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SECOND ORDER UNCONDITIONALLY STABLE AND CONVERGENT LINEARIZED SCHEME FOR A FLUID-FLUID INTERACTION MODEL*

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Abstract

In this paper, a fully discrete finite element scheme with second-order temporal accuracy is proposed for a fluid-fluid interaction model, which consists of two Navier-Stokes equations coupled by a linear interface condition. The proposed fully discrete scheme is a combination of a mixed finite element approximation for spatial discretization, the second-order backward differentiation formula for temporal discretization, the second-order Gear's extrapolation approach for the interface terms and extrapolated treatments in linearization for the nonlinear terms. Moreover, the unconditional stability is established by rigorous analysis and error estimate for the fully discrete scheme is also derived. Finally, some numerical experiments are carried out to verify the theoretical results and illustrate the accuracy and efficiency of the proposed scheme.

Mathematics subject classification: 65M15, 65M60.
Key words: Fluid-fluid interaction model, Unconditional stability, Second order temporal accuracy, Error estimate.

1. Introduction

Numerical simulation of multi-domain and multi-physics coupling of one fluid with another fluid is an important aspect in many industrial applications. In fact, the fluid-fluid interaction model can be seen as one of them arises in many important scientific, engineering and industrial applications, such as heterogeneous of blood flow [8] and atmosphere-ocean interaction [20–22]. Due to the practical importance of the fluid-fluid interaction problem, there has been a lot of attention recently paid to the development of accurate and efficient numerical methods; see, e.g., [5,16–19,23] among many others. Besides, Bresch and Koko [4] have presented a numerical simulation of the considered model by using an operator-splitting method and optimization-based nonoverlapping domain decomposition methods. Based on implicit-explicit scheme for the nonlinear interface conditions, Connors et al. [7] have presented a decoupled time stepping method, which is conditionally stable proved by Zhang et al. [25]. Recently, Aggul et al. [2] have developed a predictor-corrector-type method that is an unconditionally stable scheme with second order time accuracy.

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In this paper, we study the following governing equations of a fluid-fluid interaction model [9, 26]. Let a bounded domain $\Omega \subset \mathbb{R}^2$ consist of two sub-domains Ω_1 and Ω_2 coupled across their shared interface I, for times $t \in [0, T]$. For $i = 1, 2$, given the kinematic viscosities $\nu_i > 0$, the friction coefficients $\kappa > 0$, the body forces $f_i : [0, T] \to H^1(\Omega_i)^2$, and initial values $u_{i,0} \in H^1(\Omega_i)^2$, find the fluid velocities $u_i : [0, T] \times \Omega_i \to \mathbb{R}^2$ and pressures $p_i : [0, T] \times \Omega_i \to \mathbb{R}$ satisfying (for $t \in (0, T]$)

$$
\begin{aligned}
& u_{i,t} - \nu_i \Delta u_i + u_i \cdot \nabla u_i + \nabla p_i = f_i && \text{in } \ \Omega_i, \\
& - \nu_i n_i \cdot \nabla u_i \cdot \tau = \kappa (u_i - u_j) \cdot \tau && \text{on } I, \ \text{for } i, j = 1, 2, \ \text{and } i \neq j, \\
& u_i \cdot n_i = 0 && \text{on } I, \\
& \nabla \cdot u_i = 0 && \text{in } \ \Omega_i, \\
& u_i(0, x) = u_{i,0}(x) && \text{in } \ \Omega_i, \\
& u_i = 0 && \text{on } \Gamma_i := \partial \Omega_i \backslash I.
\end{aligned} \tag{1.1}
$$

The vectors n_i are the unit normals on $\partial \Omega_i$, and τ is any vector on I such that $\tau \cdot n_i = 0$. Note that the linear interface conditions are considered on the interface I, which have been studied in past score years. Lions et al. [22] and Friedlander and Serre [9] have proved the existence, uniqueness and regularity of the solution of the problem (1.1). Recently, Zhang et al. [26] have proved that the error estimates of a decoupled scheme for the velocities in H^1 norm and pressures in L^2 norm are $\Delta t^{\frac{7}{8}} + h$ and $\Delta t^{\frac{3}{4}} + h$, respectively. However, the decoupled scheme is conditionally convergent with $\Delta t \leq c h^{\frac{1}{2}}$. Besides, for the same interface condition as problem (1.1), Connors et al. [6] have proposed a partitioned time stepping method for a parabolic two-domain problem and analyzed the error estimates.

In this paper, the purpose of the current efforts is to propose and investigate a fully discrete finite element scheme with second order temporal accuracy for the fluid-fluid interaction model (1.1). We discretize the system in time via a combination of second order backward differentiation formula (BDF) for the temporal terms, second order Gear's extrapolation approach for the interface terms and extrapolated treatments in linearization for the nonlinear terms. The coupling terms in the interface conditions are treated explicitly in our scheme so that only two decoupled Navier-Stokes equations are solved at each time step.

The rest of the paper is arranged as follows: In the next section, we introduce some mathematical preliminaries and provide the corresponding variational form for the problem (1.1). In Section 3, we propose a fully discrete finite element scheme for the fluid-fluid interaction model. Besides, the unconditional stability of the presented scheme is proven. Then in Section 4, we derive and prove the error estimates for the considered scheme. In Section 5, some numerical experiments are implemented to verify the theoretical results and efficiency of the proposed scheme. Consequently, we end our paper by drawing a conclusion in the last section.

2. Notation and Preliminaries

In this section, we describe some necessary definitions and inequalities, which will be frequently applied to the following sections. We introduce the usual $L^2(\Omega_i)$ norm and its inner product by $\|\cdot\|_0$ and $(\cdot, \cdot)_{\Omega_i}$, respectively. The $L^p(\Omega_i)$ norms and the Sobolev $W_p^m(\Omega_i)$ norms are denoted by $\|\cdot\|_{L^p(\Omega_i)}$ and $\|\cdot\|_{W_p^m(\Omega_i)}$ for $m \in \mathbb{N}^+$, $1 \leq p \leq \infty$. In particular, $H^m(\Omega_i)$ is used to represent the Sobolev space $W_2^m(\Omega_i)$ and $\|\cdot\|_m$ denotes the norm in $H^m(\Omega_i)$. For

X_i being a normed function space in Ω_i, $L^p(0,T;X_i)$ is the space of all functions defined on $[0,T]\times\Omega_i$ for which the norm

$$\|u\|_{L^p(0,T;X_i)}=\left(\int_0^T\|u\|_{X_i}^p\mathrm{dt}\right)^{\frac{1}{p}},p\in[1,\infty)$$

is finite. For $p=\infty$, the usual modification is used in the definition of this space.

For the mathematical setting of the fluid-fluid interaction model (1.1), we introduce the following function spaces:

$$X_i=\{v_i\in H^1(\Omega_i)^2;v_i|_{\Gamma_i}=0;v_i\cdot n_i=0\text{ on }I\},\quad M_i=\{q_i\in L^2(\Omega_i);(q_i,1)=0\ \}.$$

For f_i an element in the dual space of X_i, its norm is defined by

$$\|f_i\|_{-1}=\sup_{v_i\in X_i}\frac{|(f_i,v_i)|}{\|\nabla v_i\|_0}.$$

In particular, all of the above notations are adaptable to the sub-domain Ω_j.

Based on the above definitions of the function spaces, the corresponding variational formulation of the problem (1.1) is given as follows: Find $(u_i,p_i)\in L^2(0,T;X_i)\times L^2(0,T;M_i)$ for all $(v_i,q_i)\in X_i\times M_i$, $i,j=1,2$, $i\neq j$ such that

$$(u_{i,t},v_i)+a(u_i,v_i)-d(v_i,p_i)+d(u_i,q_i)+b(u_i,u_i,v_i)+\int_I\kappa(u_i-u_j)v_i\mathrm{ds}=(f_i,v_i),\qquad(2.1)$$

where $(u_{i,t},v_i)=\int_{\Omega_i}\frac{\partial u_i}{\partial t}v_i\mathrm{d}\Omega_i$, the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are defined on $X_i\times X_i$ and $X_i\times M_i$, respectively, by

$$\begin{aligned}a(u_i,v_i)&=\nu_i(\nabla u_i,\nabla v_i), && u_i,v_i\in X_i,\\ d(v_i,q_i)&=-(v_i,\nabla q_i)=(\nabla\cdot v_i,q_i), && v_i\in X_i,\ q_i\in M_i,\end{aligned}$$

and the trilinear term $b(\cdot,\cdot,\cdot)$ are defined on $X_i\times X_i\times X_i$ by

$$\begin{aligned}b(u_i,v_i,w_i)&=((u_i\cdot\nabla)v_i,w_i)+\tfrac{1}{2}((\nabla\cdot u_i)v_i,w_i)\\&=\tfrac{1}{2}((u_i\cdot\nabla)v_i,w_i)-\tfrac{1}{2}((u_i\cdot\nabla)w_i,v_i),\qquad\forall u_i,v_i,w_i\in X_i.\end{aligned}$$

Some properties of this skew-symmetric trilinear term will be used in the next analysis and given in the following lemma.

Lemma 2.1 ([12, 15, 24]). *For $u_i,v_i,w_i\in X_i$, $i=1,2$, we have*

$$\begin{aligned}&b(u_i,v_i,w_i)=-b(u_i,w_i,v_i),\\&|b(u_i,v_i,w_i)|\le c_0\|\nabla u_i\|_0\|\nabla v_i\|_0\|\nabla w_i\|_0.\end{aligned}$$

Besides, if $v_i\in H^2(\Omega_i)^2$, then we have

$$|b(u_i,v_i,w_i)|\le c_1\|u_i\|_0\|v_i\|_2\|\nabla w_i\|_0,$$

where c_0,c_1 are two positive constants depending on Ω_i.

As is known, the discrete Gronwall's inequality will play an important rule in convergence's analysis, so we introduce it in the following lemma.

Lemma 2.2 ([14]). *Let C, k and a_n, b_n, d_n, for integers $n_1 \le n \le m$, be nonnegative numbers such that*

$$a_m + k\sum_{n=n_1}^{m} b_n \le k\sum_{n=n_1}^{s} a_n d_n + C, \qquad \forall m \ge n_1.$$

If $s = m-1$, then

$$a_m + k\sum_{n=n_1}^{m} b_n \le \exp\left(k\sum_{n=n_1}^{m-1} d_n\right)C, \qquad \forall m \ge n_1.$$

Finally, we recall the Poincaré inequality and the trace inequality, which are useful in the following analysis. There exist some positive constants C_p and C_{tr}, which depend on Ω_i, such that [1,10]

$$\|v_i\|_0 \le C_p\|\nabla v_i\|_0, \quad \|v_i\|_{L^2(I)} \le C_{tr}\|v_i\|_0^{\frac{1}{2}}\|\nabla v_i\|_0^{\frac{1}{2}}. \tag{2.2}$$

3. A Fully Discrete Scheme with Second Order Temporal Accuracy

From now on, given $N > 0$, let $\{t_n\}_{n=0}^{N}$ be a uniform partition of $[0,T]$ with time step $\Delta t = T/N$, and $t_n = n\Delta t$. Next, for $i = 1,\ 2$, let π_i^h be a triangulation of Ω_i and $\pi^h = \pi_1^h \cup \pi_2^h$. The mesh size h is the largest diameter of the element in π^h. Accordingly, we consider the finite element spaces on π_i^h by $X_i^h \subset X_i$ for velocity and $M_i^h \subset M_i$ for pressure. The finite element discrete subspaces are given as follows:

$$\begin{aligned}
X_i^h &= \{v_{i,h} \in C^0(\Omega_i)^2 \cap X_i : v_{i,h}|_{K_i} \in P_2(K_i)^2, \forall K_i \in \pi_i^h\},\\
M_i^h &= \{q_{i,h} \in C^0(\Omega_i) \cap M_i : q_{i,h}|_{K_i} \in P_1(K_i), \forall K_i \in \pi_i^h\},
\end{aligned}$$

where $P_l(K_i)$ $(l = 1,2)$ denote the space of the polynomials on K_i of degree at most l for every $K_i \in \pi_i^h$. It is well known that the finite element spaces M_i^h and X_i^h satisfy the discrete Ladyzenskaja-Babuška-Brezzi (LBB) condition

$$\sup_{0\neq v_{i,h}\in X_i^h} \frac{|d(v_{i,h}, q_{i,h})|}{\|\nabla v_{i,h}\|_0} \ge \beta\|q_{i,h}\|_0, \qquad \forall q_{i,h} \in M_i^h,$$

where $\beta > 0$ is only dependent on Ω_i. Furthermore, $(u_{i,h}^n, p_{i,h}^n)$ will denote the fully discrete approximation to the solution (u_i, p_i) of the problem (1.1) at $t = t_n$. Besides, we set $f_i^n = f_i(t_n)$.

