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LIE-POISSON NUMERICAL METHOD FOR A CLASS OF STOCHASTIC LIE-POISSON SYSTEMS

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Abstract. We propose a numerical method based on the Lie-Poisson reduction for a class of stochastic Lie-Poisson systems. Such system is transformed to SDE on the dual $\mathfrak{g}^*$ of the Lie algebra related to the Lie group manifold where the system is located, which is also the reduced form of a stochastic Hamiltonian system on the cotangent bundle of the Lie group by momentum mapping. Stochastic Poisson integrators are obtained by discretely reducing stochastic symplectic methods on the cotangent bundle to integrators on $\mathfrak{g}^*$. Stochastic generating functions creating stochastic symplectic methods are used to construct the schemes. An application to the stochastic rigid body system illustrates the theory and provides numerical validation of the method.

Key words. Stochastic Lie-Poisson systems, structure-preserving algorithms, Poisson integrators, Lie-Poisson reduction, Poisson structure, Casimir functions.

1. Introduction

Stochastic Poisson systems (SPSs) are stochastic differential equation systems (SDEs) of the following form ([12]):

$$\begin{aligned} dy(t) &= B(y(t))\left(\nabla H_0(y(t))dt + \sum_{r=1}^{s} \nabla H_r(y(t)) \circ dW_r(t)\right), \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where $y_0 \in \mathbb{R}^m$, $H_r : \mathbb{R}^m \to \mathbb{R}$ $(r = 0, \ldots, s)$ are smooth functions, $\{\mathcal{W}_r(t)\}_{t\geq 0}$ $(r = 0, \ldots, s)$ are independent standard real valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, '$\circ$' indicates that the SDEs are of Stratonovich sense. $B(y) = (b_{ij}(y))$ is called the structure matrix of the SPS, which is is a smooth $m \times m$ matrix–valued function of y with the skew-symmetry $b_{ij}(y) = -b_{ji}(y)$, and satisfies

$$\sum_{l=1}^{m} \left(\frac{\partial b_{ij}(y)}{\partial y^l} b_{lk}(y) + \frac{\partial b_{jk}(y)}{\partial y^l} b_{li}(y) + \frac{\partial b_{ki}(y)}{\partial y^l} b_{lj}(y) \right) = 0, \tag{2}$$

for all $i, j, k \in \{1, \ldots, m\}$. These properties of $B(y)$ guarantee that it induces the Poisson bracket of two smooth functions $K(y)$ and $L(y)$ by

$$\{K, L\}(y) = \nabla K(y)^T B(y) \nabla L(y), \tag{3}$$

which satisfies the skew-symmetry, Jacobi identity and the Leibniz' rule, as the case for canonical Poisson bracket of Hamiltonian systems ([9]).

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In this sense, the SPSs can be considered as generalizations of stochastic Hamiltonian systems (SHSs) ([5, 12, 22]) :

$$
\begin{aligned}
& dy(t) = J^{-1}\left(\nabla H_0(y(t))dt + \sum_{r=1}^{s} \nabla H_r(y(t)) \circ dW_r(t)\right), \\
(4) \qquad & y(0) = y_0,
\end{aligned}
$$

where $J^{-1} = \begin{pmatrix} \mathbf{0}_d & -\boldsymbol{I}_d \\ \boldsymbol{I}_d & \mathbf{0}_d \end{pmatrix}$ and $\boldsymbol{I}_d$ is the d-dimensional identity matrix. When the dimension of a SPS is even, i.e. $m = 2d$, and $B(y) \equiv J^{-1}$, the SPS degenerates to a SHS. If the diffusion part vanishes, i.e. $\nabla H_r \equiv \mathbf{0}$, (1) are deterministic Poisson systems which have got attention since 19th century (see e.g. [9] and references therein). The Poisson and Hamiltonian systems can transform to each other by coordinate transformations or Poisson reductions ([9, 18] and references therein). Numerical methods for SPSs can be constructed using these properties, such as those based on the Darboux-Lie theorem ([9]) which transform symplectic methods for SHSs to Poisson integrators for SPSs via coordinate transformations ([12]). In this paper, however, we attempt another way, to construct Poisson integrators via Poisson reductions for a class SPSs, generalizing the deterministic Lie-Poisson reduction numerical approach ([4, 9, 13, 26]) to stochastic cases.

