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Partial Topology Identification of Stochastic Multi-Weighted Complex Networks Based on Graph-Theoretic Method and Adaptive Synchronization

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Abstract. This article aims to identify the partial topological structures of delayed complex network. Based on the drive-response concept, a more universal model, which includes nonlinear couplings, stochastic perturbations and multi-weights, is considered into drive-response networks. Different from previous methods, we obtain identification criteria by combining graph-theoretic method and adaptive synchronization. After that, the partial topological structures of stochastic multi-weighted complex networks with or without time delays can be identified successfully. Moreover, response network can reach synchronization with drive network. Ultimately, the effectiveness of the proposed theoretical results is validated through numerical simulations.

AMS subject classifications: 60H10, 93D05, 93E12

Key words: Partial topology identification, graph-theoretic method, multi-weighted complex networks, adaptive pinning control, nonlinear coupling.

1 Introduction

Complex networks, including industrial networks, financial networks, transportation networks and neural networks, are penetrated into almost all aspects of real world [1–7]. Till now, the investigation of complex networks has been extremely extensive. In addition to synchronization and stability [8–17], the research of topology identification has attached special attention [18–23].

In the literature, topological structures of many complex networks are assumed to be known [24–26]. However, the reality is that only a small section of topological structures are known or even completely unknown. Particularly, topological structures will constantly change with increasing and decreasing of network vertices or arcs. Therefore, it is practical to figure out the unknown topological structures of complex networks.

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In [18–23], they identified the whole topological structures of complex network. However, in many practical networks, only partial topological structures are what we need. For instance, in a social network, if the message is only linked to work, we only need to transmit it to colleagues in the list of friends. Similarly, when searching articles in Web of Science, we may only want to know articles in the same research field. Considering the circumstances mentioned above, it will result in high control cost if we identify the whole topological structures. To reduce the control cost, pinning control is a powerful technique since only a part of vertices can receive the control input directly. Therefore, it is essential and indispensable to address the problem of partial topology identification with pinning control.

Existing research recognises the critical role played by partial topology identification. In [27–30], the authors studied the partial topology identification of complex dynamical networks via a pinning mechanism. However, the above literatures [27–30] focus on partial topology identification of single weighted complex networks. Nevertheless, many real world networks can be modelled as coupled systems with multi-weights. They have different coupling forms among vertices. Examples of this kind of multi-weighted networks are ubiquitous. For instance, a social network includes the relationship among friends, relatives and colleagues. A traffic network includes the transportation among cars, planes and bikes. As we all know, in recent years, though some articles contribute to researching the whole topological structures of multi-weighted complex network [20, 31], there are only few results about partial topology identification of multi-weighted complex networks [32]. However, in [32], the model is linearly coupled. Therefore, it is of great significance to research partial topology identification of more general multi-weighted complex networks. Furthermore, in [27–31], the models considered have been largely restricted to deterministic ordinary differential equations. In fact, multi-weighted complex networks are inevitably affected by various types of environmental noise [33–37]. However, it should be noted that there are few papers about partial topology identification [32] of stochastic multi-weighted complex networks. Hence, to fill the gap, this paper attempts to identify partial topological structures of multi-weighted complex networks with stochastic disturbance and nonlinear couplings.

For the previous research of topology identification, researchers usually propose identification criteria based on Lyapunov method. Although this method is quite classical and widespread, it is still a challenge to construct an appropriate global Lyapunov function directly. In recent years, many scholars used graph theory to construct global Lyapunov function indirectly and further investigated stability and boundedness of complex networks [38–44]. This method uses some results about graph theory, so it is always called graph-theoretic method. It is an effective technique to systematically construct a global Lyapunov function by using the weighted summation of vertex Lyapunov functions. Till now, the new graph-theoretic method has rarely been applied on the partial topology identification of stochastic multi-weighted complex networks with nonlinear couplings.

Motivated by aforementioned discussions, this paper attempts to use graph-theoretic method to study partial topology identification of stochastic multi-weighted complex networks with nonlinear couplings. The main contributions of this paper are summarized as follows:

- By the concept of drive-response, adaptive pinning controllers are considered into response system, which are cost-effective.
- The network model is much more general, which includes nonlinear couplings, stochastic perturbations and multi-weights.
- A novel graph-theoretic method is proposed to solve partial topology identification problem.
- Drive-response networks achieve synchronization under adaptive pinning control. Furthermore, the partial topological structures can be identified successfully by LaSalle-type invariance principle for stochastic differential equations.

The reminder of this paper is organized as follows. Some mathematical preliminaries are introduced in Section 2. Partial topology identification for stochastic multi-weighted complex networks with nonlinear couplings is provided in Section 3. Moreover, the identification for networks with time delays is shown in Section 4. In Section 5, numerical simulations are provided to illustrate the effectiveness of proposed method. Some conclusions are drawn in Section 6. Finally, some basic concepts of graph theory are introduced in Appendix.

2 Preliminaries

In order to present our main results, some necessary notations and lemmas are presented. $\mathcal{R}_+$, $\mathcal{R}^n$ and $\mathcal{R}^{n\times m}$ represent the set of nonnegative real numbers, n-dimensional Euclidean space and $n\times m$-dimensional real matrices, respectively. $\|\cdot\|$ is the Euclidean norm for vectors or the trace norm for matrices. The superscript $^{\mathrm{T}}$ denotes the transpose of a vector or a matrix.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets. $\mathbb{E}(\cdot)$ is the mathematical expectation. $\mathbb{B}(\cdot)$ is a scalar standard Brownian motion defined on the given probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$. $\mathbb{C}^{1,2}(\mathcal{R}_+\times\mathcal{R}^n;\mathcal{R}_+)$ denotes the family of all non-negative functions $\mathbb{V}(t,x)$ on $\mathcal{R}_+\times\mathcal{R}^n$, in which $\mathbb{V}(t,x)$ is continuously once differentiable in t and twice differentiable in x. $\mathbb{C}^b_{\mathcal{F}_0}([-\tau,0];\mathcal{R}^n)$ is the family of all the $\mathcal{F}_0$-measurable bounded $\mathbb{C}([-\tau,0];\mathcal{R}^n)$-valued random variables. $\mathbb{L}^p(\mathcal{R}_+;\mathcal{R}_+)$ denotes the family of positive random variables ξ with $\mathbb{E}(\|\xi\|^p)<\infty$.

First, consider a non-autonomous n-dimensional stochastic differential equation

$$dx(t)=\psi(t,x(t))dt+\phi(t,x(t))d\mathbb{B}(t), \tag{2.1}$$

where $t \geq 0$ and initial value $x(0)=x_0 \in \mathcal{R}^n$. The measurable functions $\psi(t,x):\mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}^n$ and $\phi(t,x):\mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}^n$ satisfy local Lipschitz condition and linear growth condition. For any initial value x_0, (2.1) has a unique continuous solution [45], which is denoted as $x(t;x_0)$. Moreover, if $\psi(t,0)=0$ and $\phi(t,0)=0$, (2.1) admits a trivial solution $x(t;0)\equiv 0$.

Then for $\mathbb{V}(t,x)\in C^{1,2}(\mathcal{R}_+ \times \mathcal{R}^n;\mathcal{R}_+)$, the differential operator $\mathcal{L}$ associated with equation (2.1) is

$$\mathcal{L}\mathbb{V}(t,x)=\mathbb{V}_t(t,x)+\mathbb{V}_x(t,x)\psi(t,x)+\frac{1}{2}\text{trace}\left[\phi^{\mathrm{T}}(t,x)\mathbb{V}_{xx}(t,x)\phi(t,x)\right], \tag{2.2}$$

in which

$$\begin{aligned}
&\mathbb{V}_t(t,x)=\frac{\partial \mathbb{V}(t,x)}{\partial t}, \qquad \mathbb{V}_x(t,x)=\left(\frac{\partial \mathbb{V}(t,x)}{\partial x_1},\frac{\partial \mathbb{V}(t,x)}{\partial x_2},\cdots,\frac{\partial \mathbb{V}(t,x)}{\partial x_n}\right),\\
&\mathbb{V}_{xx}(t,x)=\left(\frac{\partial^2 \mathbb{V}(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}.
\end{aligned}$$

Lemma 2.1 ([46]). *Assume that there is a function* $\mathbb{V}\in \mathbb{C}^{1,2}(\mathcal{R}_+ \times \mathcal{R}^n;\mathcal{R}_+)$*, a function* $\Upsilon \in \mathbb{L}^1(\mathcal{R}_+;\mathcal{R}_+)$ *and a continuous function* $\Theta:\mathcal{R}^n \to \mathcal{R}_+$ *such that*

$$\lim_{\|x\|\to\infty} \inf_{0\leq t<\infty} \mathbb{V}(t,x)=\infty$$

and $\mathcal{L}$ *acting on* $\mathbb{V}$ *along with the trajectories of* (2.1) *satisfies*

$$\mathcal{L}\mathbb{V}(t,x)\leq \Upsilon(t)-\Theta(x), \quad (t,x)\in \mathcal{R}_+ \times \mathcal{R}^n.$$

Moreover, ϕ *is bounded. Then, for every* $x_0 \in \mathcal{R}^n$*,* $\lim_{t\to\infty}\mathbb{V}(t,x(t;x_0))$ *exists and is finite almost surely and*

$$\lim_{t\to\infty}\Theta(x(t;x_0))=0 \quad a.s. \tag{2.3}$$

Next, a non-autonomous n-dimensional stochastic differential equation with time delay is presented by

$$dx(t)=\psi(t,x(t),x(t-\tau))dt+\phi(t,x(t),x(t-\tau))d\mathbb{B}(t), \tag{2.4}$$

in which $t\geq 0$ and initial value $\xi \in \mathbb{C}^b_{\mathcal{F}_0}([-\tau,0];\mathcal{R}^n)$, $\psi:\mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ and $\phi:\mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ are assumed to satisfy local Lipschitz condition and linear growth condition. For any initial data ξ, (2.4) has a unique solution on $t \geq -\tau$, which can be denoted as $x(t;\xi)$ [47].

