Journal of Nonlinear Modeling and Analysis
Volume 4, Number 3, September 2022, 502–513
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.502

Asymptotics of a Class of Singularly Perturbed Weak Nonlinear Boundary Value Problem with a Multiple Root of the Degenerate Equation*

Qian Yang[1] and Mingkang Ni[1,2,†]

Abstract A singularly perturbed boundary value problem with weak nonlinearity in the case when the degenerate equation has a multiple root is studied. The asymptotic approximation of the solution is constructed by the modified boundary layer function method. Based on the comparison principle, there exist multizonal boundary layers in the neighborhood of the endpoints. The existence of a solution is proved by using the method of asymptotic differential inequalities.

Keywords Singularly perturbed problem, Multiple root of the degenerate equation, Asymptotic method, Upper and lower solutions.

MSC(2010) 34B15, 34b16, 34E05, 34E15.

1. Problem statement

It is well-known that many scholars [9, 10] have been attracted to the study of ordinary differential equations. With the development of the times, it has been found out that there are many mathematical models with small parameters in practical problems. In particular, the singularly perturbed reaction-advection-diffusion equation plays an important role in practical application such as the propagation, decay and chemical reaction of impurities in the atmosphere [8]. Therefore, this kind of problem has attracted the attention of a great many of mathematical experts and scholars. To the best of our knowledge, a lot of research in the case when the degenerate equation has isolated roots has been carried out [7, 12, 14–18]. When the degenerate equation has multiple roots, the critical manifold is not normally hyperbolic, and cannot meet the stability condition of Tikhonov's theorem. In this case, it is necessary to use a modified boundary layer method to resolve difficulties. As shown in [1–6, 19], the boundary layers can be decomposed into three zones, and their formal asymptotic solutions have different decay characters with respect to diverse scales in distinct regions. The singularly perturbed reaction-diffusion equa-

†the corresponding author.
Email address: 15934551791@163.com (Q. Yang), xiaovikdo@163.com (M. Ni)
[1]School of Mathematical Sciences, East China Normal University, Shanghai 200241, China
[2]Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, Shanghai 200241, China
*The authors were supported by National Natural Science Foundation of China (No. 11871217) and the Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).

tions without advection terms in the case of multiple roots have been studied in [2,3]. In this paper, the stationary problem for a class of reaction-advection-diffusion equation with weak nonlinearity and multiple root of the degenerate equation is considered:

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} - a(x)\left(\varepsilon \dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = f(u,x,\varepsilon), \quad 0 < x < 1, \\ u(0,\varepsilon) = u^0, \quad u(1,\varepsilon) = u^1, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and u is a scalar function.

This singularly perturbed problem is sophisticated, and the modified boundary layer function method shall be applied to construct the asymptotic solution. More importantly, we obtain a result that the nature of boundary layer with a transition from algebraic decay to exponential decay by comparison principle. Finally, the existence of a solution is proved by the method of upper and lower solutions [11,13].

Let the following assumptions be satisfied.

Denote

$$\bar{D} = \{(u,x,\varepsilon) |\ |u| \le l,\ 0 \le x \le 1,\ 0 \le \varepsilon \le \varepsilon_0\}.$$

Assumption 1.1. *Let*

$$f(u,x,\varepsilon) = h(u,x)(u-\varphi(x))^2 - \varepsilon f_1(u,x,\varepsilon), \tag{1.2}$$

where the functions $h(u,x)$, $\varphi(x)$ and $f_1(u,x,\varepsilon)$ are sufficiently smooth on the set $\bar{D}$, and one has the inequality

$$\bar{f}_1(x) := f_1(\varphi(x),x,0) > 0, \quad 0 \le x \le 1.$$

Moreover, $h(u,x)$ conforms to one of the following requirements:

(i)

$$h(u,x) > 0, \quad \varphi(x) \le u < \psi(x), \quad 0 \le x \le 1,$$

where $\psi(x)$, $0 \le x \le 1$ is a function that satisfies $\psi(x) > \phi(x)$, $h(\psi(x),x) = 0$;

(ii)

$$h(u,x) > 0, \quad u \ge \varphi(x), \quad 0 \le x \le 1,$$

.

To be definite, we shall consider case (ii).

Assumption 1.1 shows that the degenerate equation

$$f(u,x,0) = 0 \tag{1.3}$$

has a multiple root

$$\bar{u}(x) = \varphi(x), \quad 0 \le x \le 1. \tag{1.4}$$

To determine the leading term in the asymptotic representation of the boundary layers in the course of constructing the asymptotics of solution to problem (1.1), the following condition is needed.

