n - u^{n-1}}{k}$. Setting $v = \eta^n \in S_h(\Omega)$ in (49) gives

$$\left(\frac{\eta^n - \eta^{n-1}}{k}, \eta^n\right) + \nu\left(\eta_x^n, \eta_x^n\right) - \mu\left(\eta_{xx}^n, \eta_x^n\right) = \left(\frac{\xi^n - \xi^{n-1}}{k}, \eta^n\right) + \left(\rho^n, \eta^n\right) - \frac{\varepsilon}{2}\left(\left(u^n\right)^2 - \left(u_h^n\right)^2, \eta_x^n\right) + \nu\left(\xi_x^n, \eta_x^n\right) - \mu\left(\xi_{xx}^n, \eta_x^n\right). \tag{50}$$

Using the Young's inequalities and the inverse inequality, we have

$$\begin{aligned}&\frac{1}{2}\left\|\eta^n - \eta^{n-1}\right\|^2 + \frac{1}{2}\left\|\eta^n\right\|^2 - \frac{1}{2}\left\|\eta^{n-1}\right\|^2 + \nu k\left\|\eta_x^n\right\|^2\\ &\leq \frac{k}{2}\left\|\frac{\xi^n - \xi^{n-1}}{k}\right\|^2 + k\left\|\eta^n\right\|^2 + \frac{k}{2}\left\|\rho^n\right\|^2 + \frac{Ck\varepsilon^2}{\nu}\left(\left\|\eta^n\right\|^2 + \left\|\xi^n\right\|^2\right)\\ &+ \frac{k\nu}{16}\left\|\eta_x^n\right\|^2 + \ k\nu\left(\frac{1}{4}\left\|\eta_x^n\right\|^2 + \left\|\xi_x^n\right\|^2\right) + \frac{Ck\mu^2}{\nu h^2}\left\|\xi_x^n\right\|^2 + \frac{k\nu}{16}\left\|\eta_x^n\right\|^2.\end{aligned} \tag{51}$$

With the CauchySchwarz inequality and Taylor's formula, we obtain

$$\|\xi^n - \xi^{n-1}\|^2 \leq k\int_{t_{n-1}}^{t_n} \|\partial_t \xi\|^2 ds, \tag{52}$$

$$\left\|\rho^n\right\|^2 \leq Ck\int_{t_{n-1}}^{t_n} \left\|u_{tt}(s)\right\|^2 ds. \tag{53}$$

Combining the above inequalities, (51) is rewritten as

$$\begin{aligned}\|\eta^n\|^2 - \|\eta^{n-1}\|^2 + \nu k\|\eta_x^n\|^2 &\leq \int_{t_{n-1}}^{t_n} \|\partial_t \xi\|^2 ds + Ck^2\int_{t_{n-1}}^{t_n} \|u_{tt}\|^2 ds\\ &+ C\left(1 + \frac{\varepsilon^2}{\nu}\right)k\|\eta^n\|^2 + \frac{Ck\varepsilon^2}{\nu}\|\xi^n\|^2 + k\nu\|\xi_x^n\|^2 + \frac{Ck\mu^2}{\nu h^2}\|\xi_x^n\|^2.\end{aligned} \tag{54}$$

Summing (54) from $n = 1$ to N, we have that

$$\begin{aligned}&\|\eta^N\|^2 - \|\eta^0\|^2 + \nu k\sum_{n=1}^N \|\eta_x^n\|^2 \leq \int_0^{t_N} \|\partial_t \xi\|^2 ds + Ck^2\int_{t_0}^{t_N} \|u_{tt}\|^2 ds\\ &+ C\left(1 + \frac{\varepsilon^2}{\nu}\right)k\sum_{n=1}^N \|\eta^n\|^2 + \frac{Ck\varepsilon^2}{\nu}\sum_{n=1}^N \|\xi^n\|^2 + k\nu\sum_{n=1}^N \|\xi_x^n\|^2 + \frac{Ck\mu^2}{\nu h^2}\sum_{n=1}^N \|\xi_x^n\|^2\\ &\leq C\left(1 + \frac{\varepsilon^2}{\nu}\right)k\sum_{n=1}^N \|\eta^n\|^2 + h^{2r+2}\|u_t\|^2_{L^2([0,T];H^{r+1})} + Ck^2\|u_{tt}\|^2_{L^2([0,T];L^2)}\\ &+ \frac{CT\varepsilon^2}{\nu}h^{2r+2}\|u\|^2_{C([0,T];H^{r+1})} + C\left(T\nu h^{2r} + \frac{T\mu^2 h^{2r-2}}{\nu}\right)\|u\|^2_{C([0,T];H^{r+1})}.\end{aligned} \tag{55}$$