Now, we construct a fully discrete finite element scheme involving a second order BDF scheme and mixed finite element method as temporal-spatial discretization, where the interface terms on I are treated via a second order explicit Gear's extrapolation approach and the nonlinear terms are dealt with by the extrapolated linearization. Hence, we propose the fully discrete scheme as follows:

Given $u_{1,h}^{n-1}, u_{1,h}^n \in X_1^h$ and $u_{2,h}^{n-1}, u_{2,h}^n \in X_2^h$, for $1 \le n \le N-1$, find $(u_{1,h}^{n+1}, p_{1,h}^{n+1}) \in X_1^h \times M_1^h$ satisfying

$$\begin{aligned}
&\left(\frac{3u_{1,h}^{n+1} - 4u_{1,h}^n + u_{1,h}^{n-1}}{2\Delta t}, v_{1,h}\right) + a(u_{1,h}^{n+1}, v_{1,h}) + b(2u_{1,h}^n - u_{1,h}^{n-1}, u_{1,h}^{n+1}, v_{1,h})\\
&\quad - d(v_{1,h}, p_{1,h}^{n+1}) + d(u_{1,h}^{n+1}, q_{1,h}) + 2\int_I \kappa(u_{1,h}^n - u_{2,h}^n)v_{1,h}\mathrm{d}s\\
&\quad - \int_I \kappa(u_{1,h}^{n-1} - u_{2,h}^{n-1})v_{1,h}\mathrm{d}s = (f_1^{n+1}, v_{1,h}),
\end{aligned} \tag{3.1}$$

for all $(v_{1,h},q_{1,h})\in X_1^h\times M_1^h$. Besides, given $u_{2,h}^{n-1},u_{2,h}^n\in X_2^h$ and $u_{1,h}^{n-1},u_{1,h}^n\in X_1^h$, for $1\le n\le N-1$, find $(u_{2,h}^{n+1},p_{2,h}^{n+1})\in X_2^h\times M_2^h$ satisfying

$$\begin{aligned}
&\left(\frac{3u_{2,h}^{n+1}-4u_{2,h}^n+u_{2,h}^{n-1}}{2\Delta t},v_{2,h}\right)+a(u_{2,h}^{n+1},v_{2,h})+b(2u_{2,h}^n-u_{2,h}^{n-1},u_{2,h}^{n+1},v_{2,h})\\
&\quad-d(v_{2,h},p_{2,h}^{n+1})+d(u_{2,h}^{n+1},q_{2,h})+2\int_I\kappa(u_{2,h}^n-u_{1,h}^n)v_{2,h}\mathrm{ds}\\
&\quad-\int_I\kappa(u_{2,h}^{n-1}-u_{1,h}^{n-1})v_{2,h}\mathrm{ds}=(f_2^{n+1},v_{2,h}),
\end{aligned}\tag{3.2}$$

for all $(v_{2,h},q_{2,h})\in X_2^h\times M_2^h$.

Remark 3.1. Note that the schemes (3.1) and (3.2) require some initial values $u_{i,h}^1$ and $u_{i,h}^0$ $(i=1,2)$. For the sake of simplification, we set $u_{i,h}^1=R_iu_i(t_1)$ (see Section 4 for the definition of the projection R_i). In fact, it can obtained by the calculation of the first order scheme in [26]. Besides, we choose $u_{i,h}^0=R_iu_i(t_0)$.

In the following part of this section, we will analyze the stability of the schemes (3.1) and (3.2). We will prove that the schemes (3.1) and (3.2) are unconditionally stable in Theorem 3.1. Besides, the long-time stability of the schemes (3.1) and (3.2) will be stated in Theorem 3.2.

Theorem 3.1. *Let $f_i\in L^\infty\left(0,T;H^{-1}(\Omega_i)^2\right)$, $i=1,2$. Then the schemes* (3.1) *and* (3.2) *are unconditionally stable.*

Proof. Setting $(v_{1,h},q_{1,h})=4\Delta t(u_{1,h}^{n+1},p_{1,h}^{n+1})$ in (3.1) and $(v_{2,h},q_{2,h})=4\Delta t(u_{2,h}^{n+1},p_{2,h}^{n+1})$ in (3.2), using the equality $(2a,3a-4b+c)=|a|^2+|2a-b|^2-|b|^2-|2b-c|^2+|a-2b+c|^2$ and Lemma 2.1, and summing the ensuing equations yield

$$\begin{aligned}
&\|u_{1,h}^{n+1}\|_0^2+\|2u_{1,h}^{n+1}-u_{1,h}^n\|_0^2-\|u_{1,h}^n\|_0^2-\|2u_{1,h}^n-u_{1,h}^{n-1}\|_0^2+\|u_{1,h}^{n+1}-2u_{1,h}^n+u_{1,h}^{n-1}\|_0^2\\
&\quad+\|u_{2,h}^{n+1}\|_0^2+\|2u_{2,h}^{n+1}-u_{2,h}^n\|_0^2-\|u_{2,h}^n\|_0^2-\|2u_{2,h}^n-u_{2,h}^{n-1}\|_0^2+\|u_{2,h}^{n+1}-2u_{2,h}^n+u_{2,h}^{n-1}\|_0^2\\
&\quad+4\Delta t\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2+4\Delta t\nu_2\|\nabla u_{2,h}^{n+1}\|_0^2+8\Delta t\int_I\kappa(u_{1,h}^n-u_{2,h}^n)u_{1,h}^{n+1}\mathrm{ds}\\
&\quad-4\Delta t\int_I\kappa(u_{1,h}^{n-1}-u_{2,h}^{n-1})u_{1,h}^{n+1}\mathrm{ds}+8\Delta t\int_I\kappa(u_{2,h}^n-u_{1,h}^n)u_{2,h}^{n+1}\mathrm{ds}\\
&\quad-4\Delta t\int_I\kappa(u_{2,h}^{n-1}-u_{1,h}^{n-1})u_{2,h}^{n+1}\mathrm{ds}=4\Delta t(f_1^{n+1},u_{1,h}^{n+1})+4\Delta t(f_2^{n+1},u_{2,h}^{n+1}).
\end{aligned}\tag{3.3}$$

Next, concerning the interface terms of (3.3), applying (2.2), the Hölder inequality and the Young's inequality, there holds

$$\begin{aligned}
&2\int_I\kappa(u_{1,h}^n-u_{2,h}^n)u_{1,h}^{n+1}\mathrm{ds}-\int_I\kappa(u_{1,h}^{n-1}-u_{2,h}^{n-1})u_{1,h}^{n+1}\mathrm{ds}\\
&\le\kappa\left(\|2u_{1,h}^n-u_{1,h}^{n-1}\|_{L^2(I)}+\|2u_{2,h}^n-u_{2,h}^{n-1}\|_{L^2(I)}\right)\|u_{1,h}^{n+1}\|_{L^2(I)}\\
&\le C_{tr}^2C_P^{\frac12}\kappa\|2u_{1,h}^n-u_{1,h}^{n-1}\|_0^{\frac12}\|\nabla(2u_{1,h}^n-u_{1,h}^{n-1})\|_0^{\frac12}\|\nabla u_{1,h}^{n+1}\|_0\\
&\quad+C_{tr}^2C_P^{\frac12}\kappa\|2u_{2,h}^n-u_{2,h}^{n-1}\|_0^{\frac12}\|\nabla(2u_{2,h}^n-u_{2,h}^{n-1})\|_0^{\frac12}\|\nabla u_{1,h}^{n+1}\|_0\\
&\le\frac32C_{tr}^4C_P\kappa^2\nu_1^{-1}\|2u_{1,h}^n-u_{1,h}^{n-1}\|_0\|\nabla(2u_{1,h}^n-u_{1,h}^{n-1})\|_0+\frac{\nu_1}{3}\|\nabla u_{1,h}^{n+1}\|_0^2\\
&\quad+\frac32C_{tr}^4C_P\kappa^2\nu_1^{-1}\|2u_{2,h}^n-u_{2,h}^{n-1}\|_0\|\nabla(2u_{2,h}^n-u_{2,h}^{n-1})\|_0
\end{aligned}$$

$$
\begin{aligned}
&\le 54C_{tr}^8 C_p^2 \kappa^4 \nu_1^{-3} \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2 + 54C_{tr}^8 C_p^2 \kappa^4 \nu_1^{-2}\nu_2^{-1} \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2 \\
&\quad + \frac{\nu_1}{96}\|\nabla(2u_{1,h}^n - u_{1,h}^{n-1})\|_0^2 + \frac{\nu_2}{96}\|\nabla(2u_{2,h}^n - u_{2,h}^{n-1})\|_0^2 + \frac{\nu_1}{3}\|\nabla u_{1,h}^{n+1}\|_0^2 \\
&\le 54C_{tr}^8 C_p^2 \kappa^4 \nu_1^{-3} \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2 + 54C_{tr}^8 C_p^2 \kappa^4 \nu_1^{-2}\nu_2^{-1} \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2 + \frac{\nu_1}{12}\|\nabla u_{1,h}^n\|_0^2 \\
&\quad + \frac{\nu_1}{48}\|\nabla u_{1,h}^{n-1}\|_0^2 + \frac{\nu_2}{12}\|\nabla u_{2,h}^n\|_0^2 + \frac{\nu_2}{48}\|\nabla u_{2,h}^{n-1}\|_0^2 + \frac{\nu_1}{3}\|\nabla u_{1,h}^{n+1}\|_0^2. \qquad (3.4)
\end{aligned}
$$

Arguing in exactly the same way as (3.4), we get

$$
\begin{aligned}
&2\int_I \kappa(u_{2,h}^n - u_{1,h}^n)u_{2,h}^{n+1}ds - \int_I \kappa(u_{2,h}^{n-1} - u_{1,h}^{n-1})u_{2,h}^{n+1}\mathrm{ds} \\
&\le 54C_{tr}^8 C_p^2 \kappa^4 \nu_2^{-3} \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2 + 54C_{tr}^8 C_p^2 \kappa^4 \nu_2^{-2}\nu_1^{-1} \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2 \\
&\quad + \frac{\nu_2}{12}\|\nabla u_{2,h}^n\|_0^2 + \frac{\nu_2}{48}\|\nabla u_{2,h}^{n-1}\|_0^2 + \frac{\nu_1}{12}\|\nabla u_{1,h}^n\|_0^2 + \frac{\nu_1}{48}\|\nabla u_{1,h}^{n-1}\|_0^2 + \frac{\nu_2}{3}\|\nabla u_{2,h}^{n+1}\|_0^2. \qquad (3.5)
\end{aligned}
$$