Almost surely, the phase flow of the SPS (1) $\varphi_{t,\omega} : y \to \varphi_{t,\omega}(y)$ possesses the Poisson structure, i.e. ([3, 12])

$$
(5) \qquad \frac{\partial \varphi_{t,\omega}(y)}{\partial y} B(y) \frac{\partial \varphi_{t,\omega}(y)}{\partial y}^T = B(\varphi_{t,\omega}(y)), \quad \forall t \geq 0, \quad a.s.
$$

If the rank of $B(y)$ is not full such that there exist functions $C(y)$ yielding

$$
B(y)\nabla C(y) = \mathbf{0}
$$

almost surely, then these functions are called Casimir functions ([9]) of the SPSs, which are invariants of the systems, since almost surely

$$
dC(y) = \nabla C(y)^T dy = \nabla C(y)^T B(y)\left(\nabla H_0(y)dt + \sum_{r=1}^{s} \nabla H_r(y) \circ dW_r(t)\right) = 0.
$$

Now we consider special structure matrices $B(y)$ whose elements depend linearly on y, i.e.

$$
(6) \qquad b_{ij}(y(t)) = \sum_{k=1}^{m} C_{ji}^k y^k(t), \quad \forall i,j = 1,\ldots,m.
$$

Analog to deterministic case ([9]), SPSs (1) with $B(y)$ fulfilling (6) are called stochastic Lie-Poisson systems (SLPSs) ([3, 11, 16]). The skew-symmetry as well as properties (2) and (6) of $B(y)$ make it possible to define a Lie bracket calculation using the constants C_{ij}^k in (6) by:

$$
(7) \qquad [E_i, E_j] = \sum_{k=1}^{m} C_{ij}^k E_k, \quad i,j = 1,\ldots,m
$$

on a vector space with basis $\{E_i\}$ $(i = 1,\ldots,m)$. The vector space equipped with the Lie bracket calculation constituts a Lie algebra ([9]), denoted by $\mathfrak{g}$.

Lie-Poisson systems arise in celestial mechanics, robotics, fluid mechanics, and rigid body, etc. Typical examples include the Vlasov-Poisson equations, the Euler equations for rigid bodies ([4, 9, 13, 18, 19]). Numerical methods for deterministic Lie-Poisson systems have been developed during the last decades, including the

Lie-Poisson reduction methods ([4, 13, 26]), the splitting approach ([14, 19]), the Lie-group method ([7]), the variational approach on the dual of Lie algebra ([17]), Lie-Poisson methods on $\mathbb{R}^3$ ([20]), and so on. Methods for stochastic Lie-Poisson systems appeared in recent years, such as those in [3, 16] based on splitting or Lie group methods, etc. These methods aim to preserve the Poisson structure and/or the Casimir functions of the original systems. A numerical method $y_n \to y_{n+1}$ is called a Poisson integrator if it preserves both the Poisson structure and the Casimir functions ([9, 12]), namely for any $n \in \mathbb{N}$ it holds (in 'a.s.' sense in stochastic cases)

$$\frac{\partial y_{n+1}}{\partial y_n} B(y_n) \frac{\partial y_{n+1}}{\partial y_n}^T = B(y_{n+1}), \tag{8}$$

$$C(y_{n+1}) = C(y_n). \tag{9}$$

It has been shown theoretically and empirically that such structure-preserving integrators behave much better than general methods, especially in long time simulations([3, 9, 12, 24, 25] and references therein).

Rigid body system has been a benchmark model in geometric mechanics, so is its stochastic counterpart in stochastic geometric mechanics ([11]). In [3, 5, 12, 15, 24], the following stochastic rigid body system was studied

$$\begin{pmatrix} dy^1 \\ dy^2 \\ dy^3 \end{pmatrix} = \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix} \begin{pmatrix} y^1/I_1 \\ y^2/I_2 \\ y^3/I_3 \end{pmatrix} (dt + c \circ dW(t)), \tag{10}$$

where $(y^1, y^2, y^3)^T \in \mathbb{R}^3$, c is a constant, and I_j $(j = 1, 2, 3)$ are moments of inertia. It is easy to see that the stochastic rigid body system is a stochastic Lie-Poisson system. In the above mentioned papers, various numerical methods were constructed to preserve the Poisson structure, Casimir function, or the energy of the system. However, none of them treated the system from the Lie-Poisson reduction point of view. Generally, to our best knowledge, there is still no Lie-Poisson reduction numerical methods for stochastic Lie-Poisson systems.

In this paper, we focus on SLPSs with one noise of the following form:

$$dy = B(y)\nabla H(y)(dt + c \circ dW(t)), \tag{11}$$

where $B(y)$ is skew-symmetric and satisfies (2) and (6). We generalize the deterministic Lie-Poisson reduction method ([4, 13, 26]) to SLPSs (11), to construct Lie-Poisson integrators for them. Lie-Poisson reduction methods for more general SLPSs will be topics of further study.