Assumption 1.2. *Assume that the inequalities are satisfied:*

$$u^0 - \varphi(0) > 0, \quad u^1 - \varphi(1) > 0;$$

$$a(x) < 0, \quad 0 \le x \le 1.$$

2. Construction of the asymptotics of the solution of problem (1.1)

The asymptotic expansion of the solution $u(x,\varepsilon)$ to problem (1.1) shall be constructed on the interval $[0,1]$ as follows:

$$u(x,\varepsilon) = \bar{u}(x,\varepsilon) + \Pi(\xi,\varepsilon) + \tilde{\Pi}(\tilde{\xi},\varepsilon), \tag{2.1}$$

$$\xi = \frac{x}{\varepsilon}, \quad \tilde{\xi} = \frac{1-x}{\varepsilon},$$

where $\bar{u}(x,\varepsilon)$ is the regular part, $\Pi(\xi,\varepsilon)$ and $\tilde{\Pi}(\tilde{\xi},\varepsilon)$ are respectively the boundary layer parts in the neighborhood of endpoints $x=0$ and $x=1$. Due to the fact that the degenerate equation (1.3) of problem (1.1) has a multiple root $\varphi(x)$, the studies have shown that each term of the asymptotics can be sought in the forms of power series [2, 3]:

$$\bar{u}(x,\varepsilon) = \bar{u}_0(x) + \varepsilon^{\frac{1}{2}}\bar{u}_1(x) + \cdots + \varepsilon^{\frac{k}{2}}\bar{u}_k(x) + \cdots, \tag{2.2}$$

$$\Pi(\xi,\varepsilon) = \Pi_0(\xi) + \varepsilon^{\frac{1}{4}}\Pi_1(\xi) + \cdots + \varepsilon^{\frac{k}{4}}\Pi_k(\xi) + \cdots, \quad \xi = \frac{x}{\varepsilon}, \tag{2.3}$$

$$\tilde{\Pi}(\xi,\varepsilon) = \tilde{\Pi}_0(\tilde{\xi}) + \varepsilon^{\frac{1}{4}}\tilde{\Pi}_1(\tilde{\xi}) + \cdots + \varepsilon^{\frac{k}{4}}\tilde{\Pi}_k(\tilde{\xi}) + \cdots, \quad \tilde{\xi} = \frac{1-x}{\varepsilon}. \tag{2.4}$$

For boundary layer functions $\Pi_k(\xi)$, $\tilde{\Pi}_k(\tilde{\xi})$, $k \geq 0$, it is necessary to impose the standard boundary value conditions vanishing at infinity:

$$\Pi_k(\infty) = 0, \quad \tilde{\Pi}_k(\infty) = 0. \tag{2.5}$$

For the regular part of the asymptotics $\bar{u}(x,\varepsilon)$, one can obtain

$$\varepsilon^2 \frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2} - a(x)\left(\varepsilon\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right)^2 = h(\bar{u},x)(\bar{u}-\varphi(x))^2 - \varepsilon f_1(\bar{u},x,\varepsilon). \tag{2.6}$$

By substituting series (2.2) into series (2.6) and comparing the coefficients of like powers of ε on the left and right sides of (2.6), we have

$$\bar{u}_0(x) = \varphi(x)$$

by virtue of (1.4). Moreover, for $\bar{u}_1(x)$, we obtain the following quadratic equation

$$\bar{h}(x)(\bar{u}_1(x))^2 - \bar{f}_1(x) = 0,$$

where $\bar{h}(x) = h(\varphi(x),x)$.
Owing to the characteristic of boundary layers and Assumption 1.1, we take the positive root

$$\bar{u}_1(x) = \sqrt{\frac{\bar{f}_1(x)}{\bar{h}(x)}} > 0, \quad 0 \leq x \leq 1. \tag{2.7}$$

The subsequent coefficients $\bar{u}_k(x)$ can be represented in the form

$$\bar{u}_k(x) = \frac{f_k(x)}{2\bar{h}(x)\bar{u}_1(x)}, \quad k = 1, 2, \cdots.$$

Here, $f_k(x)$ are known functions that depend on $\bar{u}_j(x)$, $j < k$. From Assumption 1.1 and (2.7), $\bar{u}_k(x)$ can be uniquely determined.

In the following, we write out the problems for determining the boundary layer part of the asymptotics $\Pi(\xi, \varepsilon)$ in the neighborhood of $x = 0$

$$\begin{cases} \dfrac{\mathrm{d}^2\Pi}{\mathrm{d}\xi^2} - a(x)\left(\dfrac{\mathrm{d}\Pi}{\mathrm{d}\xi}\right)^2 = \Pi f, \\ \Pi(0,\varepsilon) = u^0 - \bar{u}(0,\varepsilon), \quad \Pi(+\infty,\varepsilon) = 0, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} \Pi f =& h\left(\bar{u}(\varepsilon^{\frac{3}{4}}\zeta) + \Pi(\xi,\varepsilon), \varepsilon^{\frac{3}{4}}\zeta\right) \times \left[\bar{u}(\varepsilon^{\frac{3}{4}}\zeta) + \Pi(\xi,\varepsilon) - \varphi(\varepsilon^{\frac{3}{4}}\zeta)\right]^2 - \\ &- h\left(\bar{u}(\varepsilon^{\frac{3}{4}}\zeta), \varepsilon^{\frac{3}{4}}\zeta\right)\left[\bar{u}(\varepsilon^{\frac{3}{4}}\zeta) - \varphi(\varepsilon^{\frac{3}{4}}\zeta)\right]^2 - \varepsilon\Pi f_1. \end{aligned}$$