Besides, the right-hand sides (RHSs) of (3.3) are bounded

$$
\begin{aligned}
&4\Delta t(f_1^{n+1}, u_{1,h}^{n+1}) + 4\Delta t(f_2^{n+1}, u_{2,h}^{n+1}) \\
&\le \Delta t \nu_1 \|\nabla u_{1,h}^{n+1}\|_0^2 + \Delta t \nu_2 \|\nabla u_{2,h}^{n+1}\|_0^2 + 4\Delta t \nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + 4\Delta t \nu_2^{-1}\|f_2^{n+1}\|_{-1}^2. \qquad (3.6)
\end{aligned}
$$

Moreover, set $C^* = C_{tr}^8 C_p^2 \kappa^4$ and $\nu^* = \max\{\nu_1^{-3}, \nu_2^{-3}, \nu_1^{-2}\nu_2^{-1}, \nu_1^{-1}\nu_2^{-2}\}$. Combining (3.4)-(3.6) with (3.3) yields

$$
\begin{aligned}
&\|u_{1,h}^{n+1}\|_0^2 + \|2u_{1,h}^{n+1} - u_{1,h}^n\|_0^2 + \|u_{2,h}^{n+1}\|_0^2 + \|2u_{2,h}^{n+1} - u_{2,h}^n\|_0^2 + \|u_{1,h}^{n+1} - 2u_{1,h}^n + u_{1,h}^{n-1}\|_0^2 \\
&\quad + \|u_{2,h}^{n+1} - 2u_{2,h}^n + u_{2,h}^{n-1}\|_0^2 + \Delta t\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2 + \Delta t \nu_2 \|\nabla u_{2,h}^{n+1}\|_0^2 \\
&\le 4\Delta t \nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + 4\Delta t\nu_2^{-1}\|f_2^{n+1}\|_{-1}^2 + (1 + 432C^*\nu^*\Delta t)(\|u_{1,h}^n\|_0^2 + \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2) \\
&\quad + (1 + 432C^*\nu^*\Delta t)(\|u_{2,h}^n\|_0^2 + \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2) + \frac{2\nu_1}{3}\Delta t\|\nabla u_{1,h}^n\|_0^2 + \frac{\nu_1}{6}\Delta t\|\nabla u_{1,h}^{n-1}\|_0^2 \\
&\quad + \frac{2\nu_2}{3}\Delta t\|\nabla u_{2,h}^n\|_0^2 + \frac{\nu_2}{6}\Delta t \|\nabla u_{2,h}^{n-1}\|_0^2. \qquad (3.7)
\end{aligned}
$$

Next, add $\pm\frac{\nu_1}{3}\Delta t\|\nabla u_{1,h}^n\|_0^2$ and $\pm\frac{\nu_2}{3}\Delta t\|\nabla u_{2,h}^n\|_0^2$ to (3.7), which implies that

$$
\begin{aligned}
&E_1^{n+1} + E_2^{n+1} + \frac{432C^*\nu^*\nu_1\Delta t^2}{1 + 432C^*\nu^*\Delta t}\|\nabla u_{1,h}^{n+1}\|_0^2 + \frac{432C^*\nu^*\nu_1\Delta t^2}{3(1 + 432C^*\nu^*\Delta t)}\|\nabla u_{1,h}^n\|_0^2 \\
&\quad + \frac{432C^*\nu^*\nu_2\Delta t^2}{1 + 432C^*\nu^*\Delta t}\|\nabla u_{2,h}^{n+1}\|_0^2 + \frac{432C^*\nu^*\nu_2\Delta t^2}{3(1 + 432C^*\nu^*\Delta t)}\|\nabla u_{2,h}^n\|_0^2 \\
&\quad + \|u_{1,h}^{n+1} - 2u_{1,h}^n + u_{1,h}^{n-1}\|_0^2 + \|u_{2,h}^{n+1} - 2u_{2,h}^n + u_{2,h}^{n-1}\|_0^2 \\
&\le 4\Delta t(\nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + \nu_2^{-1}\|f_2^{n+1}\|_{-1}^2) + (1 + 432C^*\nu^*\Delta t)(E_1^n + E_2^n), \qquad (3.8)
\end{aligned}
$$

where

$$
E_i^{n+1} = \|u_{i,h}^{n+1}\|_0^2 + \|2u_{i,h}^{n+1} - u_{i,h}^n\|_0^2 + \frac{\nu_i\Delta t}{1 + 432C^*\nu^*\Delta t}\|\nabla u_{i,h}^{n+1}\|_0^2 + \frac{\nu_i \Delta t}{3(1 + 432C^*\nu^*\Delta t)}\|\nabla u_{i,h}^n\|_0^2,
$$

for i=1,2. Discarding all terms on the left-hand side of (3.8), all of which are positive, except for E_1^{n+1} and E_2^{n+1}, we arrive at

$$
E_1^{n+1} + E_2^{n+1} \le 4\Delta t(\nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + \nu_2^{-1}\|f_2^{n+1}\|_{-1}^2) + (1 + 432C^*\nu^*\Delta t)(E_1^n + E_2^n).
$$

Then, by recursion

$$E_1^n + E_2^n \le \exp(432C^*\nu^*T)(E_1^1 + E_2^1) + \frac{\exp(432C^*\nu^*T)}{108C^*\nu^*}\sum_{i=1}^{2}\left(\nu_i^{-1}\max_n \|f_i^{n+1}\|_{-1}^2\right). \quad (3.9)$$

Hence, the unconditional stability of the considered schemes is proved. □

In fact, according to (3.9), the proof of the unconditionally stable bound of the considered schemes for the velocity fields results in the dependence of the bounds on the final time T of the form $O(\exp(T))$. Although being time step independent, such bound is not very practical for longer final time problem.

Now, we consider the long time stability over $0 \le t < \infty$ and show that the considered schemes are uniformly bounded for all time, without any time step restriction.

Theorem 3.2. *Assume that* $f_i \in L^2\left(0,T;H^{-1}(\Omega_i)^2\right)$, $i=1,2$, *and the viscosity coefficients hold for the condition* $160C_* \le \nu_*^2$, *where* $C_* = C_{tr}^4 C_p^2 \kappa^2$ *and* $\nu_* = \min\{\nu_1,\nu_2\}$, *then the considered schemes* (3.1) *and* (3.2) *for problem* (1.1) *are uniformly bounded on* $(0,T]$.

Proof. Note that the interface terms of (3.3) can be bounded by

$$\begin{aligned}
&\int_I \kappa\left(2(u_{1,h}^n - u_{2,h}^n) - (u_{1,h}^{n-1} - u_{2,h}^{n-1})\right) u_{1,h}^{n+1}\mathrm{ds} \qquad (3.10)\\
&\le \kappa\left(2\|u_{1,h}^n - u_{2,h}^n\|_{L^2(I)} + \|u_{1,h}^{n-1} - u_{2,h}^{n-1}\|_{L^2(I)}\right)\|u_{1,h}^{n+1}\|_{L^2(I)}\\
&\le \kappa C_{tr}^2 C_p\left(2\|\nabla(u_{1,h}^n - u_{2,h}^n)\|_0 + \|\nabla(u_{1,h}^{n-1} - u_{2,h}^{n-1})\|_0\right)\|\nabla u_{1,h}^{n+1}\|_0\\
&\le \kappa^2 C_{tr}^4 C_p^2 \nu_1^{-1}\left(8\|\nabla(u_{1,h}^n - u_{2,h}^n)\|_0^2 + 2\|\nabla(u_{1,h}^{n-1} - u_{2,h}^{n-1})\|_0^2\right) + \frac{\nu_1}{4}\|\nabla u_{1,h}^{n+1}\|_0^2\\
&\le \kappa^2 C_{tr}^4 C_p^2 \nu_1^{-1}\left(16\|\nabla u_{1,h}^n\|_0^2 + 16\|\nabla u_{2,h}^n\|_0^2 + 4\|\nabla u_{1,h}^{n-1}\|_0^2 + 4\|\nabla u_{2,h}^{n-1}\|_0^2\right) + \frac{\nu_1}{4}\|\nabla u_{1,h}^{n+1}\|_0^2,
\end{aligned}$$

as well as

$$\begin{aligned}
&\int_I \kappa\left(2(u_{2,h}^n - u_{1,h}^n) - (u_{2,h}^{n-1} - u_{1,h}^{n-1})\right) u_{2,h}^{n+1}\mathrm{ds} \qquad (3.11)\\
&\le \kappa^2 C_{tr}^4 C_p^2 \nu_2^{-1}\left(16\|\nabla u_{2,h}^n\|_0^2 + 16\|\nabla u_{1,h}^n\|_0^2 + 4\|\nabla u_{2,h}^{n-1}\|_0^2 + 4\|\nabla u_{1,h}^{n-1}\|_0^2\right) + \frac{\nu_2}{4}\|\nabla u_{2,h}^{n+1}\|_0^2.
\end{aligned}$$

Next, multiplying (3.10) and (3.11) by $4\Delta t$ and combining (3.6) and (3.3) with the ensuing inequalities, we get

$$\begin{aligned}
&\|u_{1,h}^{n+1}\|_0^2 + \|2u_{1,h}^{n+1} - u_{1,h}^n\|_0^2 - \|u_{1,h}^n\|_0^2 - \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2 + 2\Delta t\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2\\
&+ \|u_{2,h}^{n+1}\|_0^2 + \|2u_{2,h}^{n+1} - u_{2,h}^n\|_0^2 - \|u_{2,h}^n\|_0^2 - \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2 + 2\Delta t\nu_2\|\nabla u_{2,h}^{n+1}\|_0^2\\
&+ \|u_{1,h}^{n+1} - 2u_{1,h}^n + u_{1,h}^{n-1}\|_0^2 + \|u_{2,h}^{n+1} - 2u_{2,h}^n + u_{2,h}^{n-1}\|_0^2\\
\le\ & 4\Delta t\nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + 4\Delta t\nu_2^{-1}\|f_2^{n+1}\|_{-1}^2 + 64\Delta t\kappa^2 C_{tr}^4 C_p^2\nu_1^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right)\\
&+ 16\Delta t\kappa^2 C_{tr}^4 C_p^2\nu_1^{-1}\left(\|\nabla u_{1,h}^{n-1}\|_0^2 + \|\nabla u_{2,h}^{n-1}\|_0^2\right) + 64\Delta t\kappa^2 C_{tr}^4 C_p^2\nu_2^{-1}\left(\|\nabla u_{2,h}^n\|_0^2 + \|\nabla u_{1,h}^n\|_0^2\right)\\
&+ 16\Delta t\kappa^2 C_{tr}^4 C_p^2\nu_2^{-1}\left(\|\nabla u_{2,h}^{n-1}\|_0^2 + \|\nabla u_{1,h}^{n-1}\|_0^2\right). \qquad (3.12)
\end{aligned}$$