Contents are arranged as follows. In Section 2 we derive the dual Lie algebra representation of SLPSs (11). In Section 3 we give the stochastic Lie-Poisson reduction method. The method is applied to the stochastic rigid body system (10) in Section 4 with some numerical experiments. Section 5 is a brief conclusion.

2. Dual Lie algebra representation of SLPSs

Given a SLPS (11) with $B(y) = (b_{ij}(y))$ satisfying (6), namely

$$b_{ij}(y(t)) = \sum_{k=1}^{m} C_{ji}^k y^k(t), \quad \forall i, j = 1, \ldots, m.$$

Let $\{E_i\}$ $(i = 1, \ldots, n)$ be the basis of a Lie algebra $\mathfrak{g}$ on which the Lie bracket $[\cdot, \cdot]$ is defined by

$$[E_i, E_j] = \sum_{k=1}^{m} C_{ij}^k E_k, \quad i, j = 1, \ldots, m,$$

and let G be the Lie group associated with $\mathfrak{g}$, i.e. $\mathfrak{g} = T_I G$.

Denote by $\mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$, namely the vector space of all linear forms $Y : \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g}$. The duality between $\mathfrak{g}^*$ and $\mathfrak{g}$ is represented by $\langle Y, X\rangle$ for $Y \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. Let $F_i \in \mathfrak{g}^*$ $(i = 1, \ldots, m)$ satisfy

$$\langle F_i, E_j\rangle = \delta_{ij}, \tag{12}$$

where δ is the Kronecker function. It is not difficult to verify that $\{F_i\}$ $(i = 1, \ldots, m)$ form a basis of $\mathfrak{g}^*$ ([9]). We have the following theorem.

Theorem 2.1. *Let $\{y(t)\}_{t\geq 0}$ be the solution of* (11)*, define $Y(t) = \sum_{i=1}^m y^i(t)F_i \in \mathfrak{g}^*$, and let $H(Y) := H(y)$. Then the SLPSs* (11) *is equivalent to the following stochastic differential equation on $\mathfrak{g}^*$:*

$$\langle dY, X\rangle = \langle Y, [H'(Y), X]\rangle(dt + c \circ dW(t)), \quad \forall X \in \mathfrak{g}, \tag{13}$$

where $H'(Y) = \sum_{i=1}^m \frac{\partial H(y)}{\partial y^i} E_i$.

Proof. According to the definition of $H'(Y)$ and the relation (7),

$$\begin{aligned}\langle Y, [H'(Y), E_i]\rangle &= \langle Y, \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j}[E_j, E_i]\rangle \\ &= \sum_{j=1}^m \sum_{k=1}^m \frac{\partial H(y)}{\partial y^j} C_{ji}^k \langle Y, E_k\rangle.\end{aligned}$$

Due to $\langle dY, E_i\rangle = dy^i, \langle Y, E_i\rangle = y^i$, as well as the linearity of the Lie bracket and the duality calculations, (13) is equivalent to

$$\begin{aligned}dy^i &= \langle dY, E_i\rangle \\ &= \langle Y, [H'(Y), E_i]\rangle(dt + c \circ dW(t)) \\ &= \sum_{j=1}^m \left(\sum_{k=1}^m C_{ji}^k y^k\right) \frac{\partial H(y)}{\partial y^j}(dt + c \circ dW(t)) \\ &= \sum_{j=1}^m b_{ij}(y)\frac{\partial H(y)}{\partial y^j}(dt + c \circ dW(t)), \quad i = 1, \ldots, m,\end{aligned}$$

which is just the SLPSs (11). The last equality has used the relation (6). □

Thus, we found the equivalent stochastic differential equation on $\mathfrak{g}^*$ for the SLPSs (11).