Here, $\zeta = \varepsilon^{\frac{1}{4}}\xi = \frac{x}{\varepsilon^{3/4}}$. Note that $\Pi_k(\xi)$ depend on ξ and ε, but $\Pi_k(\xi,\varepsilon)$ is still denoted as $\Pi_k(\xi)$ for simplicity. In this case, the algorithm for constructing equations of the functions $\Pi_k(\xi)$ from problem (2.8) is qualitatively different from the boundary layer function method [18]. Given the specific character of boundary layers, the equation and boundary value conditions for $\Pi_0(\xi)$ have the form

$$\begin{cases} \dfrac{\mathrm{d}^2\Pi_0}{\mathrm{d}\xi^2} - a(0)\left(\dfrac{\mathrm{d}\Pi_0}{\mathrm{d}\xi}\right)^2 = h(\varphi(0) + \Pi_0, 0)\left[(\Pi_0)^2 + 2\sqrt{\varepsilon}\bar{u}_1(0)\Pi_0\right], \\ \Pi_0(0,\varepsilon) = u^0 - \varphi(0), \quad \Pi_0(\infty,\varepsilon) = 0, \end{cases} \tag{2.9}$$

which can be reduced to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\Pi_0}{\mathrm{d}\xi} = -\sqrt{2\displaystyle\int_0^{\Pi_0} h(\varphi(0)+s, 0)\left(s^2 + 2\sqrt{\varepsilon}\bar{u}_1(0)s\right)\mathrm{e}^{2a(0)(\Pi_0 - s)}\,\mathrm{d}s}, \\ \Pi_0(0,\varepsilon) = u^0 - \varphi(0). \end{cases} \tag{2.10}$$

It turns out that by virtue of Assumptions 1.1-1.2, problem (2.10) has a solution $\Pi_0(\xi)$, which satisfies the estimate

$$\Pi_{\kappa_2}(\xi) \le \Pi_0(\xi) \le \Pi_{\kappa_1}(\xi), \quad \xi \ge 0, \tag{2.11}$$

where $\kappa_1 > 0$, $\kappa_2 > 0$, and $\Pi_\kappa(\xi)$ can be represented as

$$\Pi_\kappa(\xi) = \frac{12\sqrt{\varepsilon}\bar{u}_1(0)(1 + O(\varepsilon^{1/4}))}{[1 - (1 - c_0\varepsilon^{1/4} + O(\sqrt{\varepsilon}))\exp(-\varepsilon^{1/4}k_0\kappa\xi)]^2}\exp(-\varepsilon^{1/4}k_0\kappa\xi). \tag{2.12}$$

Here, $k_0 = \sqrt{2\bar{u}_1(0)}$, $c_0 = \sqrt{12\bar{u}_1(0)/(u^0 - \varphi(0))}$. It is easy to see that estimate (2.12) provides the same behavior of $\Pi_0(\xi)$ as that of $\Pi_\kappa(\xi)$. Let us justify this fact.

Since $\Pi_0(0) = u^0 - \varphi(0) > 0$ and $h(s) > 0$ for $0 \le s \le \Pi_0(\xi)$ by Assumptions 1.1-1.2, there exist positive numbers κ_1 and κ_2 such that for all $0 \le s \le \Pi_0(\xi)$,

$$\kappa_1^2 \le h(\varphi(0) + s, 0)\mathrm{e}^{2a(0)(\Pi_0(s) - s)} \le \kappa_2^2. \tag{2.13}$$

At first, we consider the following two problems

$$\begin{cases} \dfrac{\mathrm{d}\Pi_{\kappa_i}}{\mathrm{d}\xi} = -\sqrt{2}\,\kappa_i\sqrt{\dfrac{1}{3}\Pi_{\kappa_i} + \sqrt{\varepsilon}\bar{u}_1(0)}\,\Pi_{\kappa_i}, \quad \xi > 0, \\ \Pi_{\kappa_i}(0) = u^0 - \varphi(0), \quad i = 1, 2, \end{cases}$$

whose solutions $\Pi_{\kappa_i}(\xi)$ and $\Pi_{\kappa_2}(\xi)$ can be represented as (2.12). Then, from (2.10), we have

$$-\sqrt{2}\kappa_2\sqrt{\frac{1}{3}\Pi_0+\sqrt{\varepsilon}\bar{u}_1(0)}\,\Pi_0 \le \frac{\mathrm{d}\Pi_0}{\mathrm{d}\xi} \le -\sqrt{2}\kappa_1\sqrt{\frac{1}{3}\Pi_0+\sqrt{\varepsilon}\bar{u}_1(0)}\,\Pi_0.$$

Thus, problem (2.10) has a solution $\Pi_0(\xi)$ that satisfies the inequality (2.11) by the comparison principle.