Lemma 2.2 ([47]). *Assume that there are functions $\mathbb{V}\in\mathbb{C}^{1,2}(\mathcal{R}_+\times\mathcal{R}^n;\mathcal{R}_+)$, $\Upsilon\in\mathbb{L}^1(\mathcal{R}_+;\mathcal{R}_+)$, and $\Theta_1,\Theta_2\in\mathbb{C}(\mathcal{R}^n;\mathcal{R}_+)$ such that the differential operator $\mathcal{L}$ acting on $\mathbb{V}$ along with the trajectories of (2.4) satisfies*

$$\begin{aligned}&\mathcal{L}\mathbb{V}(t,x,y)\le\Upsilon(t)-\Theta_1(x)+\Theta_2(y), && (t,x,y)\in\mathcal{R}_+\times\mathcal{R}^n\times\mathcal{R}^n,\\ &\Theta_1(x)>\Theta_2(x), && \forall x\neq 0,\\ &\lim_{\|x\|\to\infty}\inf_{0\le t<\infty}\mathbb{V}(t,x)=\infty.\end{aligned}$$

Then, for every initial value $\xi\in\mathbb{C}^b_{\mathcal{F}_0}([-\tau,0];\mathcal{R}^n)$, it holds that

$$\lim_{t\to\infty}x(t;\xi)=0\quad a.s. \tag{2.5}$$

Now, an important property about graph theory is described as follow.

Lemma 2.3 ([41]). *Assume that $q\ge 2$. Then the following identity holds:*

$$\sum_{i,j=1}^{q}p_ia_{ij}\mathbb{F}_{ij}(x_i,x_j)=\sum_{\mathcal{Q}\in\mathbb{Q}}\mathbb{W}(\mathcal{Q})\sum_{(s,r)\in\mathrm{E}(\mathcal{C}_\mathcal{Q})}\mathbb{F}_{rs}(x_r,x_s).$$

Here $\mathbb{F}_{ij}(x_i,x_j)$, $(i,j=1,2,\cdots,q)$ are arbitrary functions. $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G},A)$. $\mathbb{W}(\mathcal{Q})$ is the weight of $\mathcal{Q}$ and $\mathcal{C}_\mathcal{Q}$ denotes the directed cycle of $\mathcal{Q}$. p_i is the cofactor of the i-th diagonal element of the Laplacian matrix of $(\mathcal{G},A)$. Particularly, if $(\mathcal{G},A)$ is strongly connected, then $p_i>0$ for $i=1,2,\cdots,q$.

3 Partial topology identification of stochastic multi-weighted complex networks

In this section, a general model of stochastic multi-weighted complex network with N vertices and m kinds of weights is characterized by

$$dx_i(t)=\left[f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{N}a_{ij}^{(k)}h_k(x_j(t))\right]dt+g_i(t,x_i(t))d\mathbb{B}(t),\quad i=1,2,\cdots,N, \tag{3.1}$$

where $x_i=(x_{i1},x_{i2},\cdots,x_{in})^{\mathrm{T}}\in\mathcal{R}^n$ is the state vector of the i-th vertex. $f_i(t,x_i(t)):\mathcal{R}_+\times\mathcal{R}^n\to\mathcal{R}^n$ is a smooth nonlinear function determining the dynamics of i-th vertex. $A^{(k)}=(a_{ij}^{(k)})_{N\times N}$ is the unknown or uncertain k-th weighted configuration matrix. If there is a connection from vertex j to vertex i ($j\neq i$) in the k-th weight, then $a_{ij}^{(k)}>0$ presents the weight, and 0 otherwise. $A^{(k)}$ is not necessary symmetric, but the boundedness of the network should be ensured. $h_k(\cdot):\mathcal{R}^n\to\mathcal{R}^n$ is the k-th inner coupling function. $g_i(t,x_i(t))$: $\mathcal{R}_+\times\mathcal{R}^n\to\mathcal{R}^n$ is the noise intensity function of the i-th vertex.

Remark 3.1. It is well known that a number of existing networks are more accurately represented by multi-weighted complex networks. The feature of such complex network is that the coupling forms among vertices are multiple. Moreover, it is more practical to take nonlinear couplings among vertices. In addition, multi-weighted complex networks are inevitably affected by stochastic disturbance. Therefore, our model is much more general, which includes nonlinear couplings, stochastic perturbations and multi-weights. Here, we use stochastic differential equations driven by Brownian motion to describe the considered networks. In fact, multi-weighted complex network model is generally used and can model many real networks such as transportation network and communication network. For example, according to different modes of transportation, the transportation network can be regarded as a multi-weighted complex network, which is the coupling of bus network, bike network and taxi network. In this case, multi-weighted complex network can better describe the dynamics of transportation network.

Without loss of generality, suppose that the first q, $(q=2,3,\cdots,N)$ vertices are controlled (if it is not the case, it can be done by reordering the vertices [27]). Next, we take (3.1) as the drive network, one can construct the corresponding response network, which is described by

$$\begin{aligned} dy_i(t)=&\Big[f_i(t,y_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{q}b_{ij}^{(k)}h_k(y_j(t))+\sum_{k=1}^{m}\sum_{j=q+1}^{N}b_{ij}^{(k)}h_k(x_j(t))+u_i(t)\Big]dt\\ &+g_i(t,y_i(t))d\mathbb{B}(t),\quad i=1,2,\cdots,q, \end{aligned}\tag{3.2}$$

where $y_i=(y_{i1},y_{i2},\cdots,y_{in})^{\mathrm{T}}\in\mathcal{R}^n$ is the response state vector of the i-th vertex. $b_{ij}^{(k)}$ is the estimation of the weight $a_{ij}^{(k)}$. Then

$$\big(\mathcal{G},A_{q\times N}^{(k)}\big)=\big(\mathcal{G},(a_{ij}^{(k)})_{q\times N}\big)$$

are the partial topological structures needing to be identified by

$$\big(\mathcal{G},B_{q\times N}^{(k)}\big)=\big(\mathcal{G},(b_{ij}^{(k)})_{q\times N}\big).$$

$u_i(t)$ is the controller to be designed.

Assume that all coefficients of drive system (3.1) and response system (3.2) satisfy the linear growth condition and local Lipschitz condition. Hence, given arbitrary initial values $x_0\in\mathcal{R}^{Nn}$ and $y_0\in\mathcal{R}^{qn}$, the solution of Eqs. (3.1) and (3.2) are existent and unique through [45]. They can be represented by

$$x(t)=(x_1^{\mathrm{T}}(t),x_2^{\mathrm{T}}(t),\cdots,x_N^{\mathrm{T}}(t))^{\mathrm{T}}\quad\text{and}\quad y(t)=(y_1^{\mathrm{T}}(t),y_2^{\mathrm{T}}(t),\cdots,y_q^{\mathrm{T}}(t))^{\mathrm{T}},$$

respectively. Define

$$e_i(t)=y_i(t)-x_i(t),\quad c_{ij}^{(k)}=b_{ij}^{(k)}-a_{ij}^{(k)},\quad i=1,2,\cdots,q,\quad j=1,2,\cdots,N,\quad k=1,2,\cdots,m.$$

Then the error network can be described as

$$
\begin{aligned}
de_i(t)=&\Big[f_i(t,y_i(t))-f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}\Big(h_k(y_j(t))-h_k(x_j(t))\Big)\\
&+\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}h_k(y_j(t))+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}h_k(x_j(t))+u_i(t)\Big]dt\\
&+\Big[g_i(t,y_i(t))-g_i(t,x_i(t))\Big]d\mathbb{B}(t),\quad i=1,2,\cdots,q. \qquad (3.3)
\end{aligned}
$$

In order to obtain main results, two definitions and some hypotheses are introduced.

Definition 3.1 ([45]). *The zero solution of a system* (3.3)*, or simply, system* (3.3) *is said to be asymptotically stable with probability one if for all* $e_0\in\mathcal{R}^{qn}$

$$\lim_{t\to\infty}e(t;e_0)=0\quad a.s.$$

The zero solution of error equation (3.3) is asymptotically stable, which means that the response system (3.2) reaches synchronization with the drive system (3.1).

Definition 3.2. *The uncertain partial topological structures* $\big(\mathcal{G},A_{q\times N}^{(k)}\big)$*,* $(k=1,2,\cdots,m)$ *can be identified by* $\big(\mathcal{G},B_{q\times N}^{(k)}\big)$*,* $(k=1,2,\cdots,m)$ *with probability one if*

$$\lim_{t\to\infty}b_{ij}^{(k)}(t)=a_{ij}^{(k)},\quad i=1,2,\cdots,q,\quad j=1,2,\cdots,N,\quad k=1,2,\cdots,m.$$

Hypothesis 3.1. Suppose that there exist nonnegative constants α_i, $(i=1,2,\cdots,q)$ such that

$$\|f_i(t,y_i)-f_i(t,x_i)\|\le\alpha_i\|y_i-x_i\|,\quad x_i,y_i\in\mathcal{R}^n,\quad i=1,2,\cdots,q.$$

Hypothesis 3.2. Suppose that for each $k=1,2,\cdots,m$, $\{h_k(x_i(t))\}_{i=1}^N$ are linearly independent on the orbit $\{x_i(t)\}_{i=1}^N$ of the outer synchronization manifold $\{x_i(t)=y_i(t)\}_{i=1}^q$, and there exists a positive constant β_k such that

$$\|h_k(y_i)-h_k(x_i)\|\le\beta_k\|y_i-x_i\|,\quad x_i,y_i\in\mathcal{R}^n,\quad i=1,2,\cdots,N.$$

Hypothesis 3.3. For every $i=1,2,\cdots,q$, there exists a nonnegative constant σ_i such that

$$\|g_i(t,y_i)-g_i(t,x_i)\|\le\sigma_i\|y_i-x_i\|.$$

Denote

$$D_{ij}=\max_{1\le k\le m}\Big\{\beta_k a_{ij}^{(k)}\Big\},\quad i,j=1,2,\cdots,q.$$

Built the following adaptive pinning controllers and updating laws

$$u_i(t)=-d_i(t)e_i(t), \quad \dot{d}_i(t)=r_ie_i^{\mathrm{T}}(t)e_i(t), \quad i=1,2,\cdots,q, \tag{3.4a}$$

$$\dot{b}_{ij}^{(k)}=\begin{cases} -e_i^{\mathrm{T}}(t)h_k(y_j(t)), & i,j=1,2,\cdots,q, \quad k=1,2,\cdots,m, \\ -e_i^{\mathrm{T}}(t)h_k(x_j(t)), & i=1,2,\cdots,q, \quad j=q+1,\cdots,N, \quad k=1,2,\cdots,m, \end{cases} \tag{3.4b}$$

where r_i is an arbitrary positive number for $i=1,2,\cdots,q$.