Note that $\nu_* = \min\{\nu_1, \nu_2\}$ and $C_* = C_{tr}^4 C_p^2 \kappa^2$. Thus, it is easy to get

$$\begin{aligned}
&64\Delta t\kappa^2 C_{tr}^4 C_p^2 \nu_1^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right) + 64\Delta t\kappa^2 C_{tr}^4 C_p^2 \nu_2^{-1}\left(\|\nabla u_{2,h}^n\|_0^2 + \|\nabla u_{1,h}^n\|_0^2\right)\\
&\quad\le 128\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right),\\
&16\Delta t\kappa^2 C_{tr}^4 C_p^2 \nu_1^{-1}\left(\|\nabla u_{1,h}^{n-1}\|_0^2 + \|\nabla u_{2,h}^{n-1}\|_0^2\right) + 16\Delta t\kappa^2 C_{tr}^4 C_p^2 \nu_2^{-1}\left(\|\nabla u_{2,h}^{n-1}\|_0^2 + \|\nabla u_{1,h}^{n-1}\|_0^2\right)\\
&\quad\le 32\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^{n-1}\|_0^2 + \|\nabla u_{2,h}^{n-1}\|_0^2\right),
\end{aligned}$$

which together with (3.12) lead to

$$\begin{aligned}
&\|u_{1,h}^{n+1}\|_0^2 + \|2u_{1,h}^{n+1} - u_{1,h}^n\|_0^2 - \|u_{1,h}^n\|_0^2 - \|2u_{1,h}^n - u_{1,h}^{n-1}\|_0^2 + 2\Delta t\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2\\
&+ \|u_{2,h}^{n+1}\|_0^2 + \|2u_{2,h}^{n+1} - u_{2,h}^n\|_0^2 - \|u_{2,h}^n\|_0^2 - \|2u_{2,h}^n - u_{2,h}^{n-1}\|_0^2 + 2\Delta t\nu_2\|\nabla u_{2,h}^{n+1}\|_0^2\\
&+ \|u_{1,h}^{n+1} - 2u_{1,h}^n + u_{1,h}^{n-1}\|_0^2 + \|u_{2,h}^{n+1} - 2u_{2,h}^n + u_{2,h}^{n-1}\|_0^2\\
&- 128\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right) - 32\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^{n-1}\|_0^2 + \|\nabla u_{2,h}^{n-1}\|_0^2\right)\\
\le\,& 4\Delta t\nu_1^{-1}\|f_1^{n+1}\|_{-1}^2 + 4\Delta t\nu_2^{-1}\|f_2^{n+1}\|_{-1}^2. \qquad (3.13)
\end{aligned}$$

Moreover, assume that the viscosity coefficients hold under the condition $160C_* \le \nu_*^2$, which implies

$$\begin{aligned}
&\Delta t\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2 + \Delta t\nu_2\|\nabla u_{2,h}^{n+1}\|_0^2 - 160\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right)\\
\ge\,& \Delta t\nu_*(\|\nabla u_{1,h}^{n+1}\|_0^2 + \|\nabla u_{2,h}^{n+1}\|_0^2) - 160\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2\right)\\
\ge\,& 160\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^{n+1}\|_0^2 + \|\nabla u_{2,h}^{n+1}\|_0^2 - (\|\nabla u_{1,h}^n\|_0^2 + \|\nabla u_{2,h}^n\|_0^2)\right). \qquad (3.14)
\end{aligned}$$

Finally, combining (3.13) with (3.14) and summing the ensuing inequality with respect to n from 1 to $N-1$, we arrive at

$$\begin{aligned}
&\|u_{1,h}^N\|_0^2 + \|2u_{1,h}^N - u_{1,h}^{N-1}\|_0^2 + \|u_{2,h}^N\|_0^2 + \|2u_{2,h}^N - u_{2,h}^{N-1}\|_0^2 + \Delta t\sum_{n=1}^{N-1}\left(\nu_1\|\nabla u_{1,h}^{n+1}\|_0^2\right.\\
&\left.+ \nu_2\|\nabla u_{2,h}^{n+1}\|_0^2\right) + \sum_{n=1}^{N-1}\left(\|u_{1,h}^{n+1} - 2u_{1,h}^n + u_{1,h}^{n-1}\|_0^2 + \|u_{2,h}^{n+1} - 2u_{2,h}^n + u_{2,h}^{n-1}\|_0^2\right)\\
&+ 160\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^N\|_0^2 + \|\nabla u_{2,h}^N\|_0^2\right) + 32\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^{N-1}\|_0^2 + \|\nabla u_{2,h}^{N-1}\|_0^2\right)\\
\le\,& 4T\sum_{i=1}^2\left(\nu_i^{-1}\max_n\|f_i^{n+1}\|_{-1}^2\right) + \|u_{1,h}^1\|_0^2 + \|2u_{1,h}^1 - u_{1,h}^0\|_0^2 + \|u_{2,h}^1\|_0^2\\
&+ \|2u_{2,h}^1 - u_{2,h}^0\|_0^2 + 160\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^1\|_0^2 + \|\nabla u_{2,h}^1\|_0^2\right)\\
&+ 32\Delta t C_* \nu_*^{-1}\left(\|\nabla u_{1,h}^0\|_0^2 + \|\nabla u_{2,h}^0\|_0^2\right).
\end{aligned}$$

This completes the proof of the theorem. □

4. Error Analysis

In this section, we mainly explore the errors arising from the schemes (3.1) and (3.2) for the model (1.1). In order to establish error equations, set $(v_i, q_i) = (v_{i,h}, q_{i,h})$ in (2.1) with $t = t_{n+1}$

to get

$$
\begin{aligned}
&\left(\frac{3u_i(t_{n+1})-4u_i(t_n)+u_i(t_{n-1})}{2\Delta t}, v_{i,h}\right) + a(u_i(t_{n+1}), v_{i,h}) - d(v_{i,h}, p_i(t_{n+1}))\\
&\quad + d(u_i(t_{n+1}), q_{i,h}) + b(u_i(t_{n+1}), u_i(t_{n+1}), v_{i,h}) + \int_I \kappa\left(u_i(t_{n+1}) - u_j(t_{n+1})\right) v_{i,h}\mathrm{ds}\\
&= (f_i^{n+1}, v_{i,h}) + (\mathcal{E}_i^{n+1}, v_{i,h}),
\end{aligned}
\tag{4.1}
$$

where $\mathcal{E}_i^{n+1} = \frac{3u_i(t_{n+1})-4u_i(t_n)+u_i(t_{n-1})}{2\Delta t} - u_{i,t}(t_{n+1})$ is the truncation error. From (4.1) and the fully discrete schemes (3.1) and (3.2), we get the error equations

$$
\begin{aligned}
&\left(\frac{3e_i^{n+1}-4e_i^n+e_i^{n-1}}{2\Delta t}, v_{i,h}\right) + a(e_i^{n+1}, v_{i,h}) - d(v_{i,h}, e_{p,i}^{n+1}) + b(u_i(t_{n+1}), u_i(t_{n+1}), v_{i,h})\\
&- b(2u_{i,h}^n - u_{i,h}^{n-1}, u_{i,h}^{n+1}, v_{i,h}) + d(e_i^{n+1}, q_{i,h}) + \int_I \kappa(u_i(t_{n+1}) - u_j(t_{n+1}))v_{i,h}\mathrm{ds}\\
&- 2\int_I \kappa(u_{i,h}^n - u_{j,h}^n)v_{i,h}\mathrm{ds} + \int_I \kappa(u_{i,h}^{n-1} - u_{j,h}^{n-1})v_{i,h}\mathrm{ds} = (\mathcal{E}_i^{n+1}, v_{i,h}),
\end{aligned}
\tag{4.2}
$$

where $e_i^n = u_i(t_n) - u_{i,h}^n$ and $e_{p,i}^n = p_i(t_n) - p_{i,h}^n$.

Moreover, we recall the Stokes-Stokes projection [11,12,24]: Find $(R_i u_i, T_i p_i) \in (X_i^h, M_i^h)$, $i = 1,2$ such that

$$
\begin{aligned}
&a(u_i - R_i u_i, v_{i,h}) - d(v_{i,h}, p_i - T_i p_i) = 0 && \forall v_{i,h} \in X_i^h,\\
&d(R_i u_i, q_{i,h}) = 0 && \forall q_{i,h} \in M_i^h.
\end{aligned}
\tag{4.3}
$$

Besides, this projection has the following properties [12,13,24]. If $u_i \in H^3(\Omega_i)^2$ and $p_i \in H^2(\Omega_i)$, then we have

$$
\|u_i - R_i u_i\|_0 + h(\|\nabla(u_i - R_i u_i)\|_0 + \|p_i - T_i p_i\|_0) \le Ch^3(\|u_i\|_3 + \|p_i\|_2),
\tag{4.4}
$$

where $C > 0$ is a constant independent of Δt and h.

Furthermore, let us split several errors as $e_i^n = \eta_i^n + \phi_i^n, e_{p,i}^n = \varphi_{p,i}^n + \psi_{p,i}^n$, for $i,j = 1,2$, and $1 \le n \le N$, where $\eta_i^n = u_i(t_n) - R_i u_i(t_n)$, $\phi_i^n = R_i u_i(t_n) - u_{i,h}^n$, $\varphi_{p,i}^n = p_i(t_n) - T_i p_i(t_n)$ and $\psi_{p,i}^n = T_i p_i(t_n) - p_{i,h}^n$. From Remark 3.1, we notice that $\phi_{i,h}^1 = \phi_{i,h}^0 = 0$.

Hereafter, we always assume that the solution of the initial/boundary value problem (1.1) satisfies $u_i \in L^\infty(0,T;H^3(\Omega_i)^2)$, $u_{i,t} \in L^2(0,T;H^2(\Omega_i)^2)$, $u_{i,tt} \in L^2(0,T;H^1(\Omega_i)^2)$, $u_{i,ttt} \in L^2(0,T;H^{-1}(\Omega_i)^2)$ and $p_i \in L^\infty(0,T;H^2(\Omega_i))$.

We now state error estimates for velocities.

Theorem 4.1. *Let $u_i(t_{n+1})$ and $u_{i,h}^{n+1}$ be the exact solutions of the system* (1.1) *at t_{n+1} and the full-discrete approximated solutions of the schemes* (3.1) *and* (3.2) $i = 1,2$, $0 \le n \le N-1$, *respectively. Then, based on the regularity assumptions of the exact solutions, we have*

$$
\sum_{n=1}^{N-1} \Delta t \left(\nu_1 \|\nabla(u_1(t_{n+1}) - u_{1,h}^{n+1})\|_0^2 + \nu_2 \|\nabla(u_2(t_{n+1}) - u_{2,h}^{n+1})\|_0^2\right) \le C(\Delta t^4 + h^4),
$$

where $C > 0$ is a constant independent of Δt and h.

Proof. See Appendix A.1. □

Next, we state and prove error estimates for pressures.

Theorem 4.2. *Let $p_i(t_{n+1})$ and $p_{i,h}^{n+1}$ be the exact solutions of the system* (1.1) *at t_{n+1} and the full-discrete approximated solutions of the schemes* (3.1) *and* (3.2) *$i=1,2$, $0 \le n \le N-1$, respectively. Then, based on the regularity assumptions of the exact solutions, we have*

$$\sum_{n=1}^{N-1} \Delta t \left(\|p_1(t_{n+1}) - p_{1,h}^{n+1}\|_0^2 + \|p_2(t_{n+1}) - p_{2,h}^{n+1}\|_0^2 \right) \le C(\Delta t^4 + h^4),$$

where $C > 0$ is a constant independent of Δt and h.