A simple analysis of the expression (2.12) shows that different decay nature of the boundary layer function $\Pi_\kappa(\xi)$ (i.e., $\Pi_0(\xi)$) appears in three distinct regions. If $0 \le \xi \le \varepsilon^{-\gamma}(0 \le \gamma < \frac{1}{4})$, then $\Pi_\kappa(\xi) = O(1/(1+\xi^2))$ decreases by a power function, as the variable $\xi \to +\infty$. If $\varepsilon^{-\gamma} \le \xi \le \varepsilon^{-1/4}$, then the characteristic of $\Pi_\kappa(\xi)$ changes from power law decay to exponential decay. Moreover, if $\xi \ge \varepsilon^{-1/4}$, then $\Pi_\kappa(\xi) = O(\sqrt{\varepsilon})\exp(-k_0\kappa\zeta)$ has an exponential decay with respect to the new variable $\zeta = x/\varepsilon^{3/4}$.

For $\Pi_k(\xi)$, $k = 1, 2, \cdots$, one can obtain

$$\begin{cases} \dfrac{\mathrm{d}^2\Pi_k}{\mathrm{d}\xi^2} - 2a(0)\dfrac{\mathrm{d}\Pi_0}{\mathrm{d}\xi}\dfrac{\mathrm{d}\Pi_k}{\mathrm{d}\xi} = \alpha(\xi,\varepsilon)\Pi_k + \pi_k(\xi,\varepsilon), \\ \Pi_k(0) = \begin{cases} -\bar{u}_{\frac{k}{2}}(0), & k = 2n, \\ 0, \quad k = 2n-1, & n = 1, 2, \cdots, \end{cases} \\ \Pi_k(\infty,\varepsilon) = 0, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} \alpha(\xi,\varepsilon) =& h_u(\varphi(0)+\Pi_0(\xi),0)\left[(\Pi_0(\xi))^2 + 2\sqrt{\varepsilon}\bar{u}_1(0)\Pi_0(\xi)\right] \\ &+ 2h(\varphi(0)+\Pi_0(\xi),0)\left[\Pi_0(\xi)+\sqrt{\varepsilon}\bar{u}_1(0)\right]. \end{aligned}$$

Here, $\pi_k(\xi,\varepsilon)$ depends on $\Pi_j(\xi), j < k$. In particular, $\pi_1(\xi,\varepsilon) = 0$,

$$\begin{aligned} \pi_2(\xi,\varepsilon) =& \sqrt{\varepsilon}\left(h(\varphi(0)+\Pi_0(\xi),0) - \bar{h}(0)\right)(\bar{u}_1(0))^2 \\ &+ 2\sqrt{\varepsilon}h(\varphi(0)+\Pi_0(\xi),0)\bar{u}_2(0)\Pi_0(\xi) \\ &+ h_u(\xi)\bar{u}_1(0)\left[(\Pi_0(\xi))^2 + 2\sqrt{\varepsilon}\bar{u}_1(0)\Pi_0(\xi)\right] + \sqrt{\varepsilon}\Pi_0 f_1. \end{aligned}$$

Here,

$$\Pi_0 f_1 = f_1(\varphi(0)+\Pi_0(\xi),0,0) - f_1(\varphi(0),0,0).$$

Problem (2.14) has solutions $\Pi_k(\xi)$ that can be written as

$$\Pi_k(\xi) = \Pi_k(0)\Phi(\xi)\Phi^{-1}(0) - \Phi(\xi)\int_0^\xi p^{-1}(\eta)\Phi^{-2}(\eta)J_k(\eta)\,\mathrm{d}\eta, \tag{2.15}$$

where

$$J_k(\eta) = \int_\eta^\infty p(s)\Phi(s)\pi_k(s)\,ds, \quad \Phi(\xi) = \frac{\mathrm{d}\Pi_0}{\mathrm{d}\xi}(\xi), \quad p(\xi) = \mathrm{e}^{-2a(0)(\Pi_0(\xi)-\Pi_0(0))}.$$

Taking (2.15) into consideration and the proof of lemma in [2], it can be concluded that all boundary layer functions $\Pi_k(\xi),\ k \ge 0$ admit the estimate

$$|\Pi_k(\xi)| \le C\Pi_\kappa(\xi), \quad \xi \ge 0. \tag{2.16}$$

Finally, the right boundary layer part of the asymptotics $\tilde{\Pi}(\tilde{\xi},\varepsilon)$ near the right endpoint $x = 1$ can be determined by analogy with the series $\Pi(\xi,\varepsilon)$, and $\tilde{\Pi}_k(\tilde{\xi})$, $k \ge 0$ satisfy the estimate similar to (2.16). It is worth mentioning that $\tilde{\Pi}_0(\tilde{\xi}) > 0$ and $\Pi_0(\xi) > 0$. Moreover, $\Pi_1(\xi) = 0$ and $\tilde{\Pi}_1(\tilde{\xi}) = 0$.