Theorem 3.1. *Assume that Hypotheses* 3.1-3.3 *hold and weighted digraph* $(\mathcal{G},(D_{ij})_{q\times q})$ *is strongly connected, adaptive pinning controllers and updating laws are designed in* (3.4a) *and* (3.4b)*, respectively. Then: (i) drive network* (3.1) *and response network* (3.2) *can reach synchronization and (ii) for each* $k=1,2,\cdots,m$*, the unknown partial topological structures* $(\mathcal{G},A_{q\times N}^{(k)})$ *of drive network* (3.1) *can be estimated by* $(\mathcal{G},B_{q\times N}^{(k)})$ *with probability one.*

Proof. Three steps make up this proof. First, a suitable vertex Lyapunov function is constructed. Then, by using graph theory, the global Lyapunov function is established indirectly. Finally, the partial topology identification can be obtained from Lemma 2.1 and LaSalle's invariance principle.

First, choose a vertex Lyapunov function as

$$\mathbb{V}_i(t,e_i)=\frac{1}{2}e_i^{\mathrm{T}}e_i+\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{m}\big(c_{ij}^{(k)}\big)^2+\frac{1}{2r_i}\big(d_i-d_i^*\big)^2, \quad i=1,2,\cdots,q,$$

where d_i^* is a large positive constant to be determined. Through (3.3)-(3.4b), it yields that

$$\begin{aligned}\mathcal{L}\mathbb{V}_i(t,e_i(t))=&e_i^{\mathrm{T}}(t)\bigg[f_i(t,y_i(t))-f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}h_k(y_j(t))+u_i(t)\\&+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}h_k(x_j(t))+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}\Big(h_k(y_j(t))-h_k(x_j(t))\Big)\bigg]\\&+\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}\dot{c}_{ij}^{(k)}+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}\dot{c}_{ij}^{(k)}+\frac{1}{r_i}(d_i(t)-d_i^*)\dot{d}_i(t)\\&+\frac{1}{2}\mathrm{trace}\bigg\{\Big[g_i(t,y_i(t))-g_i(t,x_i(t))\Big]^{\mathrm{T}}\Big[g_i(t,y_i(t))-g_i(t,x_i(t))\Big]\bigg\},\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{L}\mathbb{V}_i(t,e_i(t))\le&\alpha_ie_i^{\mathrm{T}}(t)e_i(t)+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}e_i^{\mathrm{T}}(t)\Big(h_k(y_j(t))-h_k(x_j(t))\Big)+\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}e_i^{\mathrm{T}}(t)h_k(y_j(t))\\&+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}e_i^{\mathrm{T}}(t)h_k(x_j(t))-\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}e_i^{\mathrm{T}}(t)h_k(x_j(t))-\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}e_i^{\mathrm{T}}(t)h_k(y_j(t))\\&-d_i(t)e_i^{\mathrm{T}}(t)e_i(t)+(d_i(t)-d_i^*)e_i^{\mathrm{T}}(t)e_i(t)+\frac{1}{2}\sigma_i^2e_i^{\mathrm{T}}(t)e_i(t)\end{aligned}$$

$$
\begin{aligned}
&=\alpha_i e_i^{\mathrm{T}}(t)e_i(t)-d_i^* e_i^{\mathrm{T}}(t)e_i(t)+\frac{1}{2}\sigma_i^2 e_i^{\mathrm{T}}(t)e_i(t)+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}e_i^{\mathrm{T}}(t)\Big(h_k(y_j(t))-h_k(x_j(t))\Big)\\
&\le\Big(\alpha_i-d_i^*+\frac{1}{2}\sigma_i^2\Big)\|e_i(t)\|^2+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}\beta_k\left(\frac{\|e_i(t)\|^2}{2}+\frac{\|e_j(t)\|^2}{2}\right)\\
&=\Big(\alpha_i-d_i^*+\frac{1}{2}\sigma_i^2\Big)\|e_i(t)\|^2+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}\beta_k\left(\frac{\|e_i(t)\|^2}{2}+\frac{\|e_j(t)\|^2}{2}-\frac{\|e_i(t)\|^2}{2}+\frac{\|e_i(t)\|^2}{2}\right)\\
&=\left(\alpha_i+\sum_{j=1}^{q}mD_{ij}-d_i^*+\frac{1}{2}\sigma_i^2\right)\|e_i(t)\|^2+\sum_{j=1}^{q}mD_{ij}\left(\frac{\|e_j(t)\|^2}{2}-\frac{\|e_i(t)\|^2}{2}\right).
\end{aligned}
\tag{3.5}
$$

Second, by using graph theory, one can define global Lyapunov function as

$$\mathbb{V}(t,e)=\sum_{i=1}^{q}p_i\mathbb{V}_i(t,e_i),$$

in which p_i is the cofactor of the i-th diagonal element of the Laplacian matrix of $(\mathcal{G},(D_{ij})_{q\times q})$. Then, it is easy to obtain that

$$
\begin{aligned}
&\mathcal{L}\mathbb{V}(t,e(t))=\sum_{i=1}^{q}p_i\mathcal{L}\mathbb{V}_i(t,e_i(t))\\
&\le\sum_{i=1}^{q}p_i\left(\alpha_i+\sum_{j=1}^{q}mD_{ij}-d_i^*+\frac{1}{2}\sigma_i^2\right)\|e_i(t)\|^2+m\sum_{i=1}^{q}\sum_{j=1}^{q}p_iD_{ij}\left(\frac{\|e_j(t)\|^2}{2}-\frac{\|e_i(t)\|^2}{2}\right)\\
&\triangleq \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}
\tag{3.6}
$$

Define

$$\eta_i=d_i^*-\alpha_i-\frac{\sigma_i^2}{2}-\sum_{j=1}^{q}mD_{ij},$$

where d_i^* is large enough to ensure $\eta_i>0$, $i=1,2,\cdots,q$. Therefore, we can get that

$$\mathrm{I}_1=\sum_{i=1}^{q}p_i\left(\alpha_i+\sum_{j=1}^{q}mD_{ij}-d_i^*+\frac{1}{2}\sigma_i^2\right)\|e_i(t)\|^2=-\sum_{i=1}^{q}p_i\eta_i\|e_i(t)\|^2.$$

By applying Lemma 2.3, it follows that

$$
\begin{aligned}
\mathrm{I}_2&=\sum_{i=1}^{q}\sum_{j=1}^{q}p_iD_{ij}\left(\frac{m\|e_j(t)\|^2}{2}-\frac{m\|e_i(t)\|^2}{2}\right)\\
&=\sum_{\mathcal{Q}\in\mathbb{Q}}\mathbb{W}(\mathcal{Q})\sum_{(s,r)\in E(\mathcal{C}_{\mathcal{Q}})}\left(\frac{m\|e_s(t)\|^2}{2}-\frac{m\|e_r(t)\|^2}{2}\right)=0,
\end{aligned}
$$

in which $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $(\mathcal{G},(D_{ij})_{q\times q})$, $\mathcal{C}_{\mathcal{Q}}$ is the cycle of $\mathcal{Q}$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$. It implies that

$$\mathcal{L}\mathbb{V}(t,e(t))\leq -\sum_{i=1}^{q}p_i\eta_i\|e_i(t)\|^2\leq -\bar{\eta}e^{\mathrm{T}}(t)e(t)\triangleq -\Theta(e(t)).$$

Here $\bar{\eta}>0$, which is determined by p_i and η_i, $i=1,2,\cdots,q$.

Third, the partial topology identification can be obtained from Lemma 2.1 and LaSalle's invariance principle.

By the definition of $\mathbb{V}$, it indicates that

$$\lim_{\|e\|\to\infty}\inf_{0\leq t\leq\infty}\mathbb{V}(t,e)=\infty.$$

Since

$$\|g_i(t,y_i(t))-g_i(t,x_i(t))\|\leq\sigma_i\|y_i(t)-x_i(t)\|$$

and the solution $x_i(t)$ and $y_i(t)$ to drive-response networks (3.1) and (3.2) are bounded, $g_i(t,y_i(t))-g_i(t,x_i(t))$ is bounded for each $i=1,2,\cdots,q$. According to Lemma 2.1, it yields that $\lim_{t\to\infty}\mathbb{V}(t;e_i;c_{ij}^k;d_i)$ exists and is finite almost surely, and $\lim_{t\to\infty}\Theta(e(t))=0$ a.s. Obviously, $\lim_{t\to\infty}e(t)=0$ a.s. By virtue of LaSalle's invariance principle, Hypothesis 3.2 and coupled systems (3.3), one can get

$$M'=\{e=0,\ c_{ij}^{(k)}=0,\ d_i=d_i^*,\ i=1,2,\cdots,q,\ j=1,2,\cdots,N,\ k=1,2,\cdots,m\}$$

is the largest invariant set of $M=\{e=0\}$. It yields that the solutions regarding equations (3.3)-(3.4b) starting from $\Omega=\mathcal{R}^{q(n+N+1)}$ will asymptotically stabilize at M' with probability one. This means that under the adapted pinning controllers (3.4a) and updating laws (3.4b), the uncertain topological structures $(\mathcal{G},A_{q\times N}^{(k)})$ can be identified by $(\mathcal{G},B_{q\times N}^{(k)})$ with probability one. Meanwhile, for arbitrary initial values, the solution of error network (3.3) is asymptotically stable. That is, the response network (3.2) is synchronized with the drive network (3.1). This completes the proof. $\square$

Remark 3.2. From Theorem 3.1, a global Lyapunov function $\mathbb{V}$ of error system (3.3) are successfully constructed through combining vertex-Lyapunov functions $\mathbb{V}_i$ together in the way of

$$\mathbb{V}(t,e)=\sum_{i=1}^{q}p_i\mathbb{V}_i(t,e_i),$$

in which p_i is the cofactor of the i-th diagonal element of Laplacian matrix $(\mathcal{G},(D_{ij})_{q\times q})$. The strong connectedness of graph $(\mathcal{G},(D_{ij})_{q\times q})$ is sufficient to guarantee the synchronization and partial topology identification successfully. We do not need the strong connectedness of all subgraphs $(\mathcal{G},(a_{ij}^{(k)})_{q\times q})$, $k=1,2,\cdots,m$.

When $q=1$, the following corollary emerges naturally. Suppose that the first vertex to be controlled without loss of generality. The corresponding response network can be built as

$$\begin{aligned} dy_1(t)=&\Big[f_1(t,y_1(t))+\sum_{k=1}^{m}b_{11}^{(k)}h_k(y_1(t))+\sum_{k=1}^{m}\sum_{j=2}^{N}b_{1j}^{(k)}h_k(x_j(t))+u_1(t)\Big]dt\\ &+g_1(t,y_1(t))d\mathbb{B}(t). \end{aligned} \tag{3.7}$$

Define the following controller and updating laws

$$u_1(t)=-d_1(t)e_1(t), \qquad \dot{d}_1(t)=r_1e_1^{\mathrm{T}}(t)e_1(t), \tag{3.8a}$$

$$\dot{b}_{1j}^{(k)}=\begin{cases}-e_1^{\mathrm{T}}(t)h_k(y_1(t)), & k=1,2,\cdots,m,\\ -e_1^{\mathrm{T}}(t)h_k(x_j(t)), & j=2,\cdots,N, \quad k=1,2,\cdots,m,\end{cases} \tag{3.8b}$$

where r_1 is an arbitrary positive number.