With the application of the discrete Gronwall Lemma 1.4, we have

$$\|\eta^N\|^2 + \nu k\sum_{n=1}^N \|\eta_x^n\|^2 \leq \exp(CT)(h^{2r-2} + k^2). \tag{56}$$

Finally, with the triangle inequality $\|u_h^N - u^N\| \leq \|\eta^N\| + \|\xi^N\|$, we complete the remainder of the proof. □

3. Model order reduction of finite element model

3.1. Proper orthogonal decomposition. In this part, we consider model order reduction based on the POD technique, in which the reduced basis functions are obtained from a sequence of snapshots $u_h^n \in S_h(\Omega)$, $n = 1, 2, \cdots, N$ from Subsection 2.2. A finite dimensional space $\mathcal{W}$ is then defined as

$$\mathcal{W} = \mathrm{span}\{u_h^1, u_h^2, \ldots, u_h^N\}. \tag{57}$$

Let$\{\psi_i\}_{i=1}^d$ denote an orthonormal basis of $\mathcal{W}$ with $d = \dim \mathcal{W} \le N$. Then, each member of the collection $\{u_h^n\}$ can be written in $H^2(\Omega)$ as:

$$u_h^n = \sum_{i=1}^{d} (u_h^n, \psi_i)_{H^2}\, \psi_i, \quad n = 1, 2, \ldots, N \tag{58}$$

with $(u_h^n, \psi_i)_{H^2} = (u_{hxx}^n, (\psi_i)_{xx}) + (u_{hx}^n, (\psi_i)_x) + (u_h^n, \psi_i)$. The procedure of the POD method can be defined as: for every $l \in \{1, 2, \ldots, d\}$, seek orthonormal basis functions $\psi_1, \ldots, \psi_l \in H^2(\Omega)$ such that the following minimization problem is satisfied

$$\begin{aligned} &\min_{\{\psi_i\}_{i=1}^l} \frac{1}{N} \sum_{n=1}^{N} \left\| u_h^n - \sum_{i=1}^{l} (u_h^n, \psi_i)_2\, \psi_i \right\|_2^2 \\ &\text{subject to } (\psi_i, \psi_j)_{H^2} = \delta_{ij} \text{ for } i, j = 1, \ldots, l. \end{aligned} \tag{59}$$

The solution of the optimal system (59) can be solved by the method of an eigenvalue decomposition of the matrix which consists of the mutual H^2-inner products of $u_h^1, u_h^2, \ldots, u_h^N$, see [18] for details.

Proposition 3.1 ([18]). *Given $u_h^1, u_h^2, \ldots, u_h^N$, assume that the positive eigenvalues $\lambda_1 \ge \cdots \ge \lambda_d > 0$ and the associated eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^N$ satisfy*

$$G\mathbf{v}_i = \lambda_i \mathbf{v}_i, i = 1, 2, \ldots, d \tag{60}$$

with $G = (G_{ij})_{N\times N}$ and $G_{ij} = \frac{1}{N}(u_h^i, u_h^j)_{H^2}$. Then a POD basis of rank $l \le d$ is given by

$$\psi_i = \frac{1}{\sqrt{\lambda_i d}} \sum_{j=1}^{N} \mathbf{v}_i^j u_h^j, \quad i = 1, \ldots, l, \tag{61}$$

where $\mathbf{v}_i^j$ is the j-the component of the eigenvector $\mathbf{v}_i$. Moreover, the following error estimate holds for $l \le d$,

$$\frac{1}{N} \sum_{n=1}^{N} \left\| u_h^n - \sum_{i=1}^{l} (u_h^n, \psi_i)_{H^2}\, \psi_i \right\|_2^2 = \sum_{j=l+1}^{d} \lambda_j. \tag{62}$$