Proof. Choosing $(v_{i,h}, q_{i,h}) = (e_i^{n+1}, e_{p,i}^{n+1})$ in (4.2), it follows that

$$\begin{aligned}
&\left(\frac{3e_i^{n+1} - 4e_i^n + e_i^{n-1}}{2\Delta t}, e_i^{n+1} \right) + \nu_i \|e_i^{n+1}\|_0^2 + b(u_i(t_{n+1}), u_i(t_{n+1}), e_i^{n+1}) \\
&\quad - b(2u_{i,h}^n - u_{i,h}^{n-1}, u_{i,h}^{n+1}, e_i^{n+1}) + \int_I \kappa(u_i(t_{n+1}) - u_j(t_{n+1})) e_i^{n+1} \mathrm{ds} \\
&\quad - 2\int_I \kappa(u_{i,h}^n - u_{j,h}^n) e_i^{n+1} \mathrm{ds} + \int_I \kappa(u_{i,h}^{n-1} - u_{j,h}^{n-1}) e_i^{n+1} \mathrm{ds} \\
=&(\mathcal{E}_i^{n+1}, e_i^{n+1}),
\end{aligned}$$

which combines Lemma 2.1 and (2.2) to get

$$\begin{aligned}
&\frac{1}{2\Delta t} \|3e_i^{n+1} - 4e_i^n + e_i^{n-1}\|_{-1} \\
\le&\nu_i \|e_i^{n+1}\|_0 + c_0 \|\nabla(2e_i^n - e_i^{n-1})\|_0 \|\nabla u_i(t_{n+1})\|_0 \\
&+ c_0 \|\nabla\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\|_0 \|\nabla u_i(t_{n+1})\|_0 \\
&+ C_{tr} C_p^{\frac{1}{2}} \kappa \|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)} \\
&+ C_{tr} C_p^{\frac{1}{2}} \kappa \|u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\|_{L^2(I)} \\
&+ C_{tr}^2 C_p \kappa \left(\|\nabla(2e_i^n - e_i^{n-1})\|_0 + \|\nabla(2e_j^n - e_j^{n-1})\|_0 \right) + \|\mathcal{E}_i^{n+1}\|_{-1}.
\end{aligned} \tag{4.5}$$

Furthermore, setting $q_{i,h} = 0$ in (4.2) and applying the discrete inf-sup condition yield

$$\begin{aligned}
&\beta \|e_{p,i}^{n+1}\|_0 \\
\le&\frac{1}{2\Delta t} \|3e_i^{n+1} - 4e_i^n + e_i^{n-1}\|_{-1} + \nu_i \|\nabla e_i^{n+1}\|_0 + c_0 \|\nabla(2e_i^n - e_i^{n-1})\|_0 \|\nabla u_i(t_{n+1})\|_0 \\
&+ c_0 \|\nabla\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\|_0 \|\nabla u_i(t_{n+1})\|_0 + \|\mathcal{E}_i^{n+1}\|_{-1} \\
&+ C_{tr} C_p^{\frac{1}{2}} \kappa \|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)} + C_{tr}^2 C_p \kappa \|\nabla(2e_i^n - e_i^{n-1})\|_0 \\
&+ C_{tr} C_p^{\frac{1}{2}} \kappa \|u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\|_{L^2(I)} + C_{tr}^2 C_p \kappa \|\nabla(2e_j^n - e_j^{n-1})\|_0 \\
\le&2\nu_i \|\nabla e_i^{n+1}\|_0 + 2c_0 \|\nabla(2e_i^n - e_i^{n-1})\|_0 \|\nabla u_i(t_{n+1})\|_0 + 2\|\mathcal{E}_i^{n+1}\|_{-1} \\
&+ 2c_0 \|\nabla\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\|_0 \|\nabla u_i(t_{n+1})\|_0 \\
&+ 2C_{tr} C_p^{\frac{1}{2}} \kappa \|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)} + 2C_{tr}^2 C_p \kappa \|\nabla(2e_i^n - e_i^{n-1})\|_0 \\
&+ 2C_{tr} C_p^{\frac{1}{2}} \kappa \|u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\|_{L^2(I)} + 2C_{tr}^2 C_p \kappa \|\nabla(2e_j^n - e_j^{n-1})\|_0,
\end{aligned}$$

where we have used (4.5). Multiplying above equation by Δt and then summing respect to n

from 1 to $N-1$ and $i=1,2$ lead to

$$
\begin{aligned}
\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\|e_{p,i}^{n+1}\|_0^2 \le\ & C\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\Big(\nu_i\|\nabla e_i^{n+1}\|_0^2 + c_0\|\nabla(2e_i^n - e_i^{n-1})\|_0^2\|\nabla u_i(t_{n+1})\|_0^2 \\
& + c_0\|\nabla\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\|_0^2\|\nabla u_i(t_{n+1})\|_0^2 \\
& + C_{tr}C_p^{\frac{1}{2}}\kappa\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)}^2 + \|\mathcal{E}_i^{n+1}\|_{-1}^2 \\
& + C_{tr}C_p^{\frac{1}{2}}\kappa\|u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\|_{L^2(I)}^2 \\
& + C_{tr}^2C_p\kappa\|\nabla(2e_i^n - e_i^{n-1})\|_0^2 + C_{tr}^2C_p\kappa\|\nabla(2e_j^n - e_j^{n-1})\|_0^2\Big) \\
\le\ & C(\Delta t^4 + h^4),
\end{aligned}
$$

where we have used Theorem 4.1. $\square$

5. Numerical Experiments

In this section, some numerical experiments are presented to test the stability and convergence of the schemes (3.1) and (3.2). Besides, we compare the effectiveness of the presented schemes with the first order schemes [26]. Furthermore, by a practical problem (submarine mountain problem), which has been proposed in [23], the performance of the schemes (3.1) and (3.2) is illustrated. Finally, the coast mountain or cliff problem [3] is applied to illustrate the performance of the presented schemes.

For the numerical tests in Subsection 5.1-5.3, we consider the problem (1.1) on the domain $\Omega = \Omega_1\cup\Omega_2$, where $\Omega_1 = [0,1]\times[0,1]$ and $\Omega_2 = [0,1]\times[-1,0]$. Obviously, the $I = (0,1)\times\{0\}$ in the experiment. Then, $n_1 = [0,-1]^T$ and $n_2 = [0,1]^T$ on I.

5.1. Stability

We take $f_{1,1} = f_{1,2} = \cos(x)\sin(y)$, $f_{2,1} = f_{2,2} = \cos(y)\sin(x)$ and initial values for velocity $u_{1,1} = u_{1,2} = u_{2,1} = u_{2,2} = 0$. Moreover, we choose $\kappa = 1$, $\nu_1 = 1$, $\nu_2 = 1$ and denote the energy by $\|u_{1,1}\|_0^2 + \|u_{1,2}\|_0^2 + \|u_{2,1}\|_0^2 + \|u_{2,2}\|_0^2$.

First, we set $\Delta t = h$ and take mesh step $h = \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}$ and $\frac{1}{60}$ subsequently. In Fig. 5.1, it is easy to see that the energy keeps uniformly bounded by a constant with different mesh scale h. Second, we choose $T=3$, $h=\frac{1}{30}$ and set $N = 350, 700, 1400, 2800$. Fig. 5.2 can also

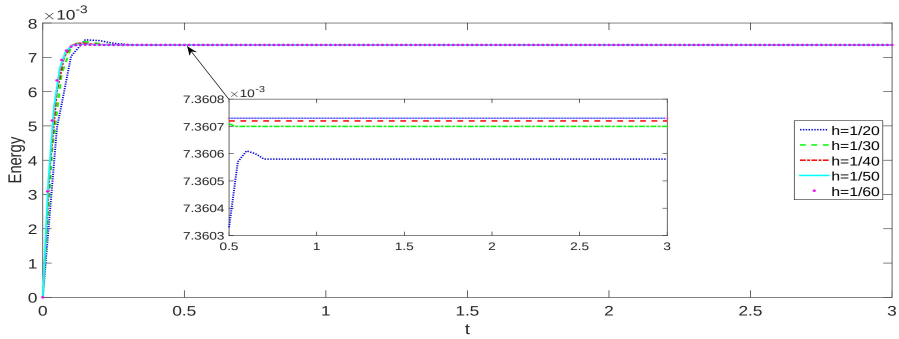


Fig. 5.1. Stability of the presented schemes for the decreasing h.

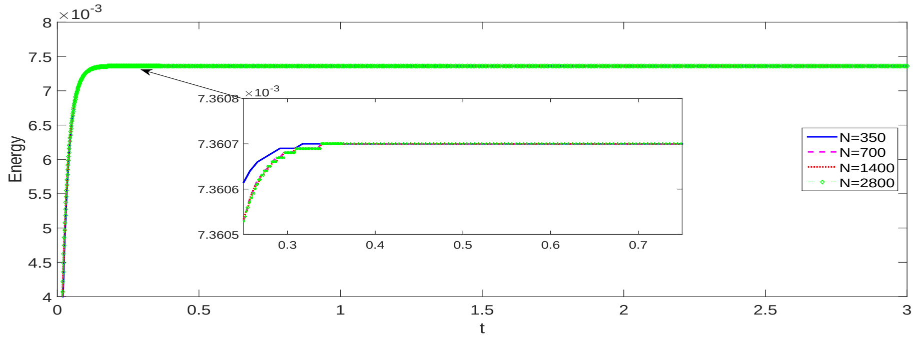


Fig. 5.2. Stability of the presented schemes for the increasing N.

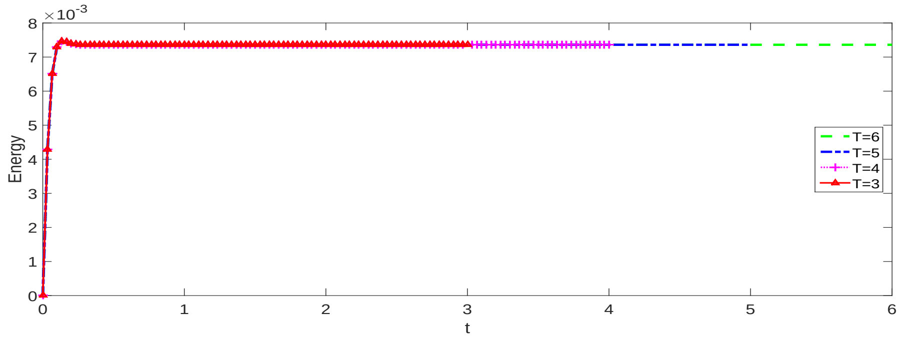


Fig. 5.3. Stability of the presented schemes for the increasing T.

demonstrate that the corresponding energy can be controlled by a constant with the increasing N. Finally, we fix $h = \frac{1}{30}$, $\Delta t = h$, and choose $T = 3, 4, 5, 6$. From Fig. 5.3, we can find that the energy is stable with these final time.

5.2. Convergence

Give the analytic solutions of the problem (1.1) as follows:

$$\begin{aligned}
u_{1,1}(t,x,y) &= -x^2 \exp(-t)(x-1)^2(y-1),\\
u_{1,2}(t,x,y) &= xy \exp(-t)(6x + y - 3xy + 2x^2y - 4x^2 - 2),\\
u_{2,1}(t,x,y) &= (1/\kappa - y + 1)\, x^2 (x-1)^2 \exp(-t),\\
u_{2,2}(t,x,y) &= \left((y - 1 - 1/\kappa)^2 - (1 + 1/\kappa)^2\right)(x^2 - x)(2x-1)\exp(-t),\\
p_1(t,x,y) &= p_2(t,x,y) = \exp(-t)\cos(\pi x)\sin(\pi y).
\end{aligned}$$

The chosen RHSs $f_1 = (f_{1,1}(t,x,y), f_{1,2}(t,x,y))$ and $f_2 = (f_{2,1}(t,x,y), f_{2,2}(t,x,y))$ are obliged to satisfy that (u_1, p_1) and (u_2, p_2) are the solutions of the original problem (1.1), respectively.

Let $Err(u_i)$ and $Err(p_i)$, $i = 1, 2$, denote the errors by

$$Err(u_i) = \left(\Delta t \sum_{n=1}^{N} \|u_i(t_n) - u_{i,h}^n\|_0^2\right)^{\frac{1}{2}}, \qquad Err(p_i) = \left(\Delta t \sum_{n=1}^{N} \|p_i(t_n) - p_{i,h}^n\|_0^2\right)^{\frac{1}{2}}.$$

We now implement the numerical tests to verify the convergent rate with respect to h by the schemes (3.1) and (3.2). Set $\Delta t = 0.01, 0.001$ with the final time $T = 0.1$ and take

Table 5.1: Convergence orders with respect to h with $\Delta t = 0.001$.