Corollary 3.1. *Assume that Hypotheses* 3.1-3.3 *hold. Then the partial topological structure* $(\mathcal{G},A_{1\times N}^{(k)})$, $(k=1,2,\cdots,m)$ *of drive network* (3.1) *can be identified by* $(\mathcal{G},B_{1\times N}^{(k)})$, $(k=1,2,\cdots,m)$ *with probability one under the pinning controller* (3.8a) *and updating laws* (3.8b).

Similar with the deducting process from (3.5) to (3.6), it yields that

$$\mathcal{L}\mathbb{V}=\mathcal{L}\mathbb{V}_1\leq\left(\alpha_1+\frac{1}{2}\sigma_1^2-d_i^*+mD_{11}\right)\|e_1\|^2.$$

Therefore, we have the identification result if

$$d_i^*>\alpha_1+\frac{1}{2}\sigma_1^2+mD_{11}.$$

Remark 3.3. Corollary 3.1 indicates that $a_{1j}^{(k)}$, $(j=1,2,\cdots,N,\ k=1,2,\cdots,m)$ can be identified successfully through response network (3.7). Because of the arbitrariness of vertex′ selection, one can add controller to any vertex (l-th vertex, $l=1,2,\cdots,N$). Therefore, $a_{lj}^{(k)}$, $(j=1,2,\cdots,N,\ k=1,2,\cdots,m)$ can be identified successfully through pinning one vertex. Furthermore, for each $k=1,2,\cdots,m$, $(\mathcal{G},A_{q\times N}^{(k)})$ can be identified successfully by using this pinning control strategy q times. If $q=N$, the whole topological structures $(\mathcal{G},A_{N\times N}^{(k)})$ can be identified by $(\mathcal{G},B_{N\times N}^{(k)})$ with probability one.

Remark 3.4. As is well known, a multi-weighted complex network is composed with a great amount of vertices and different weights. It is extremely difficult and expensive to add controllers to all vertices. Pinning control, as we all know, is a technical strategy. This paper combines pinning control and adaptive control, which reduces control cost to a large degree. Therefore, our control mechanism is more universal and cost-effective.

Remark 3.5. The problem of topology identification of complex networks has received considerable interest in recent years. The detailed descriptions are given in [18, 27–31]. Three important features emerge from the aforementioned studies. First, they usually discuss the topology identification of whole topological structures [18, 31]. Second, the models are usually single weight [27–30]. Third, most of these articles construct global Lyapunov function directly [18, 27–31]. In comparing with above work, the model in this paper is stochastic multi-weighted complex network, which is more suitable for practical applications. Furthermore, a novel graph-theoretic method is used to obtain theoretical results about partial topology identification, where global Lyapunov function can be obtained indirectly. Up to now, there are few papers about partial topology identification of stochastic multi-weighted complex networks based on graph-theoretic method and adaptive synchronization.

4 Partial topology identification of stochastic multi-weighted complex networks with time delays

Time delays are general and indispensable in control system. Moderate time delays can improve stability and dynamic performance of system. Therefore, it is necessary to identify the topological structures of stochastic multi-weighted complex networks with time delays. In what follows, partial topology identification of complex networks with time delays is further studied.

A general model for stochastic multi-weighted dynamical network with time delays is described by

$$\begin{aligned} dx_i(t)=&\Big[f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{N}a_{ij}^{(k)}h_k(x_j(t-\tau_k))\Big]dt\\ &+\sum_{k=1}^{m}g_i^{(k)}\Big(t,x_i(t),x_i(t-\tau_k)\Big)d\mathbb{B}(t),\quad i=1,2,\cdots,N, \end{aligned} \tag{4.1}$$

in which $\tau_k>0$, $k=1,2,\cdots,m$ are time delays. The denotation $g_i^{(k)}\big(t,x_i(t),x_i(t-\tau_k)\big)$ represents the vector-form noise intensity function. Here the intensity of stochastic perturbations is related with time delay in each weight, which is different with (3.1). Other parameters are the same with (3.1). Let Eq. (4.1) be the drive network. One can construct the corresponding response network as follow

$$\begin{aligned} dy_i(t)=&\Big[f_i(t,y_i(t))+\sum_{k=1}^{m}\sum_{j=1}^{q}b_{ij}^{(k)}h_k(y_j(t-\tau_k))+\sum_{k=1}^{m}\sum_{j=q+1}^{N}b_{ij}^{(k)}h_k(x_j(t-\tau_k))+u_i(t)\Big]dt\\ &+\sum_{k=1}^{m}g_i^{(k)}\Big(t,y_i(t),y_i(t-\tau_k)\Big)d\mathbb{B}(t),\quad i=1,2,\cdots,q. \end{aligned} \tag{4.2}$$

Here $b_{ij}^{(k)}$ is the estimation of $a_{ij}^{(k)}$, $u_i(t)$ is the controller to be determined. Similarly,

$$(\mathcal{G},A_{q\times N}^{(k)})=(\mathcal{G},\big(a_{ij}^{(k)}\big)_{q\times N})$$

are the partial topological structures needing to be identified by

$$\Big(\mathcal{G},B_{q\times N}^{(k)}\Big)=(\mathcal{G},\big(b_{ij}^{(k)}\big)_{q\times N}).$$

For brevity, let $e_i(t)=y_i(t)-x_i(t)$, $e_i(t-\tau_k)=y_i(t-\tau_k)-x_i(t-\tau_k)$, $c_{ij}^{(k)}=b_{ij}^{(k)}-a_{ij}^{(k)}$, $i=1,2,\cdots,q$, $j=1,2,\cdots,N$, $k=1,2,\cdots,m$. Then the error dynamics between (4.1) and (4.2) can be described as

$$\begin{aligned} de_i(t)=&\bigg[f_i(t,y_i(t))-f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c_{ij}^{(k)}h_k(x_j(t-\tau_k))+u_i(t)\\ &+\sum_{k=1}^{m}\sum_{j=1}^{q}a_{ij}^{(k)}\Big(h_k(y_j(t-\tau_k))-h_k(x_j(t-\tau_k))\Big)+\sum_{k=1}^{m}\sum_{j=1}^{q}c_{ij}^{(k)}h_k(y_j(t-\tau_k))\bigg]dt\\ &+\sum_{k=1}^{m}\bigg[g_i^{(k)}\Big(t,y_i(t),y_i(t-\tau_k)\Big)-g_i^{(k)}\Big(t,x_i(t),x_i(t-\tau_k)\Big)\bigg]d\mathbb{B}(t),\quad i=1,2,\cdots,q. \end{aligned}\tag{4.3}$$

In this section, the drive-response systems (4.1) and (4.2) are said to be synchronized, if the zero solution of error system (4.3) is asymptotically stable. That is, $\lim_{t\to\infty}e(t;e_0)=0$ a.s. for all $e_0\in\mathbb{C}_{\mathcal{F}_0}^{b}([-\tau,0];\mathcal{R}^{qn})$.

In order to obtain the main result, the following hypothesis is necessary.

Hypothesis 4.1. Assume that there exist some nonnegative constants $u_i^{(k)}$, $v_i^{(k)}$ such that

$$\|g_i^{(k)}(t,\hat{x}_i,\hat{y}_i)-g_i^{(k)}(t,x_i,y_i)\|\le u_i^{(k)}\|\hat{x}_i-x_i\|+v_i^{(k)}\|\hat{y}_i-y_i\|.$$

Meanwhile, $g_i^{(k)}(t,\hat{x}_i,\hat{y}_i)$ and $g_i^{(k)}(t,x_i,y_i)$ are bounded for any $(t,\hat{x}_i,\hat{y}_i)$, $(t,x_i,y_i)\in\mathcal{R}_+\times\mathcal{R}^n\times\mathcal{R}^n$, $i=1,2,\cdots,q$.

Consider the following controllers and updating laws

$$u_i(t)=-d_i(t)e_i(t),\quad \dot{d}_i(t)=r_ie_i^{\mathrm{T}}(t)e_i(t),\quad i=1,2,\cdots,q,\tag{4.4a}$$

$$\dot{b}_{ij}^{(k)}=\begin{cases}-e_i^{\mathrm{T}}(t)h_k(y_j(t-\tau_k)), & i,j=1,2,\cdots,q,\quad k=1,2,\cdots,m,\\ -e_i^{\mathrm{T}}(t)h_k(x_j(t-\tau_k)), & i=1,2,\cdots,q,\quad j=q+1,\cdots,N,\quad k=1,2,\cdots,m,\end{cases}\tag{4.4b}$$

where r_i is an arbitrary positive number for $i=1,2,\cdots,q$.

Theorem 4.1. *Assume that Hypotheses* 3.1, 3.2, 4.1 *hold and weighted digraph* $(\mathcal{G},(D_{ij})_{q\times q})$ *is strongly connected, adaptive pinning controllers and updating laws are designed in* (4.4a) *and* (4.4b)*, respectively. Then: (i) response network* (4.2) *reaches synchronization with drive network* (4.1) *and (ii) for each* $k=1,2,\cdots,m$*, the unknown partial topological structures* $(\mathcal{G},A^{(k)}_{q\times N})$ *of drive network* (4.1) *can be estimated by* $(\mathcal{G},B^{(k)}_{q\times N})$ *with probability one.*

