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Triple Positive Solutions for a Class of Fractional Boundary Value Problem System*

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Abstract In this paper, the solvability for the following fractional boundary value problem system

$$
\begin{aligned}
{}^C D_{0+}^{\sigma_1} v_1(t) &= f_1(t, v_2(t), D_{0+}^{\mu_1} v_2(t)), \quad 0 < t < 1,\\
{}^C D_{0+}^{\sigma_2} v_2(t) &= f_2(t, v_1(t), D_{0+}^{\mu_2} v_1(t)), \quad 0 < t < 1,\\
{v_1}'(0) = b v_1(0), \quad {v_1}''(0) = 0, &\quad {}^C D_{0+}^{\theta_1} v_1(1) = a \cdot {}^C D_{0+}^{\theta_2} v_1(\eta),\\
{v_2}'(0) = b v_2(0), \quad {v_2}''(0) = 0, &\quad {}^C D_{0+}^{\theta_1} v_2(1) = a \cdot {}^C D_{0+}^{\theta_2} v_2(\eta),
\end{aligned}
$$

is studied, where $a > 0$, $-1 < b < 0$, $2 < \sigma_1, \sigma_2 \leq 3$, $0 < \eta < 1$, $0 < \mu_1, \mu_2 \leq 1$, $0 < \theta_2 \leq \theta_1 \leq 1$, f_1, f_2: $[0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous, ${}^C D_{0+}^{\sigma_1} v_1(t)$, ${}^C D_{0+}^{\sigma_2} v_2(t)$ are the Caputo fractional derivatives, and $D_{0+}^{\mu_1} v_2(t)$, $D_{0+}^{\mu_2} v_1(t)$ are the Riemann-Liouville fractional derivatives. The fixed point theorem is used to prove that there are three positive solutions to problems.

Keywords Fractional derivative, Boundary value problems, Fixed point theorem, Positive solutions.

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1. Introduction

Fractional calculus has been widely applied in abnormal diffusion, control, fluid mechanics, image processing characteristics and so on. Therefore, fractional differential equations have captured our attention and gradually become an important model to solve many practical problems. Scholars have studied the fractional boundary value problem from the local problem to the nonlocal problem [2,7,9], from the non resonance problem to the resonance problem [2], from the finite interval problem to the infinite interval problem [1,5,14], from the single equation [3,11,13] to the system of equations [3,4,9] and so on. In particular, we note the studies as follows.

Qin and Jia [9] studied the single equation boundary value problems with Caputo derivative

$$
{}^C D_{0+}^{\alpha} u(t) = h(t) f(t, u(t)), \quad 0 < t < 1, \tag{1.1}
$$

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$$u'(0) = bu(0),\ u''(0) = 0, \quad {}^C D_{0+}^{\beta_1} u(1) = a \cdot {}^C D_{0+}^{\beta_2} u(\eta), \tag{1.2}$$

where $2 < \alpha < 3$, $0 < \eta < 1$. The existence and multiplicity results are determined by the use of the Krasnosel'skill fixed point theorem.

Su [10] studied the system of equation boundary value problem with Riemann-Liouville fractional derivatives

$$D^\alpha u(t) = f(t, v(t), D^\mu v(t)), \quad 0 < t < 1, \tag{1.3}$$

$$D^\beta v(t) = g(t, u(t), D^\nu u(t)), \quad 0 < t < 1, \tag{1.4}$$

$$u(0) = u(1) = v(0) = v(1) = 0, \tag{1.5}$$

where $f,\ g$: $[0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\mu,\ \nu > 0$, $1 < \alpha, \beta < 2$, $\alpha - \nu \geq 1$, $\beta - \mu \geq 1$. With the application of the Schauder fixed-point theorem, some existence results are obtained.

Bai and Ge [3] studied the boundary value problems with containing derivatives on the nonlinearity

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{1.6}$$

$$x(0) = x(1) = 0, \tag{1.7}$$

where f: $[0,1] \times [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ is continuous. With fixed point index theory, Bai established a new fixed point theorem in a cone. Imposing some growth conditions on the nonlinearity, the above boundary value problems have three positive solutions. The interesting point is that the nonlinear term depends on the derivative of the unknown function.