3.2. POD Galerkin reduced order method. Let $S_l(\Omega) = \mathrm{span}\{\psi_1, \psi_2, \ldots, \psi_l\}$. The same time partition is used as that in Subsection 2.2. For the time interval $[0, T]$, the time increment is denoted by k such that $k = T/N$ with $t_n = nk, n = 0, 1, \ldots N$. We have the following scheme for the POD method as: Given $u_d^0 = P_d u_0$, find $u_d^n \in S_l(\Omega)$, $n = 1, \ldots, N$ such that for all $v \in S_l(\Omega)$,

$$\left(\frac{u_d^n - u_d^{n-1}}{k}, v\right) - \frac{\varepsilon}{2}\left(\left(u_d^n\right)^2, v_x\right) + \nu\left(u_{dx}^n, v_x\right) - \mu\left(u_{dxx}^n, v_x\right) = 0. \tag{63}$$

Lemma 3.1. *Let $u_d^n \in S_l(\Omega)$ be solution of (63). The L^2 boundedness of u_d^n holds*

$$\|u_d^n\| \le \|u_d^0\|. \tag{64}$$

The following bilinear form is used for the following lemma as

$$\mathcal{B}(v,w) = (v_{xx}, w_{xx}) + (v_x, w_x) + (v, w), \quad \forall v, w \in H_0^2(\Omega). \tag{65}$$

It is trivial to check that $\mathcal{B}(v,w)$ is continuous and coercive on $H_0^2(\Omega)$. Let P_d be an auxiliary projection of $u \in H_0^2$ onto $S_l(\Omega)$ as

$$\mathcal{B}(u,v) = \mathcal{B}(P_d u, v), \quad \forall v \in S_l(\Omega). \tag{66}$$

Lemma 3.2. *Assume that $u_h^n \in S_h(\Omega)$, $l \le d$ in (59), there exists a constant C such that the projection P_d satisfies*

$$\frac{1}{N}\sum_{n=1}^{N} \|u_h^n - P_d u_h^n\|_2^2 \le \sum_{j=l+1}^{d} \lambda_j. \tag{67}$$

Proof. With $v \in H_0^2(\Omega)$, $v_d, P_d v \in S_l(\Omega)$, we have

$$\begin{aligned} \|v - P_d v\|_2^2 &= (v - P_d v, v - P_d v)_{H^2} \\ &= (v - P_d v, v - v_d)_{H^2} \\ &\le \|v - P_d v\|_2 \|v - v_d\|_2. \end{aligned} \tag{68}$$

It follows that

$$\|v - P_d v\|_2 \le \|v - v_d\|_2, \quad \forall v_d \in S_d(\Omega). \tag{69}$$

With Proposition 3.1, setting $v = u_h^n \in H_0^2(\Omega)$, $v_d = \sum_{i=1}^{l} (u_h^n, \psi_i)_{H^2} \psi_i \in S_l(\Omega)$ in the above inequality gives (67). ☐

Lemma 3.3. *For $u^n \in H^{r+1}(\Omega)$ and $P_d u^n \in S_l(\Omega)$, the following error estimate holds*

$$\frac{1}{N}\sum_{n=1}^{N} \|u^n - P_d u^n\|_2^2 \le C(h^{2r-2} + \sum_{j=l+1}^{d} \lambda_j). \tag{70}$$

Proof. Since $\mathcal{B}(v,w)$ is continuous and coercive on $H_0^2(\Omega)$, the following auxiliary equation is uniquely solvable for $w \in H_0^2(\Omega)$,

$$\mathcal{B}(v^n, w) = ((u^n - P_d u^n)_{xx}, w_{xx}) + ((u^n - P_d u^n)_x, w_x) + ((u^n - P_d u^n), w). \tag{71}$$