3. Justification of the constructed asymptotic solution

Denote

$$U_n(x,\varepsilon)=\sum_{i=0}^{n}\varepsilon^{\frac{i}{2}}\bar{u}_i(x)+\sum_{i=0}^{2n+1}\varepsilon^{\frac{i}{4}}\left(\Pi_i(\xi)+\tilde{\Pi}_i(\tilde{\xi})\right).$$

The following main theorem can be derived.

Theorem 3.1. *If Assumptions 1.1-1.2 are satisfied, for sufficiently small $\varepsilon>0$, problem (1.1) has a solution $u(x,\varepsilon)$ that can be written as*

$$u(x,\varepsilon)=U_n(x,\varepsilon)+O(\varepsilon^{\frac{n+1}{2}}),\quad 0\le x\le 1. \tag{3.1}$$

Proof. To prove the theorem by using the asymptotic method of differential inequalities, we recall the definition of upper and lower solutions to problem (1.1).

Definition 3.1. The functions $\beta(x,\varepsilon)$ and $\alpha(x,\varepsilon)$ are called the ordered upper and lower solutions to problem (1.1), if they satisfy the following three conditions:

(i) $\alpha(x,\varepsilon)\le\beta(x,\varepsilon)$, $0\le x\le 1$.

(ii) $L_\varepsilon\alpha\ge 0\ge L_\varepsilon\beta$, $0<x<1$, where

$$L_\varepsilon\alpha=\varepsilon^2\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}-a(x)\left(\varepsilon\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2-f(\alpha,x,\varepsilon),$$

$$L_\varepsilon\beta=\varepsilon^2\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}-a(x)\left(\varepsilon\frac{\mathrm{d}\beta}{\mathrm{d}x}\right)^2-f(\beta,x,\varepsilon).$$

(iii) $\alpha(0,\varepsilon)\le u^0\le\beta(0,\varepsilon)$, $\alpha(1,\varepsilon)\le u^1\le\beta(1,\varepsilon)$.

We shall construct the upper and lower solutions as follows:

$$\begin{aligned}\beta(x,\varepsilon)&=U_n(x,\varepsilon)+\varepsilon^{\frac{n+1}{2}}\gamma,\\ \alpha(x,\varepsilon)&=U_n(x,\varepsilon)-\varepsilon^{\frac{n+1}{2}}\gamma,\end{aligned} \tag{3.2}$$

where γ is a positive number that is not relied on ε.

For sufficiently small $\varepsilon>0$, it is clear that $\beta(x,\varepsilon)$ and $\alpha(x,\varepsilon)$ satisfy Condition (i) and Condition (iii).

Taking $\alpha(x,\varepsilon)$ for example, we shall justify the fact that Condition (ii) is fulfilled. From the algorithm for constructing the asymptotics, one can obtain

$$L_\varepsilon U_n=O(\varepsilon^{\frac{n}{2}+1})+O(\varepsilon^{\frac{n+1}{2}})\left(\Pi_\kappa(\xi)+\tilde{\Pi}(\tilde{\xi})\right),\quad 0\le x\le 1.$$

Therefore,

$$\begin{aligned}L_\varepsilon\alpha(x,\varepsilon)&=L_\varepsilon U_n+h(U_n,x)\left[(U_n-\varphi(x))^2-(U_n-\varphi(x)-\varepsilon^{\frac{n+1}{2}}\gamma)^2\right]+\\ &\quad+h_u(U_n,x)(U_n-\varphi(x))^2-\varepsilon\left[f_1(U_n,x,\varepsilon)-f_1(U_n-\varepsilon^{\frac{n+1}{2}}\gamma,x,\varepsilon)\right]\\ &=O(\varepsilon^{\frac{n+1}{2}})\left(\Pi_\kappa+\tilde{\Pi}_\kappa+2\gamma h(\varphi(x)+O(\sqrt{\varepsilon}),x)(\Pi_0(\xi)+\tilde{\Pi}_0(\tilde{\xi}))\right)\\ &\quad+O(\varepsilon^{\frac{n}{2}+1}).\end{aligned}$$

Considering $\Pi_0(\xi) > 0$, $\tilde{\Pi}_0(\tilde{\xi}) > 0$ and $h(\varphi(x) + O(\sqrt{\varepsilon}), x) > 0$, we can conclude that for sufficiently large γ, $L_\varepsilon \alpha(x, \varepsilon) > 0$ is valid.

Similarly, if we take sufficiently large $\gamma > 0$, for sufficiently small $\varepsilon > 0$, $L_\varepsilon \beta(x, \varepsilon) < 0$ is also valid.