Inspired by the above pieces of literature, this paper mainly studies the system of the functional boundary value problems

$${}^C D_{0+}^{\sigma_1} v_1(t) = f_1(t, v_2(t), D_{0+}^{\mu_1} v_2(t)), \quad 0 < t < 1, \tag{1.8}$$

$${}^C D_{0+}^{\sigma_2} v_2(t) = f_2(t, v_1(t), D_{0+}^{\mu_2} v_1(t)), \quad 0 < t < 1, \tag{1.9}$$

$${v_1}'(0) = bv_1(0), \quad {v_1}''(0) = 0, \quad {}^C D_{0+}^{\theta_1} v_1(1) = a \cdot {}^C D_{0+}^{\theta_2} v_1(\eta), \tag{1.10}$$

$${v_2}'(0) = bv_2(0), \quad {v_2}''(0) = 0, \quad {}^C D_{0+}^{\theta_1} v_2(1) = a \cdot {}^C D_{0+}^{\theta_2} v_2(\eta), \tag{1.11}$$

where $a > 0$, $-1 < b < 0$, $2 < \sigma_1, \sigma_2 \leq 3$, $0 < \eta < 1$, $0 < \mu_1, \mu_2 \leq 1$, $0 < \theta_2 \leq \theta_1 \leq 1$ and $a\eta^{1-\theta_2}\Gamma(2-\theta_1) < \Gamma(2-\theta_2)$. The functions $f_1,\ f_2$: $[0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous, ${}^C D_{0+}^{\sigma_1} v_1(t)$, ${}^C D_{0+}^{\sigma_2} v_2(t)$ are the Caputo fractional derivatives, and $D_{0+}^{\mu_1} v_2(t)$, $D_{0+}^{\mu_2} v_1(t)$ are the Riemann-Liouville fractional derivatives. We will prove that problems (1.8)-(1.11) have three positive solutions under some growth conditions by using the fixed point theorem given in [3].

2. Preliminaries and lemmas

2.1. Theorem and lemma

Lemma 2.1 ([9]). *Given $\psi_i \in L[0,1]$ (i=1, 2), the unique solution of the problem*

$${}^C D_{0+}^{\sigma_i} v_i(t) = \psi_i(t), \quad 0 < t < 1, i = 1, 2,$$

$${v_i}'(0) = bv_i(0),\ {v_i}''(0) = 0,\ {}^C D_{0+}^{\theta_1} v_i(1) = a \cdot {}^C D_{0+}^{\theta_2} v_i(\eta)$$

is

$$v_i(t) = \int_0^1 G_i(t,s)\psi_i(s)ds, \tag{2.1}$$

where

$$G_i(t,s)=\begin{cases}\left[\delta_{1i}(1-s)^{\sigma_i-\theta_1-1}-a\delta_{2i}(\eta-s)^{\sigma_i-\theta_2-1}\right](1+bt)+\frac{(t-s)^{\sigma_i-1}}{\Gamma(\sigma_i)}, & 0\le s\le\min\{\eta,t\}\le 1\\ \left[\delta_{1i}(1-s)^{\sigma_i-\theta_1-1}-a\delta_{2i}(\eta-s)^{\sigma_i-\theta_2-1}\right](1+bt), & 0\le t\le s\le\eta\le 1\\ \delta_{1i}(1-s)^{\sigma_i-\theta_1-1}(1+bt)+\frac{(t-s)^{\sigma_i-1}}{\Gamma(\sigma_i)}, & 0\le\eta\le s\le t\le 1\\ \delta_{1i}(1-s)^{\sigma_i-\theta_1-1}(1+bt), & 0\le\max\{\eta,t\}\le s\le 1\end{cases} \tag{2.2}$$

and

$$\delta_{1i} = \frac{\Gamma(2-\theta_1)\Gamma(2-\theta_2)}{b\Gamma(\sigma_i-\theta_1)\left[a\eta^{1-\theta_2}\Gamma(2-\theta_1)-\Gamma(2-\theta_2)\right]};$$

$$\delta_{2i} = \frac{\Gamma(2-\theta_1)\Gamma(2-\theta_2)}{b\Gamma(\sigma_i-\theta_2)\left[a\eta^{1-\theta_2}\Gamma(2-\theta_1)-\Gamma(2-\theta_2)\right]}.$$

We call (G_1, G_2) the Green's function of problems (1.8)-(1.11).

Lemma 2.2 ([9]). *The functions $G_i(t,s)$ (i=1, 2) defined by (2.2) satisfy*
(1) $G_i(t,s)\in C([0,1]\times[0,1])$;
(2) $G_i(t,s) > 0$ for $t,s\in(0,1)$;
(3) $G_i(t,s)\le(\delta_{1i}+1)(1-s)^{\sigma_i-\theta_1-1}$, for $t,s\in[0,1]$;
(4) For $s\in[0,1]$, there holds

$$\inf_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} G_i(t,s) \ge \left[1+\frac{3}{4}b\right](\delta_{1i}-a\delta_{2i})(1-s)^{\sigma_i-\theta_1-1}.$$

Corollary 2.1 ([9]). *Let $\delta_i = \frac{(\delta_{1i}-a\delta_{2i})\left[1+\frac{3}{4}b\right]}{1+\delta_{1i}} > 0$. For $s\in[0,1]$, there holds*

$$\inf_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} G_i(t,s) \ge \delta_i \sup_{t\in[0,1]} G_i(t,s).$$