$1/h$	$Err(\nabla u_1)$	Rate	$Err(\nabla u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
10	1.54E−3	—	6.09E−3	—	1.26E−3	—	1.64E−3	—
20	3.81E−4	2.02	1.52E−3	2.00	3.11E−4	2.03	3.41E−4	2.28
30	1.69E−4	2.01	6.77E−4	2.00	1.37E−4	2.01	1.43E−4	2.12
40	9.50E−5	2.00	3.81E−4	2.00	7.70E−5	2.00	7.89E−5	2.06
50	6.07E−5	2.00	2.44E−4	2.00	4.96E−5	2.00	5.00E−5	2.04

Table 5.2: Convergence orders with respect to h with $\Delta t = 0.01$.

$1/h$	$Err(\nabla u_1)$	Rate	$Err(\nabla u_2)$	Rate	$Err(p_1)$	Rate	$Err(p_2)$	Rate
10	1.61E−3	—	6.35E−3	—	1.19E−3	—	1.54E-03	—
20	4.01E−4	2.01	1.59E−3	2.00	2.91E−4	2.03	3.21E−4	2.27
30	1.76E−4	2.01	7.06E−4	2.00	1.30E−4	2.01	1.35E−4	2.12
40	9.91E−5	2.00	3.97E−4	2.00	7.28E−5	2.00	7.45E−5	2.06
50	6.34E−5	2.00	2.54E−4	2.00	4.66E−5	2.00	4.73E−5	2.04

Table 5.3: Convergence orders with respect to h with $\Delta t = 0.01$ and $\Delta t = 0.001$.

	$\Delta t = 0.001$				$\Delta t = 0.01$			
$1/h$	$Err(u_1)$	Rate	$Err(u_2)$	Rate	$Err(u_1)$	Rate	$Err(u_2)$	Rate
10	5.93E−5	—	2.88E−4	—	5.93E−5	—	2.88E-04	—
20	7.23E−6	3.04	2.85E−5	3.00	7.23E−6	3.04	3.85E−5	3.00
30	2.13E−6	3.01	8.43E−6	3.00	2.14E−6	3.01	8.43E−6	3.00
40	8.97E−7	3.01	3.56E−6	3.00	9.08E−7	2.98	3.56E−6	3.00
50	4.59E−7	3.00	1.82E−6	3.00	4.77E−7	2.88	1.82E−6	3.00

Table 5.4: Convergence order with respect to Δt.

$1/\Delta t$	$Err(u_1)$	$Rate(u_1)$	$Err(u_2)$	$Rate(u_2)$	$Err(p_1)$	$Rate(p_1)$	$Err(p_2)$	$Rate(p_2)$
10	3.55E−3	—	1.41E−2	—	2.34E−3	—	3.02E−3	—
20	8.51E−4	2.05	3.42E−3	2.04	6.31E−4	1.90	6.81E−4	2.14
30	3.80E−4	2.03	1.50E−3	2.03	2.90E−4	1.94	3.01E−4	2.05
40	2.12E−4	2.02	8.41E−4	2.02	1.61E−4	1.96	1.72E−4	2.01
50	1.32E−4	2.02	5.40E−4	2.01	1.01E−4	1.97	1.10E−4	2.00
60	9.27E−5	2.01	3.72E−4	2.01	7.32E−5	1.97	7.42E−5	2.00

$h = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}$ successively. We display the convergence rates of the schemes (3.1) and (3.2) in Tables 5.1, 5.2 and 5.3 with $\Delta t = 0.001$ and $\Delta t = 0.01$, respectively. From these tables, it is easy to see that the convergence rates are $O(h^2)$ of the H^1-semi norm for the velocities and the L^2-norm for the pressures, and $O(h^3)$ of the L^2-norm for the velocities.

When it comes to the convergence rates with respect to Δt, we set $T = 1$ and $\Delta t = h$. In this test, we take $\Delta t = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}$, and $\frac{1}{60}$ successively. Table 5.4 lists the numerical results obtained by the presented schemes. From Table 5.4, the convergence orders of the velocity and pressure with respect to Δt are approximated to 2.

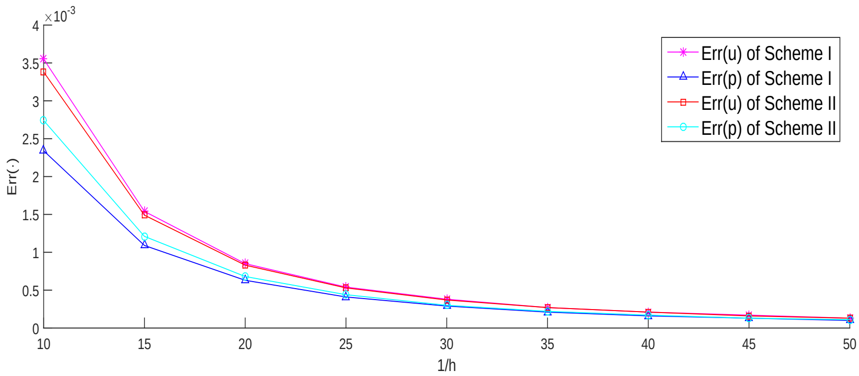
5.3. Comparison with the first order scheme

To illustrate the effectiveness of the presented scheme, we compare the presented schemes with the first order schemes [26] by the numerical example in Subsection 5.2.

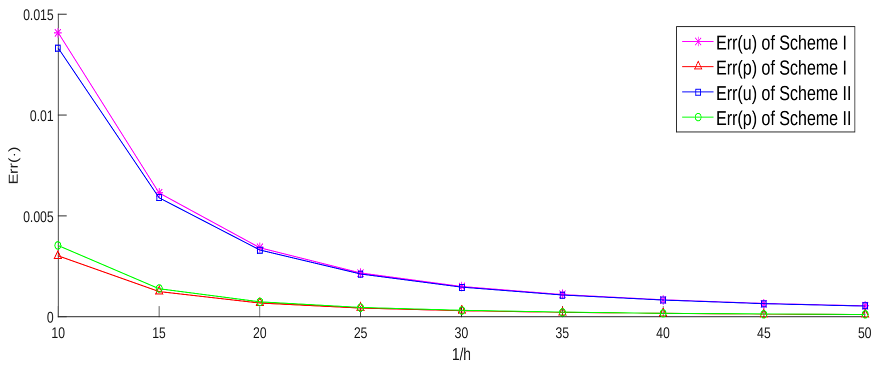
We set $\Delta t = h^2$ in the first order schemes, then the convergence order of the velocity is scale of $O(\Delta t + h^2) = O(h^2)$. When is comes to the presented schemes (3.1) and (3.2), we only choose $\Delta t = h$, which implies the same performance in convergence order aspect. Fig. 5.4 plots that the errors of both schemes with the decreasing h, and Table 5.5 collects the corresponding CPU time. As expected, the presented schemes spend less CPU time than the first order schemes [26] to get the almost the same approximated error, which is not surprising since the presented schemes have second-order temporal accuracy. Hence, its iterative step in time is far less than that of the first order Euler backward one.

5.4. Submarine mountain problem

In this example, we check the presented schemes (3.1) and (3.2) on a practical problem with a submarine mountain problem [23]. We take $\nu_1 = 0.005$ and $\nu_2 = 0.01$ in this example.



(a)



(b)

Fig. 5.4. (a): The $Err(\cdot)$ of Ω_1; (b): The $Err(\cdot)$ of Ω_2. Scheme I means the presented schemes and Scheme II means the first order schemes [26].

Table 5.5: CPU time of the schemes.

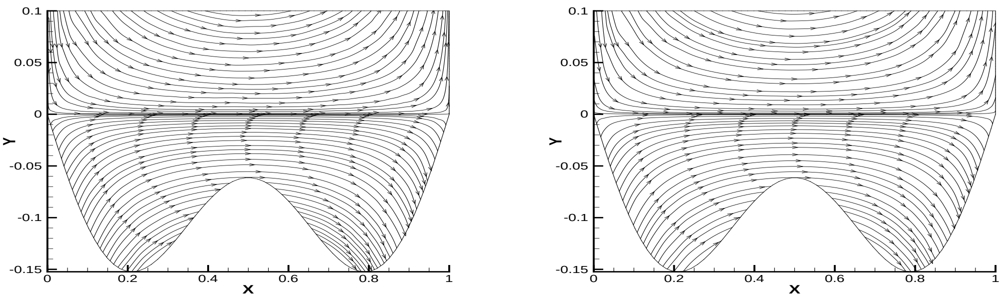
$1/h$	10	15	20	25	30	35	40	45	50
Scheme I	1.31	4.37	10.62	20.77	35.70	56.73	85.07	124.05	168.10
Scheme II	12.56	62.40	199.77	495.13	1038.29	1911.04	3311.86	5417.02	8247.96

Set $\Omega_1 = [0,1]\times[0,0.1]$ and $\Omega_2 = \{(x,y) : \frac{7}{40}\left(\sin(\frac{7}{2}) - (2x-1)\sin(7x-\frac{7}{2})\right) \le y \le 0, 0 \le x \le 1\}$. The RHSs f_1, f_2 are chosen to ensure that

$$
\begin{aligned}
&p_1(t,x,y) = p_2(t,x,y) = \cos(\pi x)\sin(\pi y),\\
&u_{1,1}(t,x,y) = x^2(1-x)^2(0.1-y),\\
&u_{1,2}(t,x,y) = xy(-0.2+y+0.6x-3xy-0.4x^2+2x^2y),\\
&u_{2,1}(t,x,y) = x^2(1-x)^2(0.1+y),\\
&u_{2,2}(t,x,y) = xy(-0.2-y+0.6x+3xy-0.4x^2-2x^2y).
\end{aligned}
$$

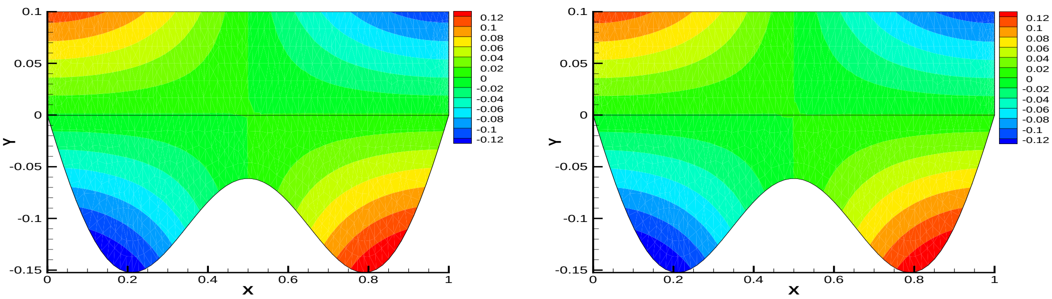
The boundary terms and initial values are chosen by the above exact solutions. We take $\Delta t = h = \frac{1}{64}$, and apply the presented schemes and the first order schemes [26] to get numerical solutions at the final time $T=1$.

Figs. 5.5 and 5.6 present profiles of the velocity streamlines and pressure contours with both schemes at the final time $T=1$ with the coefficient of friction $\kappa = 1$. From these figures, we can see that the both schemes are stable and the oscillations of the velocity streamlines do not appear. What's more, the numerical results of the two schemes are almost consistent. Hence, the proposed method gives good results and can simulate this model very well.



(a) (b)

Fig. 5.5. Velocity streamlines: (a) the presented schemes; (b) the first order schemes [26].



(a) (b)

Fig. 5.6. Pressure contours: (a) the presented schemes; (b) the first order schemes [26].

5.5. Coast mountain or cliff problem

To illustrate the long-time stability of the presented schemes, a coast mountain or cliff problem, which has been considered in [3], is tested. This problem describes a parabolic inflow in the atmosphere passing a coast mountain or cliff before it meets the ocean. The computed domain is consistent with it in [3]. On this domain, homogeneous Dirichlet boundary conditions are imposed at the coast mountain or cliff and on the bottom of the ocean. Meanwhile, the flow in the atmosphere is driven by a parabolic inflow profile with maximum inlet 1 and "do-nothing" conditions are imposed for the other boundaries.