Proof. The main proof process is also divided into three steps. First, consider the following vertex Lyapunov function

$$\mathbb{V}_i(t,e_i)=\frac{1}{2}e_i^{\mathrm{T}}e_i+\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{m}\big(c^{(k)}_{ij}\big)^2+\frac{1}{2r_i}\big(d_i(t)-d_i^*\big)^2,\quad i=1,2,\cdots,q,$$

where d_i^* is a large positive constant to be determined. From (4.3)-(4.4b), we can derive that

$$\begin{aligned}\mathcal{L}\mathbb{V}_i(t,e_i(t))=&e_i^{\mathrm{T}}(t)\bigg[f_i(t,y_i(t))-f_i(t,x_i(t))+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c^{(k)}_{ij}h_k(x_j(t-\tau_k))+u_i(t)\\&+\sum_{k=1}^{m}\sum_{j=1}^{q}a^{(k)}_{ij}\Big(h_k(y_j(t-\tau_k))-h_k(x_j(t-\tau_k))\Big)+\sum_{k=1}^{m}\sum_{j=1}^{q}c^{(k)}_{ij}h_k(y_j(t-\tau_k))\bigg]\\&+\sum_{k=1}^{m}\sum_{j=1}^{q}c^{(k)}_{ij}\dot{c}^{(k)}_{ij}+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c^{(k)}_{ij}\dot{c}^{(k)}_{ij}+\frac{1}{r_i}(d_i(t)-d_i^*)\dot{d}_i(t)\\&+\frac{1}{2}\mathrm{trace}\Bigg\{\bigg[\sum_{k=1}^{m}\Big(g_i^{(k)}\Big(t,y_i(t),y_i(t-\tau_k)\Big)-g_i^{(k)}\Big(t,x_i(t),x_i(t-\tau_k)\Big)\Big)\bigg]^{\mathrm{T}}\\&\times\bigg[\sum_{k=1}^{m}\Big(g_i^{(k)}\Big(t,y_i(t),y_i(t-\tau_k)\Big)-g_i^{(k)}\Big(t,x_i(t),x_i(t-\tau_k)\Big)\Big)\bigg]\Bigg\}.\end{aligned}$$

By using the inequality

$$(a_1+a_2+\cdots+a_m)^2\le m((a_1)^2+(a_2)^2+\cdots+(a_m)^2),$$

it yields that

$$\begin{aligned}\mathcal{L}\mathbb{V}_i(t,e_i(t))\le&\alpha_ie_i^{\mathrm{T}}(t)e_i(t)+\sum_{k=1}^{m}\sum_{j=1}^{q}c^{(k)}_{ij}e_i^{\mathrm{T}}(t)h_k(y_j(t-\tau_k))-d_i^*(t)e_i^{\mathrm{T}}(t)e_i(t)\\&+\sum_{k=1}^{m}\sum_{j=1}^{q}a^{(k)}_{ij}e_i^{\mathrm{T}}(t)\Big(h_k(y_j(t-\tau_k))-h_k(x_j(t-\tau_k))\Big)\\&+\sum_{k=1}^{m}\sum_{j=q+1}^{N}c^{(k)}_{ij}e_i^{\mathrm{T}}(t)h_k(x_j(t-\tau_k))-\sum_{k=1}^{m}\sum_{j=1}^{q}c^{(k)}_{ij}e_i^{\mathrm{T}}(t)h_k(y_j(t-\tau_k))\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=1}^{m}\sum_{j=q+1}^{N} c_{ij}^{(k)} e_i^{\mathrm{T}}(t) h_k(x_j(t-\tau_k)) + m\sum_{k=1}^{m}(u_i^{(k)})^2 e_i^{\mathrm{T}}(t)e_i(t)\\
&+m\sum_{k=1}^{m}\left(v_i^{(k)}\right)^2 e_i^{\mathrm{T}}(t-\tau_k)e_i(t-\tau_k)\\
\le&\left(\alpha_i+m^2u_i^2-d_i^*\right)\left\|e_i(t)\right\|^2+m\sum_{k=1}^{m}\left(v_i^{(k)}\right)^2\left\|e_i(t-\tau_k)\right\|^2\\
&+\sum_{k=1}^{m}\sum_{j=1}^{q}D_{ij}\left(\frac{\left\|e_i(t)\right\|^2}{2}+\frac{\left\|e_j(t-\tau_k)\right\|^2}{2}\right),
\end{aligned} \tag{4.5}
$$

where

$$u_i=\max_{1\le k\le m}\left\{u_i^{(k)}\right\}.$$

Second, define

$$\mathbb{V}(t,e)=\sum_{i=1}^{q}p_i\mathbb{V}_i,$$

where p_i is the same as Theorem 3.1. Then it can drive that

$$
\begin{aligned}
\mathcal{L}\mathbb{V}(t,e(t))\le&\sum_{i=1}^{q}p_i\left(\alpha_i+m^2u_i^2-d_i^*\right)\left\|e_i(t)\right\|^2+m\sum_{k=1}^{m}\sum_{i=1}^{q}p_i\left(v_i^{(k)}\right)^2\left\|e_i(t-\tau_k)\right\|^2\\
&+\sum_{k=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{q}p_iD_{ij}\left(\frac{\left\|e_i(t)\right\|^2}{2}+\frac{\left\|e_j(t-\tau_k)\right\|^2}{2}+\frac{\left\|e_i(t-\tau_k)\right\|^2}{2}-\frac{\left\|e_i(t-\tau_k)\right\|^2}{2}\right)\\
\le&-\sum_{i=1}^{q}p_i\left(d_i^*-\alpha_i-m^2\bar{u}^2-\frac{1}{2}\sum_{j=1}^{q}mD_{ij}\right)\left\|e_i(t)\right\|^2\\
&+\sum_{i=1}^{q}mp_i\left(m\bar{v}^2+\frac{1}{2}\sum_{j=1}^{q}D_{ij}\right)\left\|e_i(t-\tau)\right\|^2\\
&+\sum_{k=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{q}p_iD_{ij}\left(\frac{\left\|e_j(t-\tau_k)\right\|^2}{2}-\frac{\left\|e_i(t-\tau_k)\right\|^2}{2}\right)\\
\triangleq&\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3,
\end{aligned} \tag{4.6}
$$

where

$$
\begin{aligned}
&\|e_i(t-\tau)\|^2=\max\left\{\|e_i(t-\tau_1)\|^2,\|e_i(t-\tau_2)\|^2,\cdots,\|e_i(t-\tau_m)\|^2\right\},\\
&\bar{u}=\max_{1\le i\le N}\left\{u_i\right\},\qquad \bar{v}=\max_{1\le k\le m,\ 1\le i\le N}\left\{v_i^{(k)}\right\}.
\end{aligned}
$$

By applying Lemma 2.3, it yields that

$$\begin{aligned}\mathrm{I}_3=&\sum_{k=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{q}p_iD_{ij}\left(\frac{\|e_j(t-\tau_k)\|^2}{2}-\frac{\|e_i(t-\tau_k)\|^2}{2}\right)\\=&\sum_{k=1}^{m}\sum_{\mathcal{Q}\in\mathbb{Q}}\mathbb{W}(\mathcal{Q})\sum_{(u,v)\in\mathcal{E}(C_{\mathcal{Q}})}\left(\frac{\|e_u(t-\tau_k)\|^2}{2}-\frac{\|e_v(t-\tau_k)\|^2}{2}\right)=0.\end{aligned}$$

Third, the partial topology identification of stochastic multi-weighted complex network with time delay can be obtained from Lemma 2 and LaSalle's invariance principle.

Apparently, one can obtained that

$$\mathcal{L}\mathbb{V}(t,e(t))\le \mathrm{I}_1+\mathrm{I}_2\triangleq-\Theta_1(e(t))+\Theta_2(e(t-\tau)),$$

in which

$$\begin{aligned}\Theta_1(e(t))&\triangleq\sum_{i=1}^{q}p_i\Big(d_i^*-\alpha_i-m^2\bar{u}^2-\frac{1}{2}\sum_{j=1}^{q}mD_{ij}\Big)\|e_i(t)\|^2,\\\Theta_2(e(t-\tau))&\triangleq\sum_{i=1}^{q}mp_i\Big(m\bar{v}^2+\frac{1}{2}\sum_{j=1}^{q}D_{ij}\Big)\|e_i(t-\tau)\|^2.\end{aligned}$$

It is obvious that $\Theta_1(e)>\Theta_2(e)$ for sufficient large positive constant

$$d_i^*>\alpha_i+m^2(\bar{u}^2+\bar{v}^2)+\sum_{j=1}^{q}mD_{ij}$$

with $e\neq 0$. Moreover,

$$\lim_{\|e\|\to\infty}\inf_{0\le t<\infty}\mathbb{V}=\infty.$$

By Lemma 2.2, it can obtain that $\lim_{t\to\infty}\mathbb{V}(t;e_i;c_{ij}^{(k)};d_i)$ exists and is finite almost surely and $\lim_{t\to\infty}e(t)=0$, *a.s.* Under Hypothesis 3.2 and error equation (4.3), one gets $M'=\{e=0,\ c_{ij}^{(k)}=0,\ d_i=d_i^*,\ i=1,2,\cdots,q,\ j=1,2,\cdots,N,\ k=1,2,\cdots,m\}$ is the largest invariant set of $M=\{e=0\}$. Similarly, the solutions regarding Eqs. (4.3)-(4.4b) starting from Ω will asymptotically stabilize at M' with probability one. Therefore, the zero solution of network (4.3) is asymptotically stable under pinning controllers (4.4a) and updating laws (4.4b), namely the drive network and the response network reach outer synchronization. In addition, the uncertain topological structures $(\mathcal{G},A_{q\times N}^{(k)})$ can be successfully identified by $(\mathcal{G},B_{q\times N}^{(k)})$ with probability one, which completes the proof. □

Remark 4.1. In contrast to [20, 32], a graph-theoretic method is used to overcome the difficulty of constructing global Lyapunov function directly. However, there are many differences between this article and [20, 32]. First, the drive network in [20] is deterministic, while drive network and response network are stochastic in this paper. Second, in [32], the connection among vertex is linear, while nonlinear coupling is included in this paper. Third, time delays in this paper are associated with each weight. But in [20, 32], time delay is fixed, which are special cases of this paper.

Remark 4.2. In this paper, Hypothesis 3.2 is linearly independent condition, which can usually be satisfied with stochastic perturbations. Hypothesis 3.2 is used to get

$$\lim_{t\to\infty} c_{ij}^{(k)}(t)=0, \quad i=1,2,\cdots,q, \quad j=1,2,\cdots,N, \quad k=1,2,\cdots,m.$$

That is,

$$\lim_{t\to\infty} b_{ij}^{(k)}(t)=a_{ij}^{(k)}, \quad i=1,2,\cdots,q, \quad j=1,2,\cdots,N, \quad k=1,2,\cdots,m.$$

Therefore, Hypothesis 3.2 is the key for guaranteeing successful topology identification.