3. Stochastic Lie-Poisson reduction methods

3.1. From cotangent bundle to dual Lie algebra. Let the Lie group G be a subgroup of $GL(m) = \{A : A \in \mathbb{R}^{m\times m}, \det(A) \neq 0\}$ given by

$$G = \{Q : Q \in GL(m), g_i(Q) = 0, i = 1, \ldots, \kappa\}, \tag{14}$$

where $g_i : GL(m) \to \mathbb{R}$ $(i = 1, \ldots, \kappa)$ are certain constraint functions defining the group G. We consider the following SHS on G ([10, 11]):

$$\begin{cases} dP = \left(-\nabla_Q H(P, Q) - \sum_{i=1}^\kappa \lambda_i \nabla_Q g_i(Q)\right)(dt + c \circ dW(t)), \ P(t_0) = p, \\ dQ = \nabla_P H(P, Q)(dt + c \circ dW(t)), \ Q(t_0) = q, \\ g_i(Q) = 0, \quad i = 1, \ldots, \kappa, \end{cases} \tag{15}$$

where $P, Q, p, q \in \mathbb{R}^{m\times m}$, $\nabla_Q H = \left(\frac{\partial H}{\partial Q_{ij}}\right)$ $(i, j = 1\dots, m)$, λ_i $(i = 1, \dots, \kappa)$ are Lagrange parameters which can be written as functions of (P, Q) under certain conditions ([10]). The tangent space of G at Q is $T_Q G = \{v \in \mathbb{R}^{m\times m} : g'(Q)v = 0\}$, where $g = (g_1, \dots, g_\kappa)^T$, and $0 \in \mathbb{R}^{\kappa m\times m}$.

Differentiate the constraint $g(Q) = 0$ with using the Stratonovich differential chain rule, we get $g'(Q)dQ = 0$. This means that $\dot{Q} \in T_Q G$ a.s., which implies $P \in T_Q^* G$ a.s. ([9]). Therefore the SHS (15) can be seen as an equation system on the cotangent bundle T^*G of G, where

$$T^*G = \{(V, U) : U \in G, V \in T_U^* G\}.$$

A Hamiltonian function $H(P, Q)$ is said to be left-invariant under the group action of G, if ([9])

$$H(U^T P, U^{-1}Q) = H(P, Q), \quad \forall U \in G.$$

In this situation we have

$$H(P, Q) = H(Q^T P, I), \quad \nabla_P H(P, Q) = Q\nabla_P H(Q^T P, I),$$

and the second equation of (15) becomes

$$dQ = Q\nabla_P H(Q^T P, I)(dt + c \circ dW(t)). \tag{16}$$

Since $T_Q G = \{QX : X \in \mathfrak{g}\}$ and $\dot{Q} \in T_Q G$, we obtain

$$\nabla_P H(Q^T P, I)(1 + c \circ \dot{W}(t)) \in \mathfrak{g} = T_I G, \ \ a.s.,$$

where $\dot{W}(t)$ is the formal derivative of $W(t)$. This implies immediately that $Q^T P \in T_I^* G = \mathfrak{g}^*$ *a.s.*. Now we can consider $H(P, Q)$ as a function of $Y = Q^T P$, that is $H(P, Q) = H(Y)$ whereby $H : \mathfrak{g}^* \to \mathbb{R}$.

Theorem 3.1. *Consider the stochastic Hamiltonian system* (15) *on the Lie group* G (14)*. If the Hamiltonian function* H *is left-invariant under group actions of* G*, and* $(P(t), Q(t)) \in T^*G$ *solves the system* (15)*, then* $Y(t) = Q(t)^T P(t) \in \mathfrak{g}^*$ *solves the SDE* (13) *on the dual* $\mathfrak{g}^*$ *of the Lie algebra* $\mathfrak{g}$ *of* G*.*

Proof. For $Q^T P = Y \in \mathfrak{g}^*$, we have $Y = \sum_{j=1}^m y^j F_j$ so that $H(P, Q) = H(Y) = H(y)$ with $y = (y^1, \dots, y^m)^T$. The duality between $T_Q^* G$ and $T_Q G$ is given by the matrix inner product $(\cdot, \cdot)$ as

$$\langle P, V\rangle = (P, V) = trace(P^T V), \quad \forall P \in T_Q^* G, V \in T_Q G.$$

From $\nabla_A trace(A^T B) = B$, we obtain

$$\begin{aligned}\nabla_P H(P, Q) &= \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j}\nabla_P y_j = \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j}\nabla_P \langle Y, E_j\rangle \\ &= \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j}\nabla_P trace(P^T Q E_j) = \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j} Q E_j = QH'(Y),\end{aligned}$$

where we have used $\langle Y, E_j\rangle = y^j$ and $H'(Y) = \sum_{j=1}^m \frac{\partial H(y)}{\partial y^j} E_j$. Similarly, based on the equality $y^j = trace(P^T Q E_j) = trace(Q^T P E_j^T)$, we obtain

$$\nabla_Q H(P, Q) = PH'(Y)^T.$$

Thus the SHS (15) can be written as

$$dP = \left(-PH'(Y)^T - \sum_{i=1}^{\kappa} \lambda_i \nabla_Q g_i(Q)\right)(dt + c \circ dW(t)),$$