Taking $w = u^n - P_d u^n \in H_0^2(\Omega)$, $P_d v^n \in S_l(\Omega)$, we get that

$$\begin{aligned} \|u^n - P_d u^n\|_2^2 &= \mathcal{B}(v^n, u^n - P_d u^n) \\ &= \mathcal{B}(v^n - P_d v^n, u^n - P_d u^n) \\ &\le \|v^n - P_d v^n\|_2 \|u^n - P_d u^n\|_2. \end{aligned} \tag{72}$$

With $\|P_d u\|_2 \le \|u\|_2$, letting $v_h^n \in S_h(\Omega)$ be an optimal interpolation of v in $H^{r+1}(\Omega)$, it follows that for $P_d v_h^n \in S_l(\Omega)$

$$\begin{aligned} \|u^n - P_d u^n\|_2 &\le \|v^n - P_d v^n\|_2 \\ &\le \|v^n - v_h^n\|_2 + \|v_h^n - P_d v_h^n\|_2 + \|P_d v_h^n - P_d v^n\|_2 \\ &\le Ch^{r-1}\|v^n\|_{r+1} + \|v_h^n - P_d v_h^n\|_2. \end{aligned} \tag{73}$$

Then we have that

$$\|u^n - P_d u^n\|_2^2 \le C(h^{2r-2}\|v^n\|_{r+1}^2 + \|v_h^n - P_d v_h^n\|_2^2). \tag{74}$$

With Proposition 3.1, summing (74) from $n = 1$ to $n = N$ gives

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\|u^n - P_d u^n\|_2^2 &\le \frac{1}{N}\sum_{n=1}^{N} Ch^{2r-2}\|v^n\|_{r+1}^2 + \frac{1}{N}\sum_{n=1}^{N}\|v_h^n - P_d v_h^n\|_2^2 \\ &\le \frac{1}{N}\sum_{n=1}^{N} Ch^{2r-2}\|v^n\|_{r+1}^2 + \sum_{j=l+1}^{d}\lambda_j,\end{aligned}\tag{75}$$

which ends the proof. □

Theorem 3.2. *Assume that the solution $u(t) \in H^{r+1}(\Omega)\cap H_0^2(\Omega)$, is a solution of (6) with $u_{tt}, u_{ttt} \in L^2([0,T];L^2)\cap C(0,T;H^{r+1}(\Omega))$. Let u_d^n, $n = 1,\dots,N$ be the POD solutions of (63). The following error estimate holds*

$$\|u^N - u_d^N\|^2 + \nu k\sum_{n=1}^{N}\|u_x^N - u_{dx}^n\|^2 \le \exp(CT)(k^2 + h^{2r-2} + \sum_{j=l+1}^{d}\lambda_j),\tag{76}$$

with k the time step, h the mesh size, d the number of the POD basis.

Proof. Taking $v \in S_l(\Omega)$ in variational formulation (6) gives

$$(u_t^n, v) - \frac{\varepsilon}{2}\left((u^n)^2, v_x\right) + \nu\,(u_x^n, v_x) - \mu\,(u_{xx}^n, v_x) = 0.\tag{77}$$

Similar to the full discrete case in (35), we decompose the error between $u^n - u_d^n$ as

$$u_d^n - u^n = (u_d^n - P_d u^n) - (u^n - P_d u^n) = \eta_d^n - \xi_d^n.\tag{78}$$