Lemma 2.3 ([10]). *Define*

$$V_1 = \left\{v(t)|\ v(t)\in C[0,1]\ ,\ D_{0+}^{\mu_2}v(t)\in C[0,1]\right\},$$

$$V_2 = \left\{v(t)|\ v(t)\in C[0,1]\ ,\ D_{0+}^{\mu_1}v(t)\in C[0,1]\right\},$$

equipped with the norms

$$\|v\|_{V_1} = \max_{t\in[0,1]}|v(t)| + \max_{t\in[0,1]}|D_{0+}^{\mu_2}v(t)|,$$

$$\|v\|_{V_2} = \max_{t\in[0,1]}|v(t)| + \max_{t\in[0,1]}|D_{0+}^{\mu_1}v(t)|.$$

Then, $(V_1,\|\cdot\|_{V_1})$, $(V_2,\|\cdot\|_{V_2})$ are two Banach spaces.

Definition 2.1. ([3]) Let E be a Banach space, $K \subset E$ a cone, $H > 0$, $l > c > 0$ three constants. Assume γ be a nonnegative continuous concave functional on the cone K, and $\alpha,\ \beta : K \to [0, +\infty)$ be two nonnegative convex functionals satisfying

$$||v|| \le N \max\{\alpha(v),\ \beta(v)\}, \quad \text{for } (v_1, v_2) \in K, \tag{2.3}$$

where N is a positive real number, and

$$\Omega = \{v \in K | \alpha(v) < l,\ \beta(v) < H\} \ne \varnothing, \quad \text{for } l > 0,\ H > 0. \tag{2.4}$$

Define bounded convex sets

$$\begin{aligned} K(\alpha, l; \beta, H) &= \{x \in K |\ \alpha(x) < l, \beta(x) < H\}, \\ \overline{K}(\alpha, l; \beta, H) &= \{x \in K |\ \alpha(x) \le l, \beta(x) \le H\}, \\ K(\alpha, l; \beta, H; \gamma, c) &= \{x \in K |\ \alpha(x) < l, \beta(x) < H, \gamma(x) < c\}, \\ \overline{K}(\alpha, l; \beta, H; \gamma, c) &= \{x \in K |\ \alpha(x) \le l, \beta(x) \le H, \gamma(x) \le c\}. \end{aligned}$$

Lemma 2.4 ([3])**.** *Let E be a Banach space, $K \subset E$ a cone and $l_2 \ge d > k > l_1 > 0$, $H_2 \ge H_1 > 0$ be given constants. Assume that α, β are nonnegative continuous convex functionals on K, such that (2.3) and (2.4) are satisfied, and γ is a nonnegative continuous concave functional on K, such that $\gamma(x) \le \alpha(x)$ for all $x \in \overline{K}(\alpha, l_2; \beta, H_2)$. Let $L : \overline{K}(\alpha, l_2; \beta, H_2) \to \overline{K}(\alpha, l_2; \beta, H_2)$ be a completely continuous operator. Suppose*
$(C_1)\{x \in \overline{K}(\alpha, d; \beta, H_2; \gamma, k) | \gamma(x) > k\} \ne \varnothing, \gamma(Lx) > k$, for $x \in \overline{K}(\alpha, d; \beta, H_2; \gamma, k)$;
$(C_2)\alpha(Lx) < l_1, \beta(Lx) < H_1$, for all $x \in \overline{K}(\alpha, l_1; \beta, H_1)$;
$(C_3)\gamma(Lx) > k$, for all $x \in \overline{K}(\alpha, l_2; \beta, H_2; \gamma, k)$ with $\alpha(Lx) > d$.
Then, L has at least three fixed points x_1, x_2 and x_3 in $\overline{K}(\alpha, l_2; \beta, H_2)$ such that

$$x_1 \in K(\alpha, l_1; \beta, H_1), \qquad x_2 \in \left\{\overline{K}(\alpha, l_2; \beta, H_2; \gamma, k) | \gamma(x) > k\right\}$$

and

$$x_3 \in \overline{K}(\alpha, l_2; \beta, H_2) \setminus \left(\overline{K}(\alpha, l_2; \beta, H_2; \gamma, k) \cup \overline{K}(\alpha, l_1; \beta, H_1)\right).$$

For $(v_1, v_2) \in V_1 \times V_2$, let

$$\|(v_1, v_2)\|_{V_1 \times V_2} = \max\{\|v_1\|_{V_1}, \|v_2\|_{V_2}\}.$$

Evidently, $(V_1 \times V_2, \|\cdot\|_{V_1 \times V_2})$ is a Banach space.

Let

$$K_1 = \{v_1 \in V_1 |\ v_1(t) \ge 0\}, \qquad K_2 = \{v_2 \in V_2 |\ v_2(t) \ge 0\}.$$

Then, $K = K_1 \times K_2$ is a cone of the Banach space $V_1 \times V_2$.