$$dQ = QH'(Y)(dt + c \circ dW(t)). \tag{17}$$

Due to the Stratonovich differential chain rule,

$$\begin{aligned} dY = d(Q^T P) &= (P^T \circ dQ)^T + Q^T \circ dP \\ &= \left(H'(Y)^T Q^T P + Q^T(-PH'(Y)^T - \sum_{i=1}^{\kappa} \lambda_i \nabla_Q g_i(Q))\right)(dt + c \circ dW(t)) \\ &= \left(H'(Y)^T Y - YH'(Y)^T - \sum_{i=1}^{\kappa} \lambda_i Q^T \nabla_Q g_i(Q)\right)(dt + c \circ dW(t)). \end{aligned} \tag{18}$$

Note that $\forall X \in \mathfrak{g}$,

$$\begin{aligned} &\langle H'(Y)^T Y, X\rangle = trace(Y^T H'(Y) X) = \langle Y, H'(Y)X\rangle, \\ &\langle YH'(Y)^T, X\rangle = trace(H'(Y)Y^T X) = trace(Y^T X H'(Y)) = \langle Y, XH'(Y)\rangle, \\ &\langle Q^T \nabla_Q g_i(Q), X\rangle = \langle \nabla_Q g_i(Q), QX\rangle = 0, \end{aligned} \tag{19}$$

whereby the last equality is owing to $QX \in T_Q G$. From (19) and (18) we get

$$\langle dY, X\rangle = \langle Y, [H'(Y), X]\rangle (dt + c \circ dW(t)), \quad \forall X \in \mathfrak{g},$$

which is just (13). □

3.2. Discrete reduction. Consider two left-invariant smooth functions $K(P,Q)$ and $L(P,Q)$ on T^*G. By $Y = Q^T P$ they become functions $K(Y)$ and $H(Y)$ on $\mathfrak{g}^*$. Due to $Y = \sum_{j=1}^m y^j F_j$ they can be viewed as functions $K(y)$ and $L(y)$ on $\mathbb{R}^m$. In T^*G they have the canonical Poisson bracket defined by ([9])

$$\{K, L\}_{can} = \sum_{k,l=1}^{m} \left(\frac{\partial K}{\partial Q_{kl}} \frac{\partial L}{\partial P_{kl}} - \frac{\partial K}{\partial P_{kl}} \frac{\partial L}{\partial Q_{kl}}\right). \tag{20}$$

On $\mathbb{R}^m$, their Poisson bracket defined by the structure matrix $B(y)$ (where $y = (y^1, \ldots, y^m)^T$) is

$$\{K, L\} = \sum_{i,j=1}^{m} \frac{\partial K}{\partial y^i} b_{ij} \frac{\partial L}{\partial y^j}. \tag{21}$$

Lemma 3.1. *([9]) For the left-invariant smooth functions K and L, let $Q^T P = Y = \sum_{j=1}^m y^j F_j \in \mathfrak{g}^*$, we have*

$$\{K, L\}(y) = \langle Y, [L'(Y), K'(Y)]\rangle = \{K, L\}_{can}(P, Q), \tag{22}$$

where $y = (y^1, \ldots, y^m)^T$, $K'(Y) = \sum_{j=1}^m \frac{\partial K(y)}{\partial y^j} E_j \in \mathfrak{g}$.

Now we consider a symplectic integrator for a SHS on T^*G with left-invariant Hamiltonian:

$$(P_1, Q_1) = \Phi_h(P_0, Q_0). \tag{23}$$

We assume that the symplectic integrator is also left-invariant, that is:

$$\Phi_h(U^T P_0, U^{-1} Q_0) = (U^T P_1, U^{-1} Q_1), \quad \forall U \in G. \tag{24}$$

Let $Y_1 = Q_1^T P_1$, $Y_0 = Q_0^T P_0$. From (23) and by (24) it follows

$$(P_1, Q_1) = \Phi_h(P_0, Q_0) = \Phi_h(Q_0^T P_0, I) =: (\Phi_h^1(Q_0^T P_0), \Phi_h^2(Q_0^T P_0)),$$

which implies

$$Y_1 = Q_1^T P_1 = \Phi_h^2(Q_0^T P_0)^T \Phi_h^1(Q_0^T P_0) =: \Psi_h(Q_0^T P_0) = \Psi_h(Y_0).$$