Subtracting (77) from (63) gives the error equation

$$\begin{aligned}\left(\frac{\eta_d^n - \eta_d^{n-1}}{k}, v_d\right) + \nu\,(\eta_{dx}^n, v_{dx}) - \mu\,(\eta_{dxx}^n, v_{dx}) &= \left(\frac{\xi_d^n - \xi_d^{n-1}}{k}, v_d\right) + (\rho^n, v_d) \\ -\frac{\varepsilon}{2}\left((u^n)^2 - (u_d^n)^2, v_{dx}\right) &+ \nu\,(\xi_{dx}^n, v_{dx}) - \mu\,(\xi_{dxx}^n, v_{dx})\end{aligned}\tag{79}$$

with $\rho^n = u_t(t_n) - \frac{u^n - u^{n-1}}{k}$. Setting $v_d = \eta_d^n \in S_l(\Omega)$ in (79) gives

$$\begin{aligned}\left(\frac{\eta_d^n - \eta_d^{n-1}}{k}, \eta_d^n\right) + \nu\,(\eta_{dx}^n, \eta_{dx}^n) - \mu\,(\eta_{dxx}^n, \eta_{dx}^n) &= \left(\frac{\xi_d^n - \xi_d^{n-1}}{k}, \eta_d^n\right) \\ + (\rho^n, \eta_d^n) - \frac{\varepsilon}{2}\left((u^n)^2 - (u_d^n)^2, \eta_{dx}^n\right) &+ \nu\,(\xi_{dx}^n, \eta_{dx}^n) - \mu\,(\xi_{dxx}^n, \eta_{dx}^n).\end{aligned}\tag{80}$$

Using the Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\left\|\eta_d^n - \eta_d^{n-1}\right\|^2 + \frac{1}{2}\|\eta_d^n\|^2 - \frac{1}{2}\left\|\eta_d^{n-1}\right\|^2 + \nu k\,\|\eta_{dx}^n\|^2 \\ &\le \frac{k}{2}\left\|\frac{\xi_d^n - \xi_d^{n-1}}{k}\right\|^2 + k\,\|\eta_d^n\|^2 + \frac{k}{2}\,\|\rho^n\|^2 + \frac{Ck\varepsilon^2}{\nu}\left(\|\eta_d^n\|^2 + \|\xi_d^n\|^2\right) \\ &+ \frac{k\nu}{16}\,\|\eta_{dx}^n\|^2 + k\nu\left(\frac{1}{4}\,\|\eta_{dx}^n\|^2 + \|\xi_{dx}^n\|^2\right) + Ck\mu^2\,\|\xi_{dxx}^n\|^2 + \frac{k\nu}{16}\,\|\eta_{dx}^n\|^2.\end{aligned}\tag{81}$$

Combining the above inequalities of (52)-(53), (81) is rewritten as

$$\begin{aligned}\|\eta_d^n\|^2 - \|\eta_d^{n-1}\|^2 + \nu k\|\eta_{dx}^n\|^2 &\le \int_{t_{n-1}}^{t_n}\|\partial_t\xi_d\|^2 ds + Ck^2\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2 ds \\ &+C(1+\frac{\varepsilon^2}{\nu})k\|\eta_d^n\|^2 + \frac{Ck\varepsilon^2}{\nu}\|\xi_d^n\|^2 + k\nu\|\xi_{dx}^n\|^2 + Ck\mu^2\|\xi_{dxx}^n\|^2.\end{aligned}\tag{82}$$

Summing (82) from $n=1$ to $n=N$, we have

$$\|\eta_d^N\|^2 - \|\eta_d^0\|^2 + \nu k \sum_{n=1}^{N} \|\eta_{dx}^n\|^2 \le \int_0^{t_N} \|\partial_t \xi_d\|^2 ds + Ck^2 \int_{t_0}^{t_N} \|u_{tt}\|^2 ds$$
$$+ C(1+\frac{\varepsilon^2}{\nu})k \sum_{n=1}^{N} \|\eta_d^n\|^2 + \frac{Ck\varepsilon^2}{\nu} \sum_{n=1}^{N} \|\xi_d^n\|^2 + k\nu \sum_{n=1}^{N} \|\xi_{dx}^n\|^2 + Ck\mu^2 \sum_{n=1}^{N} \|\xi_{dxx}^n\|^2.$$