Consider a set of integral equations as follows:

$$\begin{cases} v_1(t) = \int_0^1 G_1(t, s) f_1(s, v_2(s), D_{0+}^{\mu_1} v_2(s)) ds \\ v_2(t) = \int_0^1 G_2(t, s) f_2(s, v_1(s), D_{0+}^{\mu_2} v_1(s)) ds. \end{cases} \tag{2.5}$$

Similar to [10], we can obtain that $(v_1, v_2) \in V_1 \times V_2$ is a solution of (1.8)-(1.11), if and only if (v_1, v_2) is a solution of problem (2.5).

Define an integral operator L: $V_1 \times V_2 \to V_1 \times V_2$ by

$$\begin{aligned} &L(v_1, v_2)(t) =: (L_1(v_1, v_2)(t), L_2(v_1, v_2)(t)) \\ &= \left(\int_0^1 G_1(t, s) f_1(s, v_2(s), D_{0+}^{\mu_1} v_2(s)) ds, \int_0^1 G_2(t, s) f_2(s, v_1(s), D_{0+}^{\mu_2} v_1(s)) ds\right). \end{aligned}$$

Then, the fixed point of operator L consists with the solution of (1.8)-(1.11).

Lemma 2.5. *L: $K \to K$ is completely continuous.*

Proof. *L is completely continuous, if and only if L_1 and L_2 are completely continuous. Therefore, we prove that L_1 is completely continuous as follows.*

The operator $L_1: K \to K_1$ is continuous on account of continuity and nonnegativeness of $G_1(t,s)$ and f.

For $\Omega \subset K$ be bounded, i.e., a positive real number $N' > 0$ can be found such that $\|(v_1,v_2)\|_{V_1\times V_2} \le N'$, for $(v_1,v_2)\in\Omega$. Thus, $\max_{t\in[0,1]}|v_2(t)| \le N'$, $\max_{t\in[0,1]}\left|D_{0+}^{\mu_1}v_2(t)\right| \le N'$. $\phi(t) := f_1(t, v_2(t), D_{0+}^{\mu_1}v_2(t))$, Then,

$$
\begin{aligned}
|L_1(v_1,v_2)(t)| &\le \left|\int_0^1 G_1(t,s) f_1(s, v_2(s), D_{0+}^{\mu_1}v_2(s))ds\right| \\
&\le M^*\int_0^1 \sup_{t\in[0,1]} G_1(t,s)ds,
\end{aligned}
$$

$$
\begin{aligned}
\left|D_{0+}^{\mu_2}L_1(v_1,v_2)(t)\right| &= \left|\frac{1}{\Gamma(1-\mu_2)}\frac{d}{dt}\int_0^t (t-s)^{-\mu_2}L_1(v_1,v_2)(s)ds\right| \\
&\le \left|\frac{d}{dt}\int_0^t \frac{(t-s)^{-\mu_2}}{\Gamma(1-\mu_2)}(\int_0^1 \delta_{1i}(1-\tau)^{\sigma_i-\theta_1-1}(1+bs)\phi(\tau)d\tau\right| \\
&\quad + \left|\int_0^s \frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}\phi(\tau)d\tau ds\right| \\
&\le \left|\frac{-\mu_2}{\Gamma(1-\mu_2)}\int_0^t (t-s)^{-\mu_2-1}(1+bs)ds\cdot\int_0^1 \delta_{1i}(1-\tau)^{\sigma_i-\theta_1-1}\phi(\tau)d\tau\right| \\
&\quad + \left|\frac{1}{\Gamma(\sigma_i-\mu_2)}\int_0^t (t-\tau)^{\sigma_i-\mu_2-1}\phi(\tau)d\tau ds\right| \\
&\le \frac{\mu_2 M^*\delta_{1i}}{\Gamma(1-\mu_2)} + \frac{M^*}{\Gamma(\sigma_i-\mu_2)},
\end{aligned}
$$

where

$$
M^* = \max_{t\in[0,1],u\in[0,N'],v\in[0,N']} |f_1(t,u,v)| + 1.
$$

Hence, $L_1(\Omega)$, $D_{0+}^{\mu_2}L_1(\Omega)$ are bounded.

On the other hand, given $\varepsilon > 0$, $G_1(t,s)$, $(t-s)^{-\mu_2-1}$, $(t-s)^{\sigma_i-\mu_2-1}$ is uniformly continuous owing to the continuity of $G_1(t,s)$, $(t-s)^{-\mu_2-1}$, $(t-s)^{\sigma_i-\mu_2-1}$ on $[0,1]\times[0,1]$, i.e., for t_1, $t_2\in[0,1]$, $t_1 < t_2$, $t_2 - t_1 < \rho$, a positive real number $\rho > 0$ can be found such that $\left|(t_2-s)^{-\mu_2-1} - (t_1-s)^{-\mu_2-1}\right| < \frac{\Gamma(1-\mu_2)\varepsilon}{2\mu_2 M^*\delta_{1i}}$, $\left|(t_2-s)^{\sigma_i-\mu_2-1} - (t_1-s)^{\sigma_i-\mu_2-1}\right| < \frac{\Gamma(\sigma_i-\mu_2)\varepsilon}{2M^*}$, $|G_1(t_2,s) - G_1(t_1,s)| < \frac{\varepsilon}{M^*}$, Thus, for $(v_1,v_2)\in\Omega$, we have