In general, Lipschitz continuity hypotheses are conditions to ensure existence and uniqueness of solutions of the considered model. In our manuscript, these hypotheses are also used to obtained inequalities (3.5) and (4.5). In addition, Lipschitz continuity hypotheses are generally employed to study synchronization [9–17], topology identification [18–23] and partial topology identification [27–30]. In fact, Lipschitz condition can be weakened to semi-Lipschitz condition [10] and the synchronization and partial topology identification can also be obtained under the semi-Lipschitz continuity hypotheses

- Semi-Lipschitz condition: For the vector-valued function $f(t,x)$, suppose the semi-Lipschitz condition with respect to t holds, i.e., for any $x, y\in\mathcal{R}^n$, there exists positive constant α such that

$$(y-x)^{\mathrm{T}}(f(t,y)-f(t,x))\le\alpha(y-x)^{\mathrm{T}}(y-x).$$

5 Numerical simulations

In what follows, numerical simulations are used to illustrate the effectiveness and correctness of the theoretical results in Sections 3 and 4. The classical chaotic Lorenz system [48] is taken as the vertex's dynamical system, which is described by

$$\dot{x}=\begin{pmatrix}-a & a & 0\\ c & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ -x_1x_3\\ x_1x_2\end{pmatrix}\triangleq Dx+G(x),$$

where $a=10$, $b=8/3$, $c=28$. It can be verified that Lorenz system satisfies Hypothesis 1 because it has bounded attractors [48].

Next, consider a stochastic multi-weighted network with 10 vertices and two kinds of weights. Without loss of generality, we can assume the first three vertices are what we needed (If it is the other three vertices, one can renumber them so that they can still be the first three vertices).

5.1 Partial topology identification of stochastic multi-weighted complex networks

Example 5.1. The drive network is described by

$$\begin{aligned} dx_i(t)=&\Big[f_i(t,x_i(t))+\sum_{j=1}^{10}a_{ij}^{(1)}h_1\left(x_j(t)\right)+\sum_{j=1}^{10}a_{ij}^{(2)}h_2\left(x_j(t)\right)\Big]dt \\ &+g_i(t,x_i(t))d\mathbb{B}(t), \quad i=1,2,\cdots,10, \end{aligned} \tag{5.1}$$

where $x_i\in\mathcal{R}^3$, $f_i(t,x_i(t))=Dx_i(t)+G(x_i(t))$, $h_1(x_j(t))=H_1x_j(t)$, $(j=1,2,\cdots,10)$, $h_2(x_j(t))=H_2x_j(t)$, $(j=1,2,\cdots,10)$. $H_1=(h_{ij}^{(1)})_{3\times3}$ and $H_2=(h_{ij}^{(2)})_{3\times3}$ are inner coupling matrices. $g_i(t,x_i(t))=0.01\sin x_i(t)$. $A^{(1)}=(a_{ij}^{(1)})_{10\times10}$ and $A^{(2)}=(a_{ij}^{(2)})_{10\times10}$ are two weighted configuration matrices. We select the elements of inner coupling matrices and weighted configuration matrices as follows and other elements not mentioned are set as zero.

- $h_{1,1}^{(1)}=0.2$, $h_{1,2}^{(1)}=0.2$, $h_{1,3}^{(1)}=0.3$, $h_{2,2}^{(1)}=0.25$, $h_{3,1}^{(1)}=0.1$, $h_{3,2}^{(1)}=0.1$,
- $h_{1,2}^{(2)}=0.2$, $h_{1,3}^{(2)}=0.2$, $h_{2,1}^{(2)}=0.11$, $h_{2,2}^{(2)}=0.2$, $h_{2,3}^{(2)}=0.1$, $h_{3,1}^{(2)}=0.15$, $h_{3,2}^{(2)}=0.1$;
- $a_{1,2}^{(1)}=a_{2,1}^{(1)}=a_{3,2}^{(1)}=a_{4,3}^{(1)}=a_{6,4}^{(1)}=a_{6,5}^{(1)}=a_{8,6}^{(1)}=a_{10,1}^{(1)}=1$,
- $a_{2,4}^{(1)}=a_{2,10}^{(1)}=a_{5,4}^{(1)}=a_{5,6}^{(1)}=a_{6,7}^{(1)}=a_{7,6}^{(1)}=a_{7,8}^{(1)}=a_{8,9}^{(1)}=a_{9,8}^{(1)}=a_{10,2}^{(1)}=a_{10,8}^{(1)}=0.5$;
- $a_{1,2}^{(2)}=a_{2,1}^{(2)}=a_{2,6}^{(2)}=a_{3,2}^{(2)}=a_{4,8}^{(2)}=a_{5,4}^{(2)}=a_{8,2}^{(2)}=a_{8,9}^{(2)}=a_{10,1}^{(2)}=a_{10,9}^{(2)}=1$,
- $a_{2,3}^{(2)}=a_{2,8}^{(2)}=a_{4,3}^{(2)}=a_{4,10}^{(2)}=a_{6,2}^{(2)}=a_{6,7}^{(2)}=a_{6,10}^{(2)}=a_{7,6}^{(2)}=a_{7,8}^{(2)}=a_{8,4}^{(2)}=a_{9,8}^{(2)}=a_{9,10}^{(2)}=a_{10,4}^{(2)}=a_{10,6}^{(2)}=0.5$.

Accordingly, the response network with adaptive controllers which just controlling the first three vertices is established as follow:

$$\begin{aligned} dy_i(t)=&\Big[f_i(t,y_i(t))+\sum_{j=1}^{3}b_{ij}^{(1)}H_1y_j(t)+\sum_{j=4}^{10}b_{ij}^{(1)}H_1x_j(t)+\sum_{j=1}^{3}b_{ij}^{(2)}H_2y_j(t) \\ &+\sum_{j=4}^{10}b_{ij}^{(2)}H_2x_j(t)+u_i(t)\Big]dt+g_i(t,y_i(t))d\mathbb{B}(t), \quad i=1,2,3. \end{aligned} \tag{5.2}$$

By some manipulation, one can get

$$\begin{aligned} &\left\|h_1(y_j(t))-h_1(x_j(t))\right\|=\left\|H_1y_j(t)-H_1x_j(t)\right\|\le\left\|H_1\right\|_\infty\left\|y_j(t)-x_j(t)\right\|, \\ &\left\|h_2(y_j(t))-h_2(x_j(t))\right\|=\left\|H_2y_j(t)-H_2x_j(t)\right\|\le\left\|H_2\right\|_\infty\left\|y_j(t)-x_j(t)\right\|, \quad (j=1,2,3). \end{aligned}$$

Here,

$$H_k=(h_{rs}^{(k)})_{3\times3}, \quad (k=1,2), \quad \left\|H_k\right\|_\infty=\max_{1\le r\le3}\sum_{s=1}^{3}|h_{rs}^{(k)}|, \quad (k=1,2).$$

Let $\beta_1=\|H_1\|_\infty$, $\beta_2=\|H_2\|_\infty$, it indicates that

$$\|h_k(y_j)-h_k(x_j)\|\leq\beta_k\|y_j-x_j\|$$

is satisfied. Moreover,

$$\|g_i(t,y_i)-g_i(t,x_i)\|=\|0.02\cos((y_i+x_i)/2)\sin((y_i-x_i)/2)\|\leq 0.01\|y_i-x_i\|.$$

Obviously they satisfy Hypothesis 3.3. Therefore, f_i and g_i satisfy the Hypotheses of Theorem 3.1, that is to say, response network (5.2) can reach synchronization with drive network (5.1) under adaptive pinning controllers (5.3a) and updating laws (5.3b)

$$u_i(t)=-d_i(t)e_i(t),\quad \dot{d}_i(t)=r_ie_i^{\mathrm{T}}(t)e_i(t),\quad i=1,2,3, \tag{5.3a}$$

$$\dot{b}_{ij}^{(k)}=\begin{cases}-e_i^{\mathrm{T}}(t)H_ky_j(t), & i,j=1,2,3,\quad k=1,2,\\ -e_i^{\mathrm{T}}(t)H_kx_j(t), & i=1,2,3,\quad j=4,\cdots,10,\quad k=1,2.\end{cases} \tag{5.3b}$$

Moreover, the unknown partial topological structures $(\mathcal{G},A^{(1)}_{3\times 10})$ and $(\mathcal{G},A^{(2)}_{3\times 10})$ can be identified successfully by $(\mathcal{G},B^{(1)}_{3\times 10})$ and $(\mathcal{G},B^{(2)}_{3\times 10})$, respectively. The initial values are arbitrary given as follows:

- $x_1=(5,5,5)^{\mathrm{T}}$, $x_2=(5,5,5)^{\mathrm{T}}$, $x_3=(6,6,6)^{\mathrm{T}}$, $x_4=(2,2,6)^{\mathrm{T}}$, $x_5=(2,6,6)^{\mathrm{T}}$,
- $x_6=(7,7,1)^{\mathrm{T}}$, $x_7=(7,7,7)^{\mathrm{T}}$, $x_8=(0,8,2)^{\mathrm{T}}$, $x_9=(8,0,8)^{\mathrm{T}}$, $x_{10}=(9,9,9)^{\mathrm{T}}$,
- $y_1=(0,0,0)^{\mathrm{T}}$, $y_2=(-2.5,-2.5,-2.5)^{\mathrm{T}}$, $y_3=(1,1,1)^{\mathrm{T}}$,
- $b_{ij}^{(1)}(0)=b_{ij}^{(2)}(0)=4.5$, $d_1(0)=d_2(0)=d_3(0)=2$, $r_1=r_2=r_3=1$.

The validity of Theorem 3.1 is illustrated in Figs. 1-3. The sample path $(y_i,x_i,i=1,2,3)$ are shown in Fig. 1, from which we can clearly see that y_{i1}, y_{i2}, y_{i3} $(i=1,2,3)$ and x_{i1}, x_{i2}, x_{i3}, $(i=1,2,3)$ coincide perfectly with time t. This means that the drive network and response network reach outer synchronization. The estimations of the uncertain topological structures $(\mathcal{G},A^{(1)}_{3\times 10})$ and $(\mathcal{G},A^{(2)}_{3\times 10})$ are displayed in Figs. 2(a) and (b), respectively. For a clearer understanding, we arbitrarily select some curves for separate presentation. In subgraph (a), it is easily viewed that the curves regarding $b_{32}^{(1)}$, $b_{21}^{(1)}$ get stabilized at 1, the curves regarding $b_{24}^{(1)}$, $b_{210}^{(1)}$ get stabilized at 0.5 and curves regarding $b_{13}^{(1)}$, $b_{14}^{(1)}$, $b_{31}^{(1)}$ tend to 0. In subgraph (b), one can clearly obtained that the curves regarding $b_{21}^{(2)}$, $b_{26}^{(2)}$ get stabilized at 1, the curves regarding $b_{23}^{(2)}$, $b_{28}^{(2)}$ get stabilized at 0.5 and curves regarding $b_{13}^{(2)}$, $b_{14}^{(2)}$, $b_{36}^{(2)}$ tend to 0. All of the curves in response system (5.2) tend to real values in drive system (5.1) perfectly. It indicates that the estimation of unknown topological structures are identified successfully by using pinning control strategies. In Fig. 3, the subgraph (a) demonstrates the feedback gains $d_i(t)$ in network (5.2), where $d_i(t)$ has an upper bound and tends to some constants for $i=1,2,3$.