Using the composite trapezoidal rule with $\|\partial_{ttt}\xi_d^n\| < C$, we have that

$$\int_0^{t_N} \|\partial_t \xi_d\|^2 ds \le \sum_{n=1}^{N} \frac{k}{2}(\|\partial_t \xi_d^n\|^2 + \|\partial_t \xi_d^{n-1}\|^2) + Ck^2. \tag{83}$$

With the discrete Gronwall Lemma 1.4, Lemma 3.3 and $k = T/N$, we have that

$$\|\eta_d^N\|^2 + k\nu \sum_{n=1}^{N} \|\eta_{dx}^n\|^2 \le \exp(CT)(k^2 + h^{2r-2} + \sum_{j=l+1}^{d} \lambda_j). \tag{84}$$

With the triangle inequality $\|u_h^N - u_d^N\| \le \|u^N - P_d u^N\| + \|P_d u^N - u_d^N\|$, we conclude the proof. □

4. Computational experiments

In this part, the numerical experiments are performed to verify the efficiency of the proposed methods. The B-spline basis functions are strongly localized functions which have the characteristics of smoothness and minimal support. Since the third order KdVB equation is considered, the higher regularity of the basis functions of the Galerkin method is required. In this paper, the C^1 quadratic B-spline basis functions are used, which have higher regularity than the C^0 quadratic Lagrange basis functions. As a comparison, the stiffness matrices correspondingly to this two basis functions differ greatly. The order of the coefficient matrix of the former is much lower than that of the latter. The coefficient matrices generated by the Galerkin method based on the B-spline functions are highly sparsity, which is easy to be implemented on computer. Based on the above consideration, the finite element method based on the quadratic B-spline has some merits and is constructed for third order KdVB equations.

Example 4.1. For testing accuracy of our schemes, we consider the following KdVB equation with an artificially exact solution $u(x,t) = e^{-t}(x^2-1)^2$ in the interval $\Omega = [-1,1]$ as

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = f(x,t), x \in \Omega, 0 \le t \le T \tag{85}$$

with the parameters $\epsilon = 0.2, \nu = 0.01, \mu = 0.1$. Then the initial condition and the term $f(x,t)$ can be correspondingly deduced as

$$u(x,0) = (x^2-1)^2, \tag{86}$$
$$f(x,t) = e^{-t}\big[24\mu x - (x^2-1)^2 - 8\nu x^2 + 4(x^2-1) + 4\epsilon x(x^2-1)^3 e^{-t}\big]. \tag{87}$$

As mentioned above, the space interval $[-L, L]$ is uniformly divided into M subintervals, with mesh size $h = 2L/M$ and knots $x_i = -L + ih$, $i = 0, 1, \ldots, M$. The time interval $[0,T]$ is discretized into N equal parts with $t_n = nk, n = 0, 1, \cdots N$ and time step $k = T/N$. Referring to [19], the quadratic B-splines $\psi_i(x)$ $(i =$

$-1, \ldots, M$) which form a basis over the interval $[-L, L]$ are defined as

$$\psi_i(x) = \frac{1}{h^2}\begin{cases} (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i - x)^2, & [x_{i-1}, x_i], \\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, & [x_i, x_{i+1}], \\ (x_{i+2}-x)^2, & [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise}. \end{cases}$$

For the space convergence tests, we chose a sequence of triangulations with $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$ and a fixed time step size $k = 10^{-9}$. For the time convergence tests, we fixed mesh size $h = \frac{2}{10^5}$ and set $k = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. To reduce the influence of the POD methods for the space and time convergence tests, we use all the basis functions generated by the POD methods without truncation. The L^2 errors and the convergence rates for the space accuracy test at time instant $t_{N_1} = 10^{-4}$ and the time accuracy tests at time instant $t_N = 1$ are presented in Table 1 and Figure 1. The simulation results show that with the quadratic B-splines basis, the schemes (35) and (63) have third-order convergence in space, which is larger than what we have obtained in the numerical analysis. The reason is that the third order term u_{xxx} leads to the reduction of the convergence order in the procedure of the analysis. We can only obtain the sub-optimal convergence rates. The presented schemes provide satisfactory numerical solutions with expected convergence order 1 in time, which are consistent with our theoretical results.