Thus the symplectic integrator Φ_h induces a mapping Ψ_h on $\mathfrak{g}^*$:

$$Y_1 = \Psi_h(Y_0), \tag{25}$$

which is a numerical scheme for (13). Further by considering the relation between $Y \in \mathfrak{g}^*$ and its coordinates $y = (y^1, \ldots, y^m)^T \in \mathbb{R}^m$, i.e. $Y_i = \sum_{j=1}^m y_i^j F_j$, $(i = 0, 1)$, one gets a mapping on $\mathbb{R}^m$:

$$y_1 = \psi_h(y_0), \tag{26}$$

where $y_i = (y_i^1, \ldots, y_i^m)^T$ $(i = 0, 1)$, which is an integrator for the SLPS (11). Analogous to the deterministic case ([9]), by using Lemma 3.1, we can prove the following result for the stochastic case.

Theorem 3.2. *If $\Phi_h(P, Q)$ is a left-invariant symplectic integrator for the SHS* (15)*, then its reduction $\psi_h(y)$ is a Poisson map for the SLPS* (11)*.*

We omit the proof since it can follow the same way as its deterministic counterpart given in [9]. This theorem provides the possibility of producing Poisson numerical schemes for SLPSs by reducing left-invariant symplectic integrators of its SHS formulation.

4. Lie-Poisson integrators for the stochastic rigid body system

In this section we apply the above method to the stochastic rigid body system (10). We mainly extend the generating function approach and exponential map used in Lie-Poisson reduction numerical methods solving the deterministic rigid body system ([4, 13, 26]) to the stochastic context.

Here we recall the stochastic rigid body system (10):

$$\begin{pmatrix} dy^1 \\ dy^2 \\ dy^3 \end{pmatrix} = \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix} \begin{pmatrix} y^1/I_1 \\ ?y^2/I_2 \\ y^3/I_3 \end{pmatrix} (dt + c \circ dW(t)), \tag{27}$$

where $H(y) = \frac{1}{2}\left(\frac{(y^1)^2}{I_1} + \frac{(y^2)^2}{I_2} + \frac{(y^3)^2}{I_3}\right)$, and $C(y) = \frac{1}{2}\left((y^1)^2 + (y^2)^2 + (y^3)^2\right)$ is a Casimir function of the system. The structure matrix is

$$B(y) = \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix}.$$

4.1. Generating function for stochastic symplectic mapping.

Given a standard SHS ([22]):

$$\begin{cases} dP = -\frac{\partial H_0(P,Q)}{\partial Q} dt - \sum_{r=1}^s \frac{\partial H_r(P,Q)}{\partial Q} \circ dW_r(t), & P(t_0) = p, \\ dQ = \frac{\partial H_0(P,Q)}{\partial P} dt + \sum_{r=1}^s \frac{\partial H_r(P,Q)}{\partial P} \circ dW_r(t), & Q(t_0) = q, \end{cases} \tag{28}$$

where $P, Q, p, q \in \mathbb{R}^d$. It is proved that the phase flow of (28) preserves symplectic structure ([22]), i.e.

$$\mathrm{d}P(t) \wedge \mathrm{d}Q(t) = \mathrm{d}p \wedge \mathrm{d}q, \quad \forall t \geq t_0.$$

A mapping $(p^T, q^T)^T \to (P^T, Q^T)^T$ is symplectic, if and only if there exists locally a smooth function $S(q, Q, t)$ such that ([6, 8, 9, 23])

$$P^T dQ - p^T dq = dS.$$

For SHS (28), the generating function $S(q, Q, t, \omega)$ for its symplectic flow can be found by solving the stochastic Hamilton-Jacobi partial differential equation (H-J PDE) ([6, 23])

$$dS(q, Q, t, \omega) = -H_0(\frac{\partial S}{\partial Q}, Q)dt - \sum_{r=1}^{s} H_r(\frac{\partial S}{\partial Q}, Q) \circ dW_r(t) \tag{29}$$

with the initial condition

$$\frac{\partial S}{\partial Q_i}(q, q, t_0) + \frac{\partial S}{\partial q_i}(q, q, t_0) = 0, \ i = 1, \cdots, d.$$

If the matrix $\left(\frac{\partial^2 S}{\partial q_i \partial Q_j}\right)$ is almost surely invertible for $t \in [t_0, \tau]$, where $\tau > t_0$ is a stopping time, then the solution S of (29) almost surely generates the flow $\phi_{t,\omega} : (p^T, q^T)^T \to (P(t, \omega)^T, Q(t, \omega)^T)^T$ of the SHS (28) via the relation ([6])

$$P(t, \omega) = \frac{\partial S(q, Q(t), t, \omega)}{\partial Q}, \quad p = -\frac{\partial S(q, Q(t), t, \omega)}{\partial q}. \tag{30}$$

One can construct a symplectic numerical scheme for the SHS (28) through the relation (30), by finding an approximate solution S to the stochastic H-J PDE ([6, 23]).