$$
\begin{aligned}
&|L_1(v_1,v_2)(t_2) - L_1(v_1,v_2)(t_1)| \\
&\le \left|\int_0^1 [G_1(t_2,s) - G_1(t_1,s)] f_1(s, v_2(s), D_{0+}^{\mu_1}v_2(s))ds\right| \\
&\le \int_0^1 \frac{\varepsilon}{M^*}\cdot M^* ds = \varepsilon.
\end{aligned}
$$

$$
\left|D_{0+}^{\mu_2}L_1(v_1,v_2)(t_2) - D_{0+}^{\mu_2}L_1(v_1,v_2)(t_1)\right|
$$

$$\begin{aligned}
&\leq \left| \int_{t_1}^{t_2} \frac{-\mu_2 \left[(t_2 - s)^{-\mu_2-1} - (t_1 - s)^{-\mu_2-1}\right](1+bs)}{\Gamma(1-\mu_2)} ds \cdot \int_0^1 \delta_{1i}(1-\tau)^{\sigma_i-\theta_1-1}\phi(\tau)d\tau \right| \\
&\quad + \left| \frac{1}{\Gamma(\sigma_i - \mu_2)} \int_{t_1}^{t_2} \left[(t_2-\tau)^{\sigma_i-\mu_2-1} - (t_1-\tau)^{\sigma_i-\mu_2-1}\right] \phi(\tau)d\tau \right| \\
&\leq \left| \frac{\mu_2 M^* \delta_{1i}}{\Gamma(1-\mu_2)} \int_{t_1}^{t_2} \left[(t_2-s)^{-\mu_2-1} - (t_1-s)^{-\mu_2-1}\right] ds \right| \\
&\quad + \left| \frac{M^*}{\Gamma(\sigma_i-\mu_2)} \int_{t_1}^{t_2} \left[(t_2-\tau)^{\sigma_i-\mu_2-1} - (t_1-\tau)^{\sigma_i-\mu_2-1}\right] d\tau \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $L_1(\Omega)$, $D_{0+}^{\mu_2} L_1(\Omega)$ are equicontinuous.

By making use of the Arzela-Ascoli theorem, L_1: $K \to K_1$ is completely continuous. Similarly, there holds that L_2: $K \to K_2$ is completely continuous, too. Hence, we obtain that L: $K \to K$ is completely continuous. □

3. Main results

For $v = (v_1, v_2) \in K$, define the following functionals:

$$\alpha(v) = \alpha(v_1, v_2) = \max_{t\in[0,1]} |v_1(t)| + \max_{t\in[0,1]} |v_2(t)|;$$

$$\beta(v) = \beta(v_1, v_2) = \max_{t\in[0,1]} \left|D_{0+}^{\mu_2} v_1(t)\right| + \max_{t\in[0,1]} \left|D_{0+}^{\mu_1} v_2(t)\right|;$$

$$\gamma(v) = \gamma(v_1, v_2) = \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |v_1(t)| + \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |v_2(t)|;$$

$$\omega = 2\min\left\{ \frac{(1+4b)(\delta_{11} - a\delta_{21})}{\sigma_1 - \theta_1}, \frac{(1+4b)(\delta_{12} - a\delta_{22})}{\sigma_2 - \theta_1} \right\},$$

$$M = \min\left\{ \frac{1+\delta_{11}}{(\delta_{11} - a\delta_{21})\left(1 + \frac{3}{4}b\right)}, \frac{1+\delta_{12}}{(\delta_{12} - a\delta_{22})\left(1 + \frac{3}{4}b\right)} \right\}.$$

Theorem 3.1. *Suppose there exist constants $H_2 \geq H_1 > 0$, $l_2 > Mk > k > l_1 > 0$ such that $\frac{k}{\omega} \leq \min\{M_1 l_2, M_2 H_2\}$, where $M_1 = \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)}, \frac{\sigma_2-\theta_1}{2(\delta_{12}+1)}\right\}$, $M_2 = \min\left\{\frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})}, \frac{(\sigma_2-\theta_1)\Gamma(2-\mu_1)}{2(\sigma_2-\theta_1+b\delta_{12})}\right\}$, the following assumptions hold:*

(A_1) $f_1(t, x_1, x_2) \leq \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)} l_1, \frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})} H_1\right\}$,
$f_2(t, x_1, x_2) \leq \min\left\{\frac{\sigma_2-\theta_1}{2(\delta_{12}+1)} l_1, \frac{(\sigma_2-\theta_1)\Gamma(2-\mu_1)}{2(\sigma_2-\theta_1+b\delta_{12})} H_1\right\}$,
for $(t, x_1, x_2) \in [0,1] \times [0, l_1] \times [-H_1, H_1]$;