By Nagumo theorem, (1.1) has a solution $u(x, \varepsilon)$, which satisfies the inequality

$$\alpha(x, \varepsilon) \leq u(x, \varepsilon) \leq \beta(x, \varepsilon), \quad 0 \leq x \leq 1.$$

It follows from expressions (3.2) of upper and lower solutions that

$$\alpha(x, \varepsilon) = U_n(x, \varepsilon) + O(\varepsilon^{\frac{n+1}{2}}),$$

$$\beta(x, \varepsilon) = U_n(x, \varepsilon) + O(\varepsilon^{\frac{n+1}{2}}).$$

Thus, formula (3.1) is satisfied. □

4. Numerical example

Consider singularly perturbed boundary value problem

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + (x + 0.2)\left(\varepsilon \dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = (y - x)^2 - \varepsilon, \quad 0 < x < 1, \\ u(0, \varepsilon) = 0.2, \quad u(1, \varepsilon) = 1.2. \end{cases} \tag{4.1}$$

Here, $h(x) = 1$, $\varphi(x) = x$, $f_1(u, x, \varepsilon) = 1$.

It is easy to verify Assumptions 1.1-1.2. The degenerate equation of problem (4.1) has a multiple root

$$\bar{u}_0(x) = \varphi(x) = x, \quad 0 \leq x \leq 1.$$

From (2.7), one can obtain

$$\bar{u}_1(x) = 1.$$

The problems for zero-order boundary layer functions $\Pi_0(\xi)$ and $\tilde{\Pi}_0(\tilde{\xi})$ have the forms

$$\begin{cases} \dfrac{\mathrm{d}\Pi_0}{\mathrm{d}\xi} = -\sqrt{2 \displaystyle\int_0^{\Pi_0} \left(s^2 + 2\sqrt{\varepsilon} s\right) \mathrm{e}^{-2(s+0.2)(\Pi_0 - s)} \, \mathrm{d}s}, \\ \Pi_0(0, \varepsilon) = 0.2 \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}\tilde{\Pi}_0}{\mathrm{d}\tilde{\xi}} = -\sqrt{2 \displaystyle\int_0^{\tilde{\Pi}_0} \left(s^2 + 2\sqrt{\varepsilon} s\right) \mathrm{e}^{-2(s+0.2)(\tilde{\Pi}_0 - s)} \, \mathrm{d}s}, \\ \tilde{\Pi}_0(0, \varepsilon) = 0.2, \end{cases}$$

whose solutions $\Pi_0(\xi)$ and $\tilde{\Pi}_0(\tilde{\xi})$ admit

$$\Pi_0(\xi) \geq \Pi_\kappa(\xi), \quad \tilde{\Pi}_0(\tilde{\xi}) \geq \tilde{\Pi}_\kappa(\tilde{\xi}).$$

Here, $\Pi_\kappa(\xi)$ and $\tilde{\Pi}_\kappa(\tilde{\xi})$ are found from the problems

$$\begin{cases} \dfrac{\mathrm{d}\Pi_\kappa}{\mathrm{d}\xi} = -\sqrt{2 \displaystyle\int_0^{\Pi_\kappa} \left(s^2 + 2\sqrt{\varepsilon} s\right) \, \mathrm{d}s}, \\ \Pi_\kappa(0, \varepsilon) = 0.2 \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}\tilde{\Pi}_\kappa}{\mathrm{d}\tilde{\xi}} = -\sqrt{2\displaystyle\int_0^{\tilde{\Pi}_\kappa} \left(s^2 + 2\sqrt{\varepsilon}s\right)\, \mathrm{d}s}, \\ \tilde{\Pi}_\kappa(0,\varepsilon) = 0.2, \end{cases}$$

whose solutions are

$$\Pi_\kappa(\xi) = \frac{12\sqrt{\varepsilon}\left(1 + 30\sqrt{\varepsilon} - 5\sqrt{2.4\sqrt{\varepsilon} + 36\varepsilon}\right)}{\left[1 - \left(1 + 30\sqrt{\varepsilon} - 5\sqrt{2.4\sqrt{\varepsilon} + 36\varepsilon}\right)\exp(-(4\varepsilon)^{1/4}\xi)\right]^2}\exp(-(4\varepsilon)^{1/4}\xi);$$

$$\tilde{\Pi}_\kappa(\tilde{\xi}) = \frac{12\sqrt{\varepsilon}\left(1 + 30\sqrt{\varepsilon} - 5\sqrt{2.4\sqrt{\varepsilon} + 36\varepsilon}\right)}{\left[1 - \left(1 + 30\sqrt{\varepsilon} - 5\sqrt{2.4\sqrt{\varepsilon} + 36\varepsilon}\right)\exp(-(4\varepsilon)^{1/4}\tilde{\xi})\right]^2}\exp(-(4\varepsilon)^{1/4}\tilde{\xi}).$$

Therefore, zero-order asymptotics $U_0(x,\varepsilon)$ of the singularly perturbed Dirichlet boundary value problem (4.1) is constructed, and Theorem 3.1 leads to

$$u(x,\varepsilon) = x + \Pi_\kappa(\xi) + \tilde{\Pi}_\kappa(\tilde{\xi}) + O(\sqrt{\mu}), \quad 0 \le x \le 1.$$

Figures 1-4 show the agreement between numerical solution and the zero-order asymptotic solution obtained by our algorithm for very sufficiently small values of ε.