4.2. Approximate generating function. Using the same way as in [26], one can prove that if the Hamiltonians $H_i, \ i = 0, \dots s$ are left-invariant, then the generating function S, i.e. the solution to the stochastic H-J PDE (29) is also left-invariant. In this situation it holds

$$S(Q, q) = S(I, Q^{-1}q) = \tilde{S}(g), \quad g = Q^{-1}q.$$

Then the relation (30) can be written as ([13])

$$\begin{aligned} p &= -\frac{\partial S(Q(t), q, t)}{\partial q} = -\frac{\partial \tilde{S}(Q^{-1}q)}{\partial q} \\ &= -\frac{\partial \tilde{S}(L_{Q^{-1}}q)}{\partial q} = -L^*_{Q^{-1}} \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q}, \end{aligned} \tag{31}$$

$$\begin{aligned} P &= \frac{\partial S(Q(t), q, t)}{\partial Q} = \frac{\partial \tilde{S}(Q^{-1}q)}{\partial Q} \\ &= \frac{\partial \tilde{S}(R_q Q^{-1})}{\partial Q} = -L^*_{Q^{-1}} R^*_{Q^{-1}} R^*_q \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q}, \end{aligned} \tag{32}$$

where for $g \in G$, the mappings $L_g : G \to G : u \to gu$, and $R_g : G \to G : u \to ug$ are left and right translations by g on $u \in G$ respectively ([1]). These translations can be transformed to left and right translations L^*_g and R^*_g on T^*G.

The mapping $J_R : T^*G \to \mathfrak{g}^*$ ([13]) is defined as $J_R(P, Q) = L^*_Q P$. For the stochastic rigid body system, it corresponds to the transformation $J_R(P, Q) = L^*_Q P = Q^T P = Y \in \mathfrak{g}^*$, the so-called momentum mapping.

Similar to the discussion in [13], for the stochastic rigid body system(10) where $G = SO(3)$ and H is left-invariant, one can reduce its stochastic H-J PDE (29) ($s=1$, $H_1 = cH_0 := cH$) on T^*G to that on $\mathfrak{g}^*$:

$$d\tilde{S}(g) = -H(-R_g^* \frac{\partial \tilde{S}}{\partial g})(dt + c \circ dW(t)), \tag{33}$$

where $g = Q^{-1}q$. We define

$$\begin{aligned} Y_0 &= J_R(p,q) = L_q^* p = -L_q^* L_{Q^{-1}}^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q} \\ &= -L_{Q^{-1}q}^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q} = -L_g^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q}, \\ Y &= J_R(P,Q) = L_Q^* P = -L_Q^* L_{Q^{-1}}^* R_{Q^{-1}}^* R_q^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q} \\ &= -R_{Q^{-1}q}^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q} = -R_g^* \frac{\partial \tilde{S}}{\partial g}|_{g=Q^{-1}q}. \end{aligned} \tag{34}$$

Thus we get the reduced mapping $Y = R_g^* L_{g^{-1}}^* Y_0 = Ad_{g^{-1}}^* Y_0$ on $\mathfrak{g}^*$ from the mapping $(p^T, q^T)^T \to (P^T, Q^T)^T$ on T^*G.

Applying the method in [4, 13], we describe the generating function approach using the exponential map. $\forall g \in G$, choose $\xi \in \mathfrak{g}$ such that $g = \exp(\xi)$. Thus the equation (33) becomes

$$\begin{aligned} d\tilde{S} &= -H(-d\tilde{S} \cdot \psi(ad_\xi))(dt + c \circ dW(t)) \\ &= -H^0(-d\tilde{S} \cdot \psi(ad_\xi))dt - H^1(-d\tilde{S} \cdot \psi(ad_\xi)) \circ dW_1(t), \end{aligned} \tag{35}$$

where $H^0 = H, H^1 = cH, W_1(t) = W(t)$, $d\tilde{S} \cdot \psi(ad_\xi) = R_g^* d\tilde{S}$, $d\tilde{S} \cdot \chi(ad_\xi) = L_g^* d\tilde{S}$. Meanwhile we have

$$Y_0 = -d\tilde{S} \cdot \chi(ad_\xi), \quad Y = -d\tilde{S} \cdot \psi(ad_\xi),$$

and

$$\begin{aligned} \chi(ad_\xi) &= I_d + \frac{1}{2} ad_\xi + \frac{1}{12} ad_\xi^2 + \cdots, \\ \psi(ad_\xi) &= \chi(ad_\xi) - ad_\xi, \end{aligned}$$

where the condition $g = Q^{-1}q$ is transformed to $\xi|_{t=0} = I_d$.