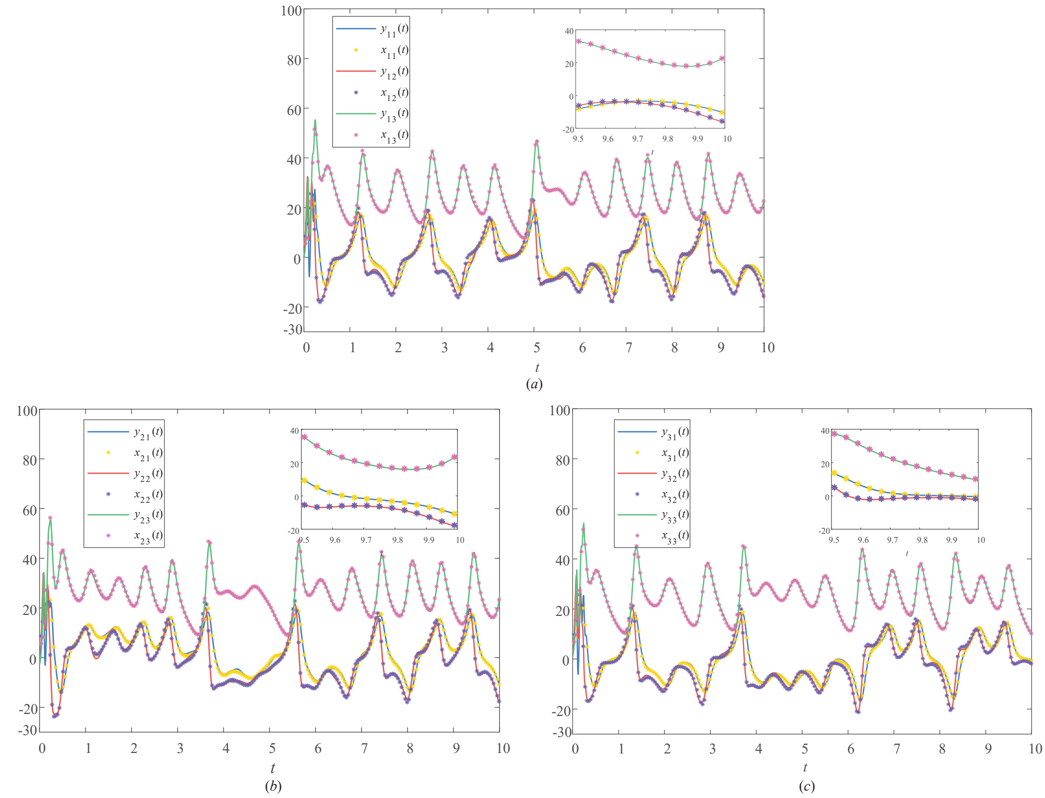

Figure 1: The sample path of drive system (5.1) and response system (5.2).

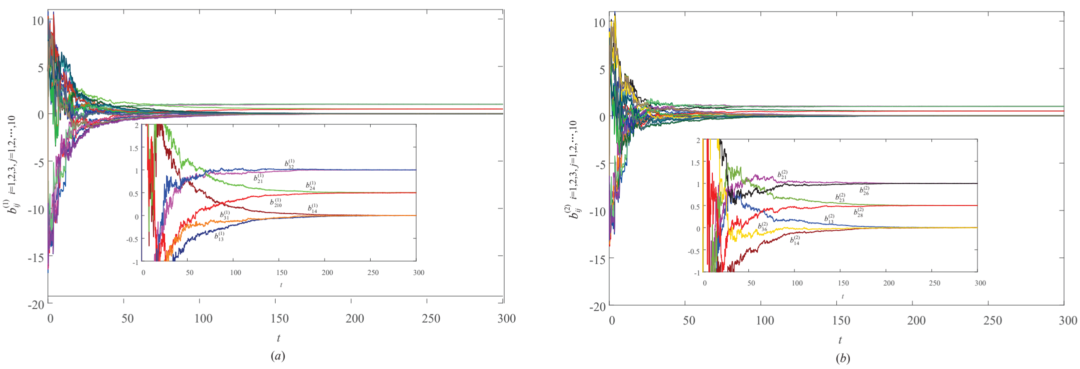

Figure 2: Partial topology identification of (5.1). It can clearly view that every curve converges to the real value.

5.2 Partial topology identification of stochastic multi-weighted complex networks with time delays

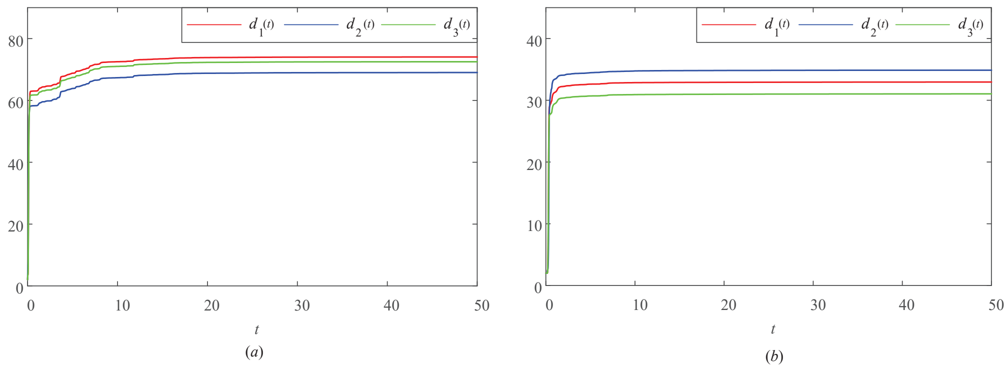

Figure 3: The subgraph (a) is feedback gains d_i in network (5.2) and (b) represents d_i in network (5.5).

Example 5.2. The drive network with time delays is characterized by

$$\begin{aligned} dx_i(t)=&\Big[f_i(t,x_i(t))+\sum_{j=1}^{10}a_{ij}^{(1)}H_1x_j(t-\tau_1)+\sum_{j=1}^{10}a_{ij}^{(2)}H_2x_j(t-\tau_2)\Big]dt\\ &+\Big(g_i^{(1)}\Big(t,x_i(t),x_i(t-\tau_1)\Big)+g_i^{(2)}\Big(t,x_i(t),x_i(t-\tau_2)\Big)\Big)d\mathbb{B}(t),\quad i=1,2,\cdots,10, \end{aligned}\tag{5.4}$$

in which the parameters of $f_i(t,x_i(t))$, $A^{(1)}$, $A^{(2)}$, H_1 and H_2 are the same as Example 5.1, $\tau_1=0.15$, $\tau_2=0.11$, $g_i^{(k)}(t,x_i(t),x_i(t-\tau_k))=0.005\sin x_i(t)+0.01\sin x_i(t-\tau_k)$, $k=1,2$. Accordingly, the response network with different time delays and stochastic perturbations is depicted as follow:

$$\begin{aligned} dy_i(t)=&\Big[f_i(t,y_i(t))+\sum_{j=1}^{3}b_{ij}^{(1)}H_1y_j(t-\tau_1)+\sum_{j=4}^{10}b_{ij}^{(1)}H_1x_j(t-\tau_1)+\sum_{j=1}^{3}b_{ij}^{(2)}H_2y_j(t-\tau_2)\\ &+u_i(t)+\sum_{j=4}^{10}b_{ij}^{(2)}H_2x_j(t-\tau_2)\Big]dt+\Big(g_i^{(1)}\Big(t,y_i(t),y_i(t-\tau_1)\Big)\\ &+g_i^{(2)}\Big(t,y_i(t),y_i(t-\tau_2)\Big)\Big)d\mathbb{B}(t),\quad i=1,2,3. \end{aligned}\tag{5.5}$$

By a simple calculation, one can have

$$\begin{aligned} &\big\|g_i^{(k)}(t,y_i(t),y_i(t-\tau_k))-g_i^{(k)}(t,x_i(t),x_i(t-\tau_k))\big\|\\ =&\big\|0.005(\sin(y_i(t))-\sin(x_i(t)))+0.01(\sin(y_i(t-\tau_k))-\sin(x_i(t-\tau_k)))\big\|\\ \le&0.005\big\|y_i(t)-x_i(t)\big\|+0.01\big\|y_i(t-\tau_k)-x_i(t-\tau_k)\big\|. \end{aligned}$$

It is apparent that they satisfy Hypothesis 4.1. According to Theorem 4.1, drive-response networks (5.4) and (5.5) are synchronized under the adaptive pinning controllers and updating laws (5.6a)-(5.6b)

$$u_i(t)=-d_i(t)e_i(t), \quad \dot{d}_i(t)=r_ie_i^{\mathrm{T}}(t)e_i(t), \quad i=1,2,3, \tag{5.6a}$$

$$\dot{b}_{ij}^{(k)}=\begin{cases}-e_i^{\mathrm{T}}(t)h_k(y_j(t-\tau_k)), & i,j=1,2,3, \quad k=1,2,\\ -e_i^{\mathrm{T}}(t)h_k(x_j(t-\tau_k)), & i=1,2,3, \quad j=4,\cdots,10, \quad k=1,2.\end{cases} \tag{5.6b}$$

Furthermore, the unknown partial topological structures $(\mathcal{G},A_{3\times10}^{(1)})$ and $(\mathcal{G},A_{3\times10}^{(2)})$ can be identified successfully by $(\mathcal{G},B_{3\times10}^{(1)})$ and $(\mathcal{G},B_{3\times10}^{(2)})$, respectively. In this subsection, all the parameters are the same as Example 5.1. Next, the initial values in this subsection are taken as

$$\begin{aligned}
&x_1=(1+0.5k_2,1+0.5k_2,1+0.5k_2)^{\mathrm{T}}, && x_2=(1+0.5k_2,1+0.5k_2,1+0.5k_2)^{\mathrm{T}},\\
&x_3=(2+0.5k_2,2+0.5k_2,2+0.5k_2)^{\mathrm{T}}, && x_4=(-2+0.5k_2,-2+0.5k_2,2+0.5k_2)^{\mathrm{T}},\\
&x_5=(-2+0.5k_2,2+0.5k_2,2+0.5k_2)^{\mathrm{T}}, && x_6=(3+0.5k_2,3+0.5k_2,-3+0.5k_2)^{\mathrm{T}},\\
&x_7=(3+0.5k_2,3+0.5k_2,3+0.5k_2)^{\mathrm{T}}, && x_8(-4+0.5k_2,4+0.5k_2,-2+0.5k_2)^{\mathrm{T}},\\
&x_9=(4+0.5k_2,-4+0.5k_2,4+0.5k_2)^{\mathrm{T}}, && x_{10}=(5+0.5k_2,5+0.5k_2,5+0.5k_2)^{\mathrm{T}},\\
&y_1=(1+0.5k_3,1+0.5k_3,1+0.5k_3)^{\mathrm{T}}, && y_2=(-1.5+0.5k_3,-1.5+0.5k_3,-1.5+0.5k_3)^{\mathrm{T}},\\
&y_3=(2+0.5k_3,2+0.5k_3,2+0.5k_3)^{\mathrm{T}}, && d_1(0)=d_2(0)=d_3(0)=2, \quad r_1=r_2=r_3=1,
\end{aligned}$$

where $k_2=k_3=\cos t$.