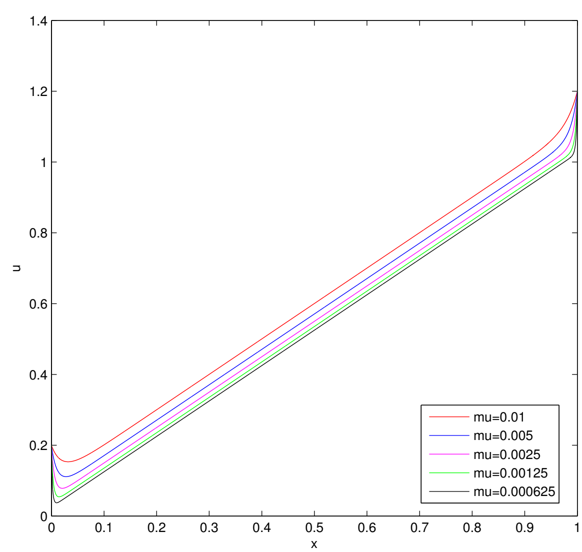

Figure 1. Numerical solution of problem (4.1)

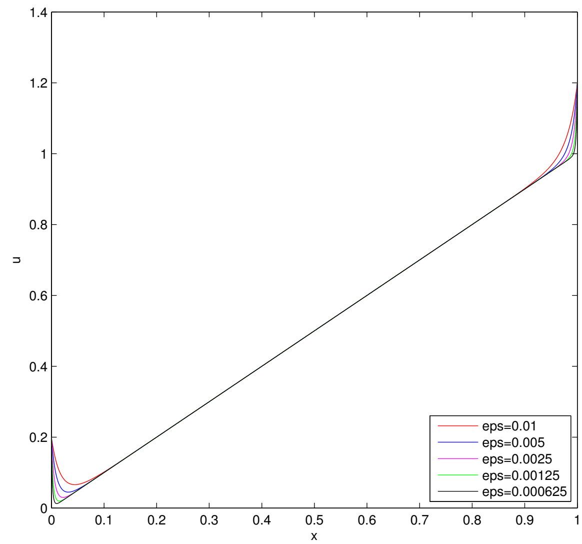

Figure 2. Zero-order asymptotic solution of problem (4.1)

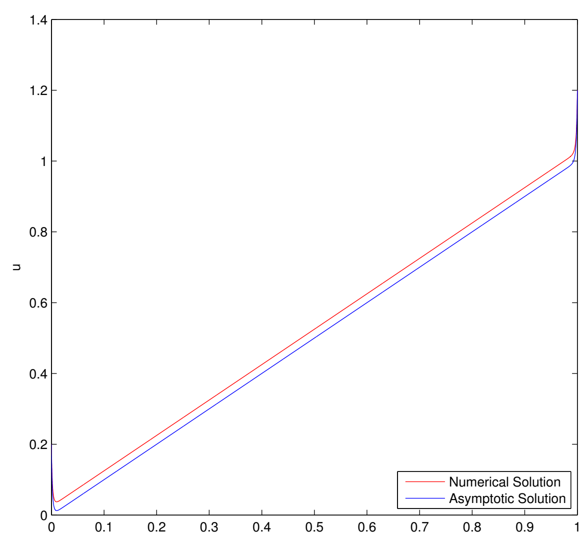

Figure 3. Zero-order asymptotic solution and numerical solution of problem (4.1) ($\varepsilon = 0.000625$)

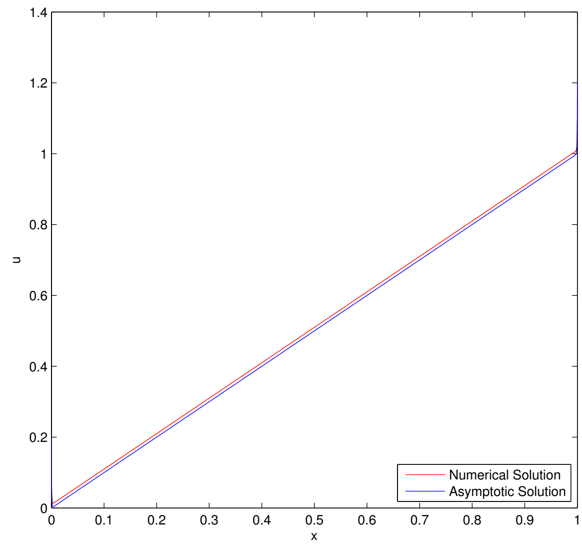

Figure 4. Zero-order asymptotic solution and numerical solution of problem (4.1) ($\varepsilon = 0.0001$)

5. Concluding remarks

In the above sections, the stationary equation of a class of reaction-advection-diffusion type with multiple root of the degenerate equation has been investigated. By using modified boundary layer function layer method and comparison principle, the asymptotics of the solution is constructed, and the character of boundary layer is obtained. Moreover, we prove the existence of a smooth solution by the asymptotic differential inequalities technique. Further, a numerical example is presented to illustrate the obtained results. Additionally, the theoretical results can be extended to reaction-advection-diffusion equation with discontinuous right-hand side and triple root of the degenerate equation.