(A_2) $\min\{f_1(t, x_1, x_2), f_2(t, x_1, x_2)\} > \frac{k}{\omega}$,
for $(t, x_1, x_2) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [k, Mk] \times [-H_2, H_2]$;

(A_3) $f_1(t, x_1, x_2) \leq \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)} l_2, \frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})} H_2\right\}$,
$f_2(t, x_1, x_2) \leq \min\left\{\frac{\sigma_2-\theta_1}{2(\delta_{12}+1)} l_2, \frac{(\sigma_2-\theta_1)\Gamma(2-\mu_1)}{2(\sigma_2-\theta_1+b\delta_{12})} H_2\right\}$,
for $(t, x_1, x_2) \in [0,1] \times [0, l_2] \times [-H_2, H_2]$;

Then, the boundary value problems (1.8)-(1.11) have at least three positive solutions (v_{11}, v_{12}), (v_{21}, v_{22}) *and* (v_{31}, v_{32}) *satisfying*

$$(v_{11}, v_{12}) \in K(\alpha, l_1; \beta, H_1), \quad (v_{21}, v_{22}) \in \left\{\overline{K}(k, l_2; \beta, H_2; \gamma, k) | \gamma(v_1, v_2) > k\right\}$$

and

$$(v_{31}, v_{32}) \in \overline{K}(\alpha, l_2; \beta, H_2) \setminus \left(\overline{K}(\alpha, l_2; \beta, H_2; \gamma, k) \cup \overline{K}(\alpha, l_1; \beta, H_1)\right).$$

Proof. For $(v_1, v_2) \in \overline{K}(\alpha, l_2; \beta, H_2)$, there are $\alpha(v_1, v_2) \leq l_2$, $\beta(v_1, v_2) \leq H_2$, and hypothesis (A_3) implies

$$f_1\left(t, v_2(t), D_{0+}^{\mu_1} v_2(t)\right) \leq \min\left\{\frac{\sigma_1 - \theta_1}{2(\delta_{11} + 1)} l_2, \frac{(\sigma_1 - \theta_1)\Gamma(2 - \mu_2)}{2(\sigma_1 - \theta_1 + b\delta_{11})} H_2\right\};$$

$$f_2\left(t, v_1(t), D_{0+}^{\mu_2} v_1(t)\right) \leq \min\left\{\frac{\sigma_2 - \theta_1}{2(\delta_{12} + 1)} l_2, \frac{(\sigma_2 - \theta_1)\Gamma(2 - \mu_1)}{2(\sigma_2 - \theta_1 + b\delta_{12})} H_2\right\}.$$

Consequently,

$$\begin{aligned}
\alpha\left(L(v_1, v_2)\right) &= \alpha(L_1(v_1, v_2), L_2(v_1, v_2)) \\
&= \max_{t \in [0,1]} |L_1(v_1, v_2)(t)| + \max_{t \in [0,1]} |L_2(v_1, v_2)(t)| \\
&= \max_{t \in [0,1]} \left| \int_0^1 G_1(t, s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds \right| \\
&\quad + \max_{t \in [0,1]} \left| \int_0^1 G_2(t, s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds \right| \\
&\leq \int_0^1 \max_{t \in [0,1]} G_1(t, s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds \\
&\quad + \int_0^1 \max_{t \in [0,1]} G_2(t, s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds \\
&\leq \sum_{i=1}^{2} \frac{\sigma_i - \theta_1}{2(\delta_{1i} + 1)} l_2 \int_0^1 (\delta_{1i} + 1)(1 - s)^{\sigma_i - \theta_1 - 1} ds \\
&= \frac{l_2}{2} + \frac{l_2}{2} = l_2.
\end{aligned}$$

For $(v_1, v_2) \in K$, there is $L(v_1, v_2) \in K$. Hence,

$$\begin{aligned}
\beta\left(L(u, v)\right) &= \beta\left(L_1(v_1, v_2), L_2(v_1, v_2)\right) \\
&= \max_{t \in [0,1]} \left| D_{0+}^{\mu_2} L_1(v_1, v_2)(t) \right| + \max_{t \in [0,1]} \left| D_{0+}^{\mu_1} L_2(v_1, v_2)(t) \right|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\max_{t \in [0,1]} \left| D_{0+}^{\mu_2} L_1(v_1, v_2)(t) \right| \\
&= \max_{t \in [0,1]} \left| \left(I^{1 - \mu_2} L_1(v_1, v_2)(t)\right)' \right| \\
&\leq \max_{t \in [0,1]} \left| \frac{1}{\Gamma(1 - \mu_2)} \int_0^t (t - s)^{-\mu_2} L_1'(v_1, v_2)(s) ds \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{t\in[0,1]}\left|\frac{(\sigma_1-\theta_1)(1-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})}H_2\int_0^t (t-s)^{-\mu_2}\int_0^1 \frac{\partial G_1(s,\tau)}{\partial s}d\tau ds\right| \\
&\leq \max_{t\in[0,1]}\left|\frac{(\sigma_1-\theta_1)(1-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})}H_2\int_0^t (t-s)^{-\mu_2}\frac{\sigma_1-\theta_1+b\delta_{11}}{\sigma_1-\theta_1}ds\right| \\
&\leq \frac{1}{2}H_2.
\end{aligned}
$$