In Fig. 5.7, we present profiles for the numerical velocity at different final time with ν_1=0.005, ν_2=0.05, κ=0.001, $h=\frac{1}{10}$ and $\tau=\frac{1}{5}$. From this figure, we can see that the presented schemes are stable and the unphysical oscillations do not appear. Besides, the numerical results of the presented method agreement with those obtained in [3].

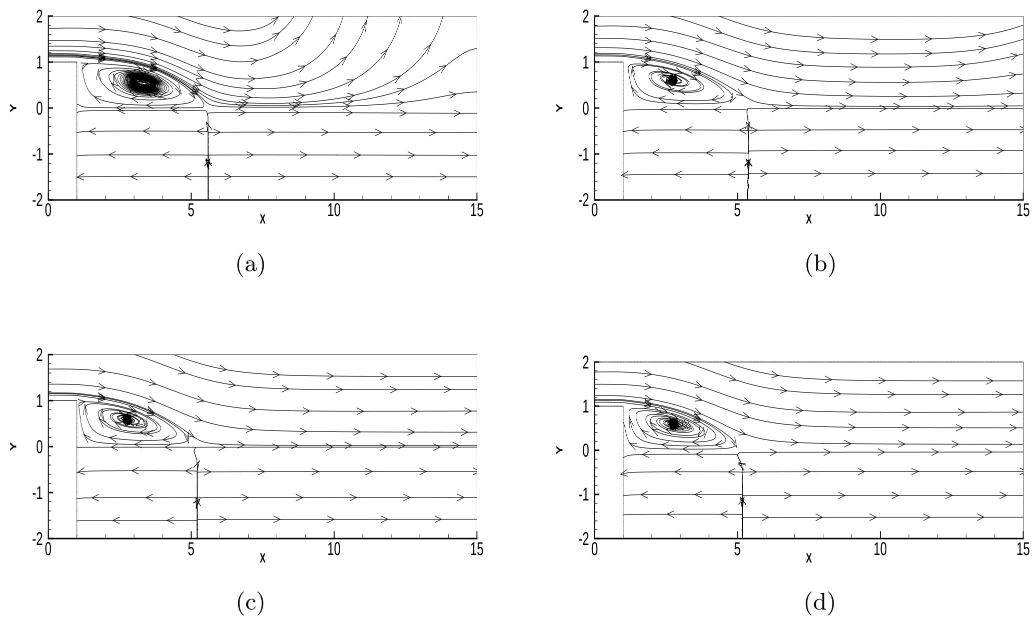


Fig. 5.7. Velocity streamlines with (a) $T=20$; (b) $T=40$; (c) $T=60$; (d) $T=80$.

6. Conclusions

In this work, we have designed and studied a second order unconditionally stable and convergent linearized scheme for a fluid-fluid interaction model. The scheme is a combination of the second order backward differentiation formula for temporal term, a extrapolated interpolation for nonlinear term and second order explicit Gear extrapolation method for interface terms. Theoretically, we have proved that the scheme is unconditionally stable and convergent, and long-time stable under the restriction of viscosity. Numerically, we validate the unconditional stability and convergence rates of this scheme. By compared with the first-order scheme, the proposed scheme is much more efficient.

A. Appendix

A.1. Proof of Theorem 4.1

Proof. Setting $(v_{i,h}, q_{i,h}) = 4\Delta t(\phi_i^{n+1}, \psi_{p,i}^{n+1})$ in (4.2) and using the Stokes-Stokes projection (4.3) result in

$$\begin{aligned}
&\|\phi_i^{n+1}\|_0^2 + \|2\phi_i^{n+1} - \phi_i^n\|_0^2 - \|\phi_i^n\|_0^2 - \|2\phi_i^n - \phi_i^{n-1}\|_0^2 + \|\phi_i^{n+1} - 2\phi_i^n + \phi_i^{n-1}\|_0^2 \\
&\quad + 4\Delta t\nu_i\|\nabla\phi_i^{n+1}\|_0^2 + 4\Delta t b(u_i(t_{n+1}), u_i(t_{n+1}), \phi_i^{n+1}) - 4\Delta t b(2u_{i,h}^n - u_{i.h}^{n-1}, u_{i,h}^{n+1}, \phi_i^{n+1}) \\
&\quad + 4\Delta t\int_I \kappa(u_i(t_{n+1}) - u_j(t_{n+1}))\phi_i^{n+1}\mathrm{ds} - 8\Delta t\int_I \kappa(u_{i,h}^n - u_{j,h}^n)\phi_i^{n+1}\mathrm{ds} \\
&\quad + 4\Delta t\int_I \kappa(u_{i,h}^{n-1} - u_{j,h}^{n-1})\phi_i^{n+1}\mathrm{ds} \\
=&2(3\eta_{i,h}^{n+1} - 4\eta_{i,h}^n + \eta_{i,h}^{n-1}, \phi_i^{n+1}) + 4\Delta t(\mathcal{E}_i^{n+1}, \phi_i^{n+1}).
\end{aligned} \tag{A.1}$$

Concerning the nonlinear terms in (A.1), noticing the definition of the trilinear terms, we have

$$\begin{aligned}
&|b(u_i(t_{n+1}), u_i(t_{n+1}), \phi_i^{n+1}) - b(2u_{i,h}^n - u_{i,h}^{n-1}, u_{i,h}^{n+1}, \phi_i^{n+1})| \\
\le& |b(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1}), u_i(t_{n+1}), \phi_i^{n+1})| + |b(2\eta_i^n - \eta_i^{n-1}, u_i(t_{n+1}), \phi_i^{n+1})| \\
&+ |b(2\phi_i^n - \phi_i^{n-1}, u_i(t_{n+1}), \phi_i^{n+1})| + |b(2u_{i,h}^n - u_{i,h}^{n-1}, \eta_i^{n+1}, \phi_i^{n+1})| \\
=:& \sum_{m=1}^4 I_m.
\end{aligned} \tag{A.2}$$

Next, using Lemma 2.1, each terms of RHS of (A.2) are bounded by

$$\begin{aligned}
I_1 \le& c_1\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_0\|u_i(t_{n+1})\|_2\|\nabla\phi_i^{n+1}\|_0 \\
\le& 9c_1^2\nu_i^{-1}\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_0^2\|u_i(t_{n+1})\|_2^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2 \\
\le& 12c_1^2\nu_i^{-1}\Delta t^3\|u_{i,tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega_i)^2)}\|u_i(t_{n+1})\|_2^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2,
\end{aligned} \tag{A.3a}$$

$$\begin{aligned}
I_2 \le& c_1\|2\eta_i^n - \eta_i^{n-1}\|_0\|u_i(t_{n+1})\|_2\|\nabla\phi_i^{n+1}\|_0 \\
\le& 9c_1^2\nu_i^{-1}\|2\eta_i^n - \eta_i^{n-1}\|_0^2\|u_i(t_{n+1})\|_2^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2 \\
\le& 72c_1^2\nu_i^{-1}\|\eta_i^n\|_0^2\|u_i(t_{n+1})\|_2^2 + 18c_1^2\nu_i^{-1}\|\eta_i^{n-1}\|_0^2\|u_i(t_{n+1})\|_2^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2,
\end{aligned} \tag{A.3b}$$

$$\begin{aligned}
I_3 \le& c_1\|2\phi_i^n - \phi_i^{n-1}\|_0\|u_i(t_{n+1})\|_2\|\nabla\phi_i^{n+1}\|_0 \\
\le& 9c_1^2\nu_i^{-1}\|2\phi_i^n - \phi_i^{n-1}\|_0^2\|u_i(t_{n+1})\|_2^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2,
\end{aligned} \tag{A.3c}$$

as well as

$$
\begin{aligned}
I_4 &\leq c_0\|\nabla(2u_{i,h}^n - u_{i,h}^{n-1})\|_0\|\nabla\eta_i^{n+1}\|_0\|\nabla\phi_i^{n+1}\|_0 \\
&\leq 9c_0^2\nu_i^{-1}\|\nabla(2u_{i,h}^n - u_{i,h}^{n-1})\|_0^2\|\nabla\eta_i^{n+1}\|_0^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2. \qquad \text{(A.4)}
\end{aligned}
$$

Moreover, we consider the interface terms in (A.1) and rewrite them as

$$
\begin{aligned}
&\int_I \kappa\left(u_i(t_{n+1}) - u_j(t_{n+1}) - 2(u_{i,h}^n - u_{j,h}^n) + u_{i,h}^{n-1} - u_{j,h}^{n-1}\right)\phi_i^{n+1}\mathrm{ds} \qquad \text{(A.5)}\\
=&\int_I \kappa\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\phi_i^{n+1}\mathrm{ds} - \int_I \kappa\left(u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\right)\phi_i^{n+1}\mathrm{ds}\\
&+\int_I \kappa\left(2\eta_i^n - \eta_i^{n-1} + 2\phi_i^n - \phi_i^{n-1}\right)\phi_i^{n+1}\mathrm{ds} - \int_I \kappa\left(2\eta_j^n - \eta_j^{n-1} + 2\phi_j^n - \phi_j^{n-1}\right)\phi_i^{n+1}\mathrm{ds}.
\end{aligned}
$$

We now estimate each terms of the RHS of (A.5) separately. Employing (2.2), we get

$$
\begin{aligned}
&\int_I \kappa\left(u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\right)\phi_i^{n+1}\mathrm{ds}\\
&\leq \kappa\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)}\|\phi_i^{n+1}\|_{L^2(I)}\\
&\leq C_{tr}C_p^{\frac{1}{2}}\kappa\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)}\|\nabla\phi_i^{n+1}\|_0\\
&\leq 9C_{tr}^2C_p\kappa^2\nu_i^{-1}\|u_i(t_{n+1}) - 2u_i(t_n) + u_i(t_{n-1})\|_{L^2(I)}^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2\\
&\leq 12C_{tr}^2C_p\kappa^2\Delta t^3\nu_i^{-1}\|u_{i,tt}\|_{L^2(t_{n-1},t_{n+1};L^2(I))}^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2, \qquad \text{(A.6)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_I \kappa\left(u_j(t_{n+1}) - 2u_j(t_n) + u_j(t_{n-1})\right)\phi_i^{n+1}\mathrm{ds}\\
&\leq 12C_{tr}^2C_p\kappa^2\Delta t^3\nu_i^{-1}\|u_{j,tt}\|_{L^2(t_{n-1},t_{n+1};L^2(I))}^2 + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2, \qquad \text{(A.7)}
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\int_I \kappa\left(2\eta_i^n - \eta_i^{n-1}\right)\phi_i^{n+1}\mathrm{ds} - \int_I \kappa\left(2\eta_j^n - \eta_j^{n-1}\right)\phi_i^{n+1}\mathrm{ds}\\
&\leq \kappa\left(\|2\eta_i^n - \eta_i^{n-1}\|_{L^2(I)} + \|2\eta_j^n - \eta_j^{n-1}\|_{L^2(I)}\right)\|\phi_i^{n+1}\|_{L^2(I)}\\
&\leq C_{tr}^2C_p\kappa\left(\|\nabla(2\eta_i^n - \eta_i^{n-1})\|_0 + \|\nabla(2\eta_j^n - \eta_j^{n-1})\|_0\right)\|\nabla\phi_i^{n+1}\|_0\\
&\leq 9C_{tr}^4C_p^2\kappa^2\nu_i^{-1}\left(\|\nabla(2\eta_i^n - \eta_i^{n-1})\|_0^2 + \|\nabla(2\eta_j^n - \eta_j^{n-1})\|_0^2\right) + \frac{\nu_i}{18}\|\nabla\phi_i^{n+1}\|_0^2\\
&\leq 18C_{tr}^4C_p^2\kappa^2\nu_i^{-1}\sum_{i=1}^2\left(4\|\nabla\eta_i^n\|_0^2 + \|\nabla\eta_i^{n-1}\|_0^2\right) + \frac{\nu_i}{18}\|\nabla\phi_i^{n+1}\|_0^2. \qquad \text{(A.8)}
\end{aligned}
$$