References

[1] V. F. Butuzov, *On periodic solutions to singularly perturbed parabolic problems in the cases of multiple roots of the degenerate equation*, Computational Mathematics and Mathematical Physics, 2011, 51(1), 40–50.

[2] V. F. Butuzov, *On the special properties of the boundary layer in singularly perturbed problems with multiple root of the degenerate equation*, Mathematical Notes, 2013, 94(1), 60–70.

[3] V. F. Butuzov, *On the stability and the attraction domain of the stationary solution of a singularly perturbed parabolic Equation with a multiple root of the degenerate equation*, Partial Differential Equations, 2015, 51(12), 1569–1582.

[4] V. F. Butuzov, *Asymptotic behaviour and stability of solutions of a singularly*

perturbed elliptic problem with a triple root of the degenerate equation, Izvestiya Mathematics, 2017, 81(3), 21–44.

[5] V. F. Butuzov, N. N. Nefedov, O. E. Omel'chenko and L. Recke, *Partly dissipative system with multizonal initial and boundary layers*, Journal of Physics: Conference Series, 2019, 1205, Article ID 012009, 7 pages.

[6] V. F. Butuzov, N. N. Nefedov, L. Recke and K. R. Schneider, *On a singularly perturbed initial value problem in the case of a double root of the degenerate equation*, Nonlinear Analysis, 2013, 83, 1–11.

[7] M. A. Davydova and S. A. Zakharova, *Singularly perturbed stationary diffusion model with a cubic nonlinearity*, Differential Equations, 2020, 56(7), 819–830.

[8] N. T. Levashova, O. A. Nikolaeva and A. D. Pashkin, *Modeling the temperature distribution at the interface "water-air" using the theory of contrast structures*, Vestnik Moskovskogo Universiteta Series III Physics Astronomy, 2015, 5, 12–16.

[9] P. Liu, Y. Fan and L. Wang, *Existence of Positive Solutions for a Nonlinear Second Order Periodic Boundary Value Problem*, Journal of Nonlinear Modeling and Analysis, 2020, 2(4), 513–524.

[10] R. Ma, S. Li and S. Guo, *Steady-state Solution for Reaction-diffusion Models with Mixed Boundary Conditions*, Journal of Nonlinear Modeling and Analysis, 2019, 1(4), 545–560.

[11] N. N. Nefedov, *The method of differential inequalities for some singularly perturbed partial differential equations*, Differential Equations, 1995, 31(4), 668–671.

[12] N. N. Nefedov and M. A. Davydova, *Contrast structures in multidimensional singularly perturbed reaction–diffusion–advection problems*, Differential Equations, 2012, 48(5), 745–755.

[13] N. N. Nefedov and A. G. Nikitin, *Development of the asymptotic method of differential inequalities for step-type solutions of singularly perturbed integro-differential equations*, Computational Mathematics and Mathematical Physics, 2001, 41(7), 1005–1014.

[14] N. N. Nefedov and E. I. Nikulin, *Existence and asymptotic stability of periodic two-dimensional contrast structures in the problem with weak linear advection*, Mathematical Notes, 2019, 106(5), 771–783.

[15] N. N. Nefedov, E. I. Nikulin and L. Recke, *On the existence and asymptotic stability of periodic contrast structures in quasilinear reaction–advection–diffusion equations*, Russian Journal of Mathematical Physics, 2019, 26(1), 55–69.

[16] N. N. Nefedov, L. Recke and K. R. Schneider, *Existence and asymptotic stability of periodic solutions with an internal layer of reaction–advection–diffusion equations*, Journal of Mathematical Analysis and Applications, 2013, 405(1), 90–103.

[17] M. Ni, Y. Pang and M. A. Davydova, *Contrast structures in problems for a stationary equation of reaction–diffusion–advection type with discontinuous nonlinearity*, Mathematical Notes, 2018, 104(5), 118–127.

[18] A. B. Vasil'eva and V. F. Butuzov, *Asymptotic Methods in Singular Perturbation Theory*, Vysshaya Shkola, Moscow, 1990.

[19] A. B. Vasil'eva and V. S. Pilyugin, *Singularly perturbed boundary value problems with a power-law boundary layer*, Ordinary Differential Equations, 2009, 45(3), 323–334.