Similarly, we have

$$\max_{t\in[0,1]}\left|D_{0+}^{\mu_1}L_2(v_1,v_2)(t)\right| \leq \frac{1}{2}H_2.$$

Hence,

$$\beta(L(v_1,v_2)) = \max_{t\in[0,1]}\left|D_{0+}^{\mu_2}L_1(v_1,v_2)(t)\right| + \max_{t\in[0,1]}\left|D_{0+}^{\mu_1}L_2(v_1,v_2)(t)\right| \leq H_2.$$

Consequently, $L:\overline{K}(\alpha,l_2;\beta,H_2)\to\overline{K}(\alpha,l_2;\beta,H_2)$. For the same reason, if $(v_1,v_2)\in\overline{K}(\alpha,l_1;\beta,H_1)$, then hypothesis (A_1) yields

$$f_1\left(t,v_2(t),D_{0+}^{\mu_1}v_2(t)\right) \leq \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)}l_1,\frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})}H_1\right\}.$$

As in the technique above, we reach $L:\overline{K}(\alpha,l_1;\beta,H_1)\to\overline{K}(\alpha,l_1;\beta,H_1)$. Therefore, condition (C_2) in Lemma 4 is satisfied.

To inspect condition (C_1) in Lemma 4, we select $(v_1^*(t),v_2^*(t))=\left(\frac{M}{2}k,\frac{M}{2}k\right)\in\overline{K}(\alpha,Mk;\beta,H_2;\gamma,k)$, $t\in[0,1]$ and $\gamma(v_1^*,v_2^*)=Mk>k$.

Therefore, $\left\{(v_1,v_2)\in\overline{K}(\alpha,Mk;\beta,H_2;\gamma,k)|\gamma(v_1.v_2)>k\right\}\neq\varnothing$.

For $(v_1,v_2)\in\overline{K}(\alpha,Mk;\beta,H_2;\gamma,k)$, we have

$$
\begin{aligned}
\gamma\left(L(v_1,v_2)\right) &= \gamma\left(L_1(v_1,v_2),L_2(v_1,v_2)\right) \\
&= \min_{t\in\left[\frac14,\frac34\right]}|L_1(v_1,v_2)(t)| + \min_{t\in\left[\frac14,\frac34\right]}|L_2(v_1,v_2)(t)| \\
&= \min_{t\in\left[\frac14,\frac34\right]}\left|\int_0^1 G_1(t,s)f_1\left(s,v_2(s),D_{0+}^{\mu_1}v_2(s)\right)ds\right| \\
&\quad + \min_{t\in\left[\frac14,\frac34\right]}\left|\int_0^1 G_2(t,s)f_2\left(s,v_1(s),D_{0+}^{\mu_2}v_1(s)\right)ds\right| \\
&\geq \int_0^1 \min_{t\in\left[\frac14,\frac34\right]} G_1(t,s)f_1\left(s,v_2(s),D_{0+}^{\mu_1}v_2(s)\right)ds \\
&\quad + \int_0^1 \min_{t\in\left[\frac14,\frac34\right]} G_2(t,s)f_2\left(s,v_1(s),D_{0+}^{\mu_2}v_1(s)\right)ds \\
&> \frac{k(1+\frac34 b)}{\omega}\sum_{i=1}^{2}\int_0^1(\delta_{1i}-a\delta_{2i})(1-s)^{\sigma_i-\theta_1-1}ds \\
&\geq \frac{k}{\omega}\cdot\omega = k.
\end{aligned}
$$

Therefore, condition (C_1) in Lemma 4 is satisfied.

Assume that $(v_1,v_2)\in\overline{K}(\alpha,l_2;\beta,H_2;\gamma,k)$ with $\alpha\left(L(v_1,v_2)\right)>Mk$. Then, we have