Next, arguing in exactly the same way as (3.4), we obtain

$$\begin{aligned}
&\int_I \kappa(2\phi_i^n - \phi_i^{n-1})\phi_i^{n+1}\mathrm{ds} - \int_I \kappa(2\phi_j^n - \phi_j^{n-1})\phi_i^{n+1}\mathrm{ds}\\
\le\ & 54C_{tr}^8 C_p^2\kappa^4\nu_i^{-3}\|2\phi_i^n - \phi_i^{n-1}\|_0^2 + 54C_{tr}^8 C_p^2\kappa^4\nu_i^{-2}\nu_j^{-1}\|2\phi_j^n - \phi_j^{n-1}\|_0^2\\
&+ \frac{\nu_i}{12}\|\nabla\phi_i^n\|_0^2 + \frac{\nu_i}{48}\|\nabla\phi_i^{n-1}\|_0^2 + \frac{\nu_j}{12}\|\nabla\phi_j^n\|_0^2 + \frac{\nu_j}{48}\|\nabla\phi_j^{n-1}\|_0^2 + \frac{\nu_i}{3}\|\nabla\phi_i^{n+1}\|_0^2. \qquad \text{(A.9)}
\end{aligned}$$

Furthermore, we consider the first term of RHS of (A.1).

$$\begin{aligned}
&2(3\eta_i^{n+1} - 4\eta_i^n + \eta_i^{n-1}, \phi_i^{n+1})\\
&\le 2\|3\eta_i^{n+1} - 4\eta_i^n + \eta_i^{n-1}\|_0\|\phi_i^{n+1}\|_0\\
&\le 96C_p^2\nu_i^{-1}\|\eta_{i,t}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega_i)^2)} + \frac{\nu_i}{6}\Delta t\|\nabla\phi_i^{n+1}\|_0^2. \qquad \text{(A.10)}
\end{aligned}$$

Besides, the truncation error in (A.1) can be bounded by

$$\begin{aligned}
&(\mathcal{E}_i^{n+1}, \phi_i^{n+1})\\
&\le \frac{1}{\Delta t}\left\|\left(\int_{t_n}^{t_{n+1}}(t-t_n)^2 u_{i,ttt}\mathrm{dt} - \frac{1}{4}\int_{t_{n-1}}^{t_{n+1}}(t-t_{n-1})^2 u_{i,ttt}\mathrm{dt}\right)\right\|_{-1}\|\nabla\phi_i^{n+1}\|_0\\
&\le 9\nu_i^{-1}\left(\frac{6}{5}\right)^2\Delta t^3\|u_{i,ttt}\|^2_{L^2(t_{n-1},t_{n+1};H^{-1}(\Omega_i)^2)} + \frac{\nu_i}{36}\|\nabla\phi_i^{n+1}\|_0^2. \qquad \text{(A.11)}
\end{aligned}$$

Furthermore, combining (A.3), (A.4), (A.6)-(A.11) with (A.1), we deduce that

$$\begin{aligned}
&\|\phi_i^{n+1}\|_0^2 + \|2\phi_i^{n+1} - \phi_i^n\|_0^2 - \|\phi_i^n\|_0^2 - \|2\phi_i^n - \phi_i^{n-1}\|_0^2\\
&\quad + \|\phi_i^{n+1} - 2\phi_i^n + \phi_i^{n-1}\|_0^2 + \frac{3\nu_i}{2}\Delta t\|\nabla\phi_i^{n+1}\|_0^2\\
&\le 48c_1^2\nu_i^{-1}\Delta t^4\|u_{i,tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega_i)^2)}\|u_i(t_{n+1})\|_2^2 + 288c_1^2\nu_i^{-1}\Delta t\|u_i(t_{n+1})\|_2^2\|\eta_i^n\|_0^2\\
&\quad + 72c_1^2\nu_i^{-1}\Delta t\|u_i(t_{n+1})\|_2^2\|\eta_i^{n-1}\|_0^2 + 36c_1^2\nu_i^{-1}\Delta t\|2\phi_i^n - \phi_i^{n-1}\|_0^2\|u_i(t_{n+1})\|_2^2\\
&\quad + 36c_0^2\nu_i^{-1}\Delta t\|\nabla(2u_{i,h}^n - u_{i,h}^{n-1})\|_0^2\|\nabla\eta_i^{n+1}\|_0^2\\
&\quad + 48C_{tr}^2 C_p\kappa^2\nu_i^{-1}\Delta t^4\left(\|u_{i,tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(I))} + \|u_{j,tt}\|^2_{L^2(t_{n-1},t_{n+1};L^2(I))}\right)\\
&\quad + 72C_{tr}^4 C_p^2\kappa^2\nu_i^{-1}\Delta t\sum_{i=1}^{2}\left(4\|\nabla\eta_i^n\|_0^2 + \|\nabla\eta_i^{n-1}\|_0^2\right)\\
&\quad + 216C_{tr}^8 C_p^2\kappa^4\nu_i^{-3}\Delta t\|2\phi_i^n - \phi_i^{n-1}\|_0^2 + 216C_{tr}^8 C_p^2\nu_i^{-2}\nu_j^{-1}\Delta t\|2\phi_j^n - \phi_j^{n-1}\|_0^2\\
&\quad + \frac{\nu_i}{3}\Delta t\|\nabla\phi_i^n\|_0^2 + \frac{\nu_i}{12}\Delta t\|\nabla\phi_i^{n-1}\|_0^2 + \frac{\nu_j}{3}\Delta t\|\nabla\phi_j^n\|_0^2\\
&\quad + \frac{\nu_j}{12}\Delta t\|\nabla\phi_j^{n-1}\|_0^2 + 96C_p^2\nu_i^{-1}\|\eta_{i,t}\|^2_{L^2(t_{n-1},t_{n+1};L^2(\Omega_i)^2)}\\
&\quad + 36\nu_i^{-1}\left(\frac{6}{5}\right)^2\Delta t^4\|u_{i,ttt}\|^2_{L^2(t_{n-1},t_{n+1};H^{-1}(\Omega_i)^2)}. \qquad \text{(A.12)}
\end{aligned}$$

Adding up (A.12) from $i=1,2$ ($j \neq i,\ j=1,2$) and $n=1,2,\cdots,N-1$ and noticing that

$$\sum_{i=1,j\neq i}^{2}\Big(\frac{\nu_i}{3}\|\nabla\phi_i^n\|_0^2+\frac{\nu_i}{12}\|\nabla\phi_i^{n-1}\|_0^2+\frac{\nu_j}{3}\|\nabla\phi_j^n\|_0^2+\frac{\nu_j}{12}\|\nabla\phi_j^{n-1}\|_0^2\Big)$$
$$=\sum_{i=1}^{2}\left(\frac{2\nu_i}{3}\|\nabla\phi_i^n\|_0^2+\frac{\nu_i}{6}\|\nabla\phi_i^{n-1}\|_0^2\right),\qquad j=1,2,$$
$$\sum_{i=1,j\neq i}^{2}\big(216C_{tr}^8C_p^2\kappa^4\nu_i^{-3}\|2\phi_i^n-\phi_i^{n-1}\|_0^2+216C_{tr}^8C_p^2\kappa^4\nu_i^{-2}\nu_j^{-1}\|2\phi_j^n-\phi_j^{n-1}\|_0^2\big)$$
$$\leq 432C^*\nu^*\sum_{i=1}^{2}\|2\phi_i^n-\phi_i^{n-1}\|_0^2,\qquad j=1,2,$$

we arrive at

$$\sum_{i=1}^{2}\|\phi_i^N\|_0^2+\sum_{i=1}^{2}\|2\phi_i^N-\phi_i^{N-1}\|_0^2+\sum_{i=1}^{2}\sum_{n=1}^{N-1}\|\phi_i^{n+1}-2\phi_i^n+\phi_i^{n-1}\|_0^2$$
$$+\frac{2}{3}\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\nu_i\|\nabla\phi_i^{n+1}\|_0^2+\frac{5}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^N\|_0^2+\frac{1}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^{N-1}\|_0^2$$
$$\leq\sum_{i=1}^{2}\|\phi_i^1\|_0^2+\sum_{i=1}^{2}\|2\phi_i^1-\phi_i^0\|_0^2+\frac{5}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^1\|_0^2+\frac{1}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^0\|_0^2$$
$$+48\sum_{i=1}^{2}c_1^2\nu_i^{-1}\Delta t^4\|u_{i,tt}\|_{L^2(0,T;L^2(\Omega_i)^2)}^2\|u_i(t_{n+1})\|_2^2$$
$$+72\sum_{i=1}^{2}\sum_{n=1}^{N-1}c_1^2\nu_i^{-1}\Delta t\|u_i(t_{n+1})\|_2^2\left(4\|\eta_i^n\|_0^2+\|\eta_i^{n-1}\|_0^2\right)$$
$$+36\sum_{i=1}^{2}\sum_{n=1}^{N-1}c_1^2\nu_i^{-1}\Delta t\|2\phi_i^n-\phi_i^{n-1}\|_0^2\|u_i(t_{n+1})\|_2^2$$
$$+36\sum_{i=1}^{2}\sum_{n=1}^{N-1}c_0^2\nu_i^{-1}\Delta t\|\nabla(2u_{i,h}^n-u_{i,h}^{n-1})\|_0^2\|\nabla\eta_i^{n+1}\|_0^2$$
$$+96\sum_{i=1}^{2}C_{tr}^2C_p\kappa^2\nu_i^{-1}\Delta t^4\|u_{i,tt}\|_{L^2(0,T;L^2(I))}^2+864C^*\nu^*\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\|2\phi_i^n-\phi_i^{n-1}\|_0^2$$
$$+72\sum_{i=1}^{2}\sum_{n=1}^{N-1}C_{tr}^4C_p^2\kappa^2\nu_i^{-1}\Delta t\sum_{k=1}^{2}\left(4\|\nabla\eta_k^n\|_0^2+\|\nabla\eta_k^{n-1}\|_0^2\right)$$
$$+96\sum_{i=1}^{2}C_p^2\nu_i^{-1}\|\eta_{i,t}\|_{L^2(0,T;L^2(\Omega_i)^2)}^2+36\sum_{i=1}^{2}\nu_i^{-1}\left(\frac{6}{5}\right)^2\Delta t^4\|u_{i,ttt}\|_{L^2(0,T;H^{-1}(\Omega_i)^2)}^2.$$

Finally, from Theorem 3.1, Remark 3.1, the regularity assumptions of the exact solutions and the properties (4.4) of the projection, we get

$$
\begin{aligned}
&\sum_{i=1}^{2}\|\phi_i^N\|_0^2+\sum_{i=1}^{2}\|2\phi_i^N-\phi_i^{N-1}\|_0^2+\sum_{i=1}^{2}\sum_{n=1}^{N-1}\|\phi_i^{n+1}-2\phi_i^n+\phi_i^{n-1}\|_0^2\\
&+\frac{2}{3}\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\nu_i\|\nabla\phi_i^{n+1}\|_0^2+\frac{5}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^N\|_0^2+\frac{1}{6}\sum_{i=1}^{2}\nu_i\Delta t\|\nabla\phi_i^{N-1}\|_0^2\\
\le\ & C\left(\Delta t^4+h^6+h^4\right)+C\sum_{i=1}^{2}\sum_{n=1}^{N-1}\Delta t\|2\phi_i^n-\phi_i^{n-1}\|_0^2,
\end{aligned}
$$

which combines with Lemma 2.2 to finish the proof. □

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