The validity of Theorem 4.1 is illustrated in Figs. 3-5. Fig. 3(b) displays the feedback gains $d_i(t)$ in network (5.5), which tends to stabilize with time t. The outer synchronization of drive network (5.4) and response network (5.5) can be clearly verified in Fig. 4. Additionally, Fig. 5 demonstrates the unknown topological structures $(\mathcal{G},A_{3\times10}^{(1)})$ and $(\mathcal{G},A_{3\times10}^{(2)})$ can be identified successfully when $\tau_1=0.15$, $\tau_2=0.11$. For a clearer view, we arbitrarily select some curves for separate presentation. In subgraph (a), it is easily viewed that the curves regarding $b_{21}^{(1)}$, $b_{32}^{(1)}$ get stabilized at 1, the curves regarding $b_{24}^{(1)}$, $b_{210}^{(1)}$ get stabilized at 0.5 and curves regarding $b_{14}^{(1)}$, $b_{15}^{(1)}$, $b_{36}^{(1)}$ tend to 0. In subgraph (b), it obvious that the curves regarding $b_{12}^{(2)}$, $b_{21}^{(2)}$, $b_{26}^{(2)}$ get stabilized at 1, the curves regarding $b_{23}^{(2)}$, $b_{28}^{(2)}$ get stabilized at 0.5 and curves regarding $b_{14}^{(2)}$, $b_{31}^{(2)}$, $b_{35}^{(2)}$ tend to 0. From Fig. 5, one can see that the estimation of unknown topological structures with time delays is successfully obtained by using pinning control strategies.

Remark 5.1. Comparing with Example 5.1, Example 5.2 has shorter time of topology identification and synchronization. A possible explanation for this case is that the drive system and response system in Example 5.2 have time delays. In Fig. 2, the uncertain topological structures are successfully identified at $t=250$, while $t=100$ in Fig. 5. In order to obtain a clear view of synchronization, we arbitrarily display some sample path

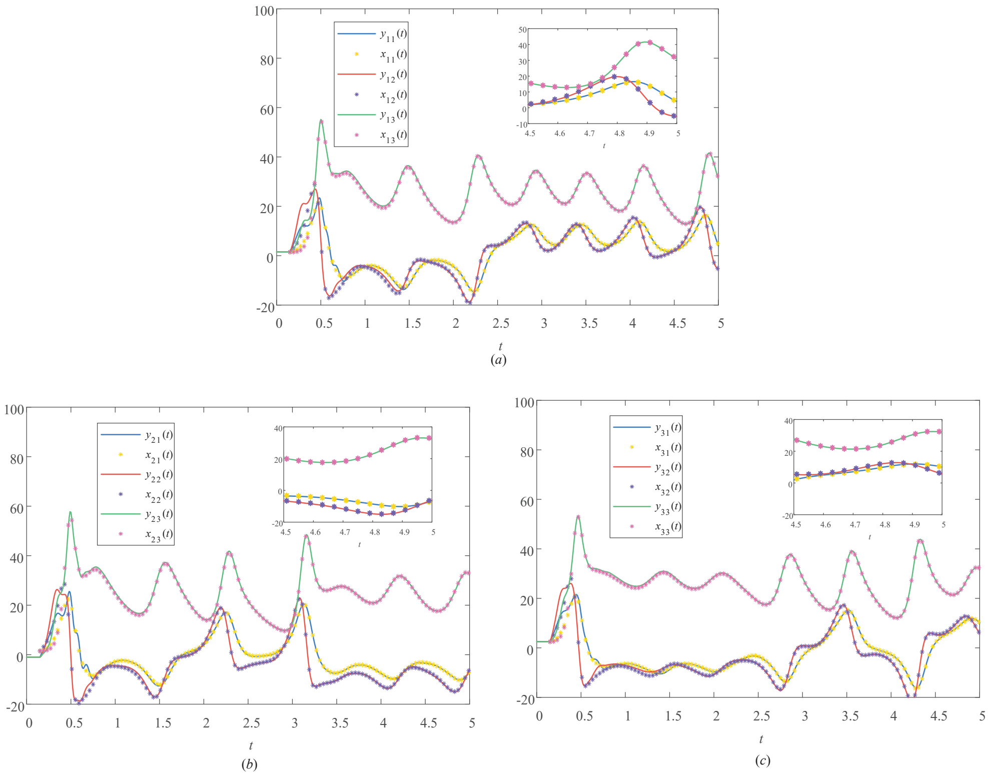

Figure 4: The sample path of drive system (5.4) and response system (5.5).

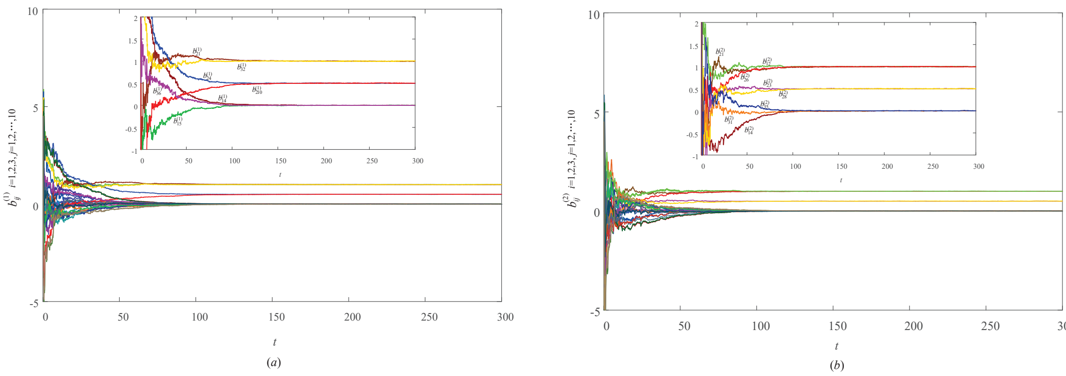

Figure 5: Partial topology identification of (5.4). It clearly shows that these couplings converge to the real values.

of drive system (5.1) and response system (5.2) in Fig. 6. One can find that sample path in Fig. 6 do not reach synchronization at $t\in[4.5,5.2]$. However, in Fig. 4, the sample path

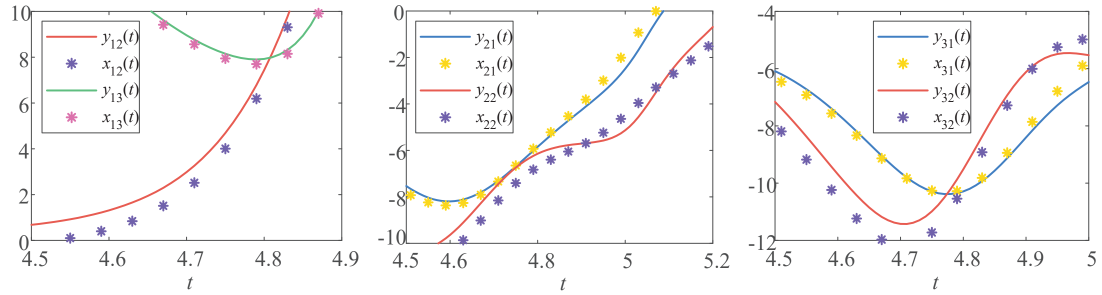

Figure 6: Some sample path of drive system (5.1) and response system (5.2).

reaches synchronization at $t \in [4.5,5]$. The results of the above two examples show that time delays may lead to shorter time of identification and synchronization.

6 Conclusions

In this paper, the network model is described for stochastic multi-weighted complex networks, both time delays and adaptive pinning controllers are considered. Based on Lyapunov method and graph theory, partial topology identification of stochastic multi-weighted complex networks has been intensively investigated. Different from most existing results, our scheme can control not only all vertices, but also a part of the whole vertices. Thus, this scheme can be designed to control the target cost-effectively. It is concluded that under proper adaptive pinning controllers, the corresponding q vertices in drive network and response network can achieve synchronization. In particular, the uncertain partial topological structures can be identified successfully. Moreover, numerical simulations have been provided to show the effectiveness of theoretical results. In addition, the issue of topology identification about stochastic multi-weighted complex networks with color noise is an intriguing one which could be usefully explored in further research.

Appendix

The following basic concepts on graph theory can be found in [41,42]. A directed digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ contains a vertex set $\mathcal{V}=\{1,2,\cdots,q\}$ and a set $\mathcal{E}$ of edges (u,v) leading from initial vertex u to terminal vertex v. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if every edge (v,u) is assigned a positive weight a_{uv}, in which $a_{uv}>0$ if and only if there exists an edge from vertex v to vertex u in $\mathcal{G}$, $A=(a_{uv})_{q\times q}$ is called the weighted matrix. The weight $\mathbb{W}(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all edges.

A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1,i_2,\cdots,i_r\}$ such that its set of edges is $\{(i_s,i_{s+1}):s=1,2,\cdots,r-1\}$. We call $\mathcal{P}$ a directed cycle if $i_1=i_r$. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex s, called the root, if s is not a terminal vertex of any edges, and each of the remaining vertices is a terminal vertex of exactly one edge. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted tree and these roots form a directed cycle. For any pair of distinct vertices in digraph $\mathcal{G}$, if there exists a directed path from one to the other, this digraph is strongly connected. Denote the digraph $\mathcal{G}$ with weighted matrix A as $(\mathcal{G},A)$, the Laplacian matrix of $(\mathcal{G},A)$ is defined as follow:

$$L=\begin{pmatrix} \sum\limits_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1q} \\ -a_{21} & \sum\limits_{k\neq 2} a_{2k} & \cdots & -a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{q1} & -a_{q2} & \cdots & \sum\limits_{k\neq q} a_{qk} \end{pmatrix}.$$

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