TABLE 1. Error estimates and convergence rates in space and time.

h	$\lVert u^{N_1} - u_h^{N_1}\rVert$	Rate	Δt	$\lVert u^N - u_h^N\rVert$	Rate
1/10	1.9337e-03	-	1/16	1.4199e-02	-
1/20	1.9041e-04	3.3441	1/32	7.3219e-03	0.9555
1/40	1.9639e-05	3.2773	1/64	3.7176e-03	0.9778
1/80	2.1956e-06	3.1610	1/128	1.8729e-03	0.9890

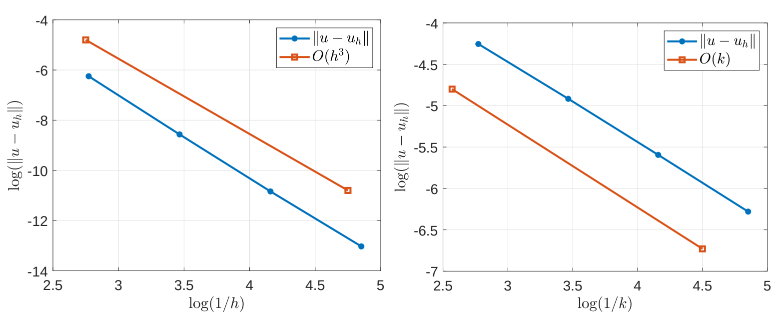

FIGURE 1. Error estimates and convergence rates in space (Left) and time (Right).

EXAMPLE 4.2. To demonstrate the efficiency of the POD method, we choose an initial condition of (1)-(4) as

$$u(x, 0) = 1 - \tanh|x|$$

in the space interval $-50 \leq x \leq 50$ and on the time interval $0 \leq t \leq 10$. We take parameters $\epsilon = 0.2, \mu = 0.1$ and viscosity constant $\nu = 0.4$. The time step $k = 0.05$

and space step $h = 0.1$ are used. In the numerical test, a group of 200 snapshots at time $t = 1k, 2k, \ldots, 200k$ are first generated using the Galerkin finite element scheme (35) by the backward Euler scheme. The evolution of numerical solutions are depicted graphically in Figure 2. Then a group of 33 snapshots from 200 are equally selected. The eigen-problem is then solved by (60). The POD orthonormal basis functions are then constructed by (61). We took 20 POD bases from the set of 33 POD bases and expanded them into subspace $S_d(\Omega)$. Then a set of POD numerical solutions at $t = 1k, 2k, \ldots, 200k$ are computed. We further test the POD finite element scheme with different number of POD basis functions at the final time instant $t = 10$. Figure 3 shows the L^2 errors between the solutions of the reduced order modeling with different number of POD bases and the solutions of the B-spline Galerkin finite element method at the final time instant $t = 10$, from which we can see the errors are gradually reduced as we expected.

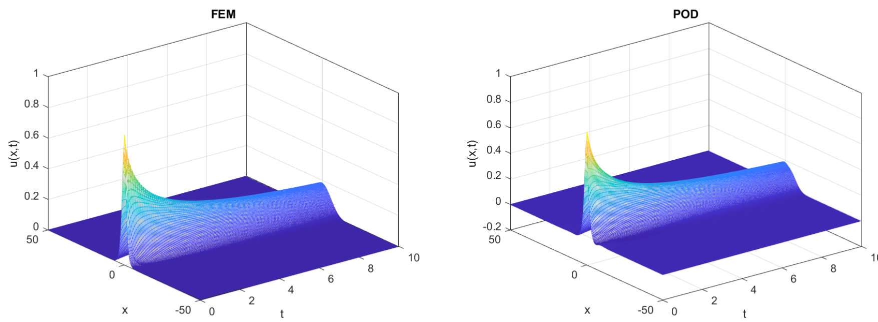

FIGURE 2. Evolutions of the B-spline solutions (left) and POD (with 20 bases) solutions (right).