$$\gamma\left(L(v_1,v_2)\right) = \gamma\left(L_1(v_1,v_2),L_2(v_1,v_2)\right)$$

$$
\begin{aligned}
&= \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |L_1(v_1,v_2)(t)| + \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |L_2(v_1,v_2)(t)| \\
&= \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left|\int_0^1 G_1(t,s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds\right| \\
&\quad + \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left|\int_0^1 G_2(t,s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds\right| \\
&\geq \int_0^1 \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} G_1(t,s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds \\
&\quad + \int_0^1 \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} G_2(t,s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds \\
&\geq \frac{1}{M} \int_0^1 \max_{t\in[0,1]} G_1(t,s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds \\
&\quad + \frac{1}{M} \int_0^1 \max_{t\in[0,1]} G_2(t,s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds \\
&\geq \frac{1}{M} \max_{t\in[0,1]} \int_0^1 G_1(t,s) f_1\left(s, v_2(s), D_{0+}^{\mu_1} v_2(s)\right) ds \\
&\quad + \frac{1}{M} \max_{t\in[0,1]} \int_0^1 G_2(t,s) f_2\left(s, v_1(s), D_{0+}^{\mu_2} v_1(s)\right) ds \\
&= \frac{1}{M}\alpha(L_1,L_2)(v_1,v_2) = \frac{1}{M}\alpha\left(L(v_1,v_2)\right) > k.
\end{aligned}
$$

Therefore, condition (C_3) in Lemma 4 is contented.

Hence, by Lemma 4, there are three solutions of the boundary value problems (1.8)-(1.11) such that

$$(v_{11}, v_{12}) \in K(\alpha, l_1; \beta, H_1), \quad (v_{21}, v_{22}) \in \left\{\overline{K}(\alpha, l_2; \beta, H_2; \gamma, k) | \gamma(v_1, v_2) > k\right\}$$

and

$$(v_{31}, v_{32}) \in \overline{K}(\alpha, l_2; \beta, H_2) \setminus \left(\overline{K}(\alpha, l_2; \beta, H_2; \gamma, k) \cup \overline{K}(\alpha, l_1; \beta, H_1)\right).$$

□

Corollary 3.1. *Suppose there are constants* $0 < H_1 \leq H_2 \leq \cdots \leq H_{m-1}$, $0 < l_1 < k_1 < Mk_1 \leq l_2 < k_2 < Mk_2 \leq \cdots \leq l_m$, $m \in N$, *such that* $\frac{k_j}{\omega} \leq \min\left\{Q_1 l_{j+1}, Q_2 H_{j+1}\right\}$ *(* $1 \leq j \leq m-1$*), where* $Q_1 = \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)}, \frac{\sigma_2-\theta_1}{2(\delta_{12}+1)}\right\}$, $Q_2 = \min\left\{\frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})}, \frac{(\sigma_2-\theta_1)\Gamma(2-\mu_1)}{2(\sigma_2-\theta_1+b\delta_{12})}\right\}$, *and the following assumptions hold:*

(A_4) $f_1(t,x_1,x_2) \leq \min\left\{\frac{\sigma_1-\theta_1}{2(\delta_{11}+1)} l_j, \frac{(\sigma_1-\theta_1)\Gamma(2-\mu_2)}{2(\sigma_1-\theta_1+b\delta_{11})} H_j\right\}$,
$f_2(t,x_1,x_2)) \leq \min\left\{\frac{\sigma_2-\theta_1}{2(\delta_{12}+1)} l_j, \frac{(\sigma_2-\theta_1)\Gamma(2-\mu_1)}{2(\sigma_2-\theta_1+b\delta_{12})} H_j\right\}$,
for $(t,x_1,x_2) \in [0,1] \times [0,l_j] \times [-H_j, H_j]$, $1 \leq j \leq m-1$*;*

(A_5) $\min\left\{f_1(t,x_1,x_2),\ f_2(t,x_1,x_2)\right\} > \frac{k_j}{\omega}$,
for $(t,x_1,x_2) \in \left[\frac{1}{4},\frac{3}{4}\right] \times [k_j, Mk_j] \times \left[-H_{j+1}, H_{j+1}\right]$, $1 \leq j \leq m-1$*;*

Therefore, problems (1.8)-(1.11) admit at least $2m-1$ *positive solutions.*

Proof. When $m = 1$, it follows from (A_1) that $L : \overline{K}(\alpha, l_1; \beta, H_1) \to K(\alpha, l_1; \beta, H_1) \subset \overline{K}(\alpha, l_1; \beta, H_1)$. By the Schauder fixed-point theorem, the boundary value problems (1.8)-(1.11) admit at least a fixed point $(v_{11}, v_{12}) \in K(\alpha, l_1; \beta, H_1)$. When $m = 2$, it meets the conditions in the Theorem 3.1, and we can get at least three positive solutions (v_{21}, v_{21}), (v_{31}, v_{32}), (v_{41}, v_{42}). By keeping on this manner, the proof can be finished in an inductive way. □

4. Conclusions

By employing the fixed-point theorem, a class of fractional boundary value problem systems (1.8)-(1.11) has at least three positive solutions on condition that (A_1), (A_2) and (A_3) exist. Finally, a corollary of Theorem 1 has been drawn. In other words, under conditions (A_4) and (A_5), the boundary value problems (1.8)-(1.11) have $2m - 1$ positive solutions, which simply prove this corollary.

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