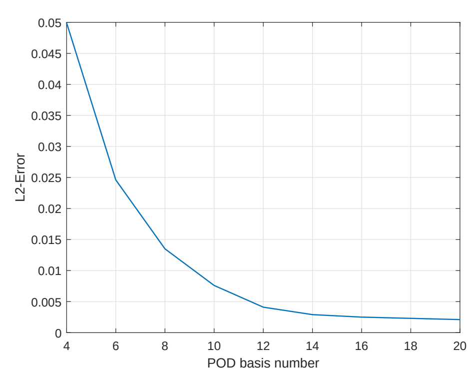

FIGURE 3. The errors between the solutions of POD method and the B-spline at time $t = 10$.

EXAMPLE 4.3. In this test, the initial condition is chosen as

$$u(x, 0) = 0.5\big(1 - \tanh\frac{|x| - 5}{5}\big)$$

in the space interval $-25 \leq x \leq 25$ and on the time interval $0 \leq t \leq 10$. We took parameters $\epsilon = 0.2, \mu = 5$ and viscosity constant $\nu = 0.00001$ with time step

$k = 0.05$ and space step $h = 0.1$. We carried out the same procedure as Example 1. The computational results are shown in Figure 4 and Figure 5, which show that the computational results for the POD method and the B-spline Galerkin finite element method are consistent. As a conclusion, the POD method is efficient to solve this kind of problem.

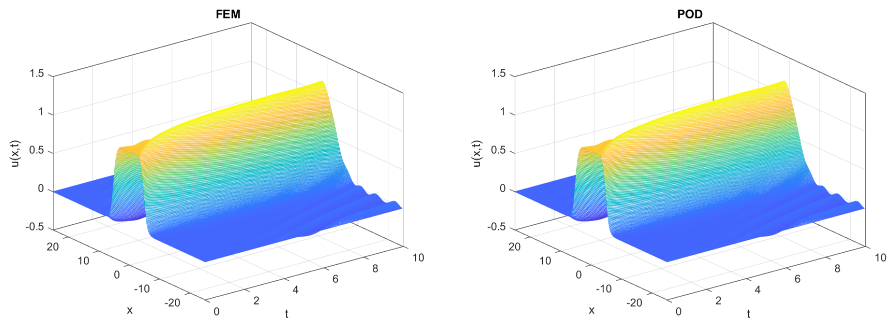

FIGURE 4. Evolutions of the B-spline solutions (left) and POD (with 20 bases) solutions (right).

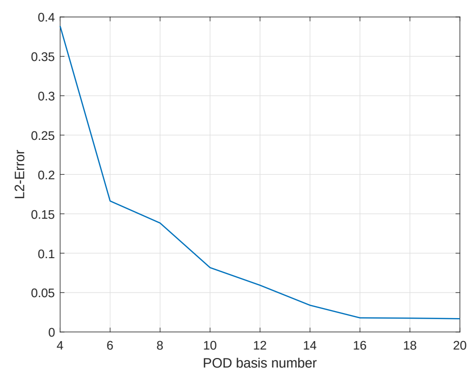

FIGURE 5. The errors between the solutions of POD method and the B-spline at time $t = 10$.

5. Conclusion

We constructed a B-spline Galerkin finite element method and POD reduced order method for the KdVB equation. This article is summarized as follows: First, we analyze the boundedness and convergence of the approximate solution for the B-spline Galerkin method to the KdVB equation. For the temporal discretization, we applied the Euler backward method and obtained L^2-error estimates. Numerical examples are performed to show the accuracy and efficiency of related scheme. Second, we have employed the POD techniques to derive a reduced POD formulation for the KdVB equation. The numerical analysis of the errors between the B-spline Galerkin solution and POD solution are performed. Third, the nonlinear term in the KdVB equation is carried out with the techniques of the POD methods for the Navier–Stokes equations as mentioned above. Fourth, in the numerical analysis, we only obtained the suboptimal convergence order in the space direction. Some new techniques should be considered and employed to get the optimal convergence order.

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