Let S_0 be the function that generates the identity map on the Lie algebra. Next we consider the expansion of the generating function $\tilde{S}(\xi, t, \omega)$ using the method in [6]:

$$\tilde{S}(\xi, t, \omega) = S_0(\xi) + \Sigma_\alpha S_\alpha(\xi) J_\alpha, \tag{36}$$

where $\alpha = \{j_1, ..., j_l\}, j_i \in \{0, 1\}$, and J_α is the Stratonovich multiple integral

$$J_\alpha = \int_0^t \int_0^{s_l} \cdots \int_0^{s_2} \circ dW_{s_1}^{j_1} \cdots \circ dW_{s_{l-1}}^{j_{l-1}} \circ dW_{s_l}^{j_l}$$

with $ds =: dW_s^0$.

For $l > 1$, denote by $\alpha = (j_1, j_2, \ldots, j_l)$ a multi-index of length l ([6]), and $\alpha- = (j_1, j_2, \ldots, j_{l-1})$. For any two multi-indices $\alpha = (j_1, j_2, \ldots, j_l)$ and $\alpha' = (j_1', j_2', \ldots, j_{l'}')$, the concatenation operation $*$ of them is defined as

$$\alpha * \alpha' = (j_1, j_2, \ldots, j_l, j_1', j_2', \ldots, j_{l'}').$$

The concatenation operation between a multi-index set Λ and a multi-index α is defined as

$$\Lambda * \alpha = \{\alpha' * \alpha | \alpha' \in \Lambda\}.$$

The multiple Stratonovich integrals possess the following property.

Lemma 4.1. *([6])*

$$J_\alpha J_{\alpha'} = \sum_{\beta \in \Lambda_{\alpha,\alpha'}} J_\beta,$$

where $\alpha = (j_1, j_2, \dots, j_l), \alpha' = (j'_1, j'_2, \dots, j'_{l'})$, and $\Lambda_{\alpha,\alpha'}$ is multi-index set relying on α and α' and given by the following formulae:

$$\Lambda_{\alpha,\alpha'} = \begin{cases} \{(j_1, j'_1), (j'_1, j_1)\}, & if\ l = 1\ and\ l' = 1, \\ \{\Lambda_{(j_1),\alpha'-} * (j'_{l'}), \alpha' * (j_l)\}, & if\ l = 1\ and\ l' \neq 1, \\ \{\Lambda_{\alpha-,(j'_1)} * (j_l), \alpha * (j'_l)\}, & if\ l \neq 1\ and\ l = 1, \\ \{\Lambda_{\alpha-,\alpha'} * (j_l), \Lambda_{\alpha,\alpha'-} * (j'_{l'})\}, & if\ l \neq 1\ and\ l \neq 1. \end{cases}$$

Substituting (36) into (35) and using the same method as in [6], as well as Lemma 4.1, we have

$$\begin{aligned} \tilde{S} &= S_0 - \sum_{r=0}^{1} \int_0^t H^r(-d\tilde{S} \cdot \psi(ad_\xi)) \circ dW_s^r(t) \\ &= S_0 - \sum_{r=0}^{1} \int_0^t \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i H^r}{\partial V}(V)(\sum_\alpha (-dS_\alpha J_\alpha) \cdot \psi(ad_\xi))^i \circ dW_s^r \\ &= S_0 - \sum_{r=0}^{1} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i H^r}{\partial V}(V) \sum_{\alpha_1,..,\alpha_i} (-dS_{\alpha_1}\psi)..(-dS_{\alpha_i}\psi) \int_0^t \prod_{k=1}^{i} J_{\alpha_k} \circ dW_s^r \\ &= S_0 - \sum_{r=0}^{1} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i H^r}{\partial V}(V) \sum_{\alpha_1,..,\alpha_i} (-dS_{\alpha_1}\psi)..(-dS_{\alpha_i}\psi) \sum_{\beta \in \Lambda_{\alpha_1,..,\alpha_i}} J_{\beta*(r)} \\ &= S_0 - \sum_{r=0}^{1} \sum_{i=0}^{\infty} \sum_{\alpha_1,..,\alpha_i} \sum_{\beta \in \Lambda_{\alpha_1,..,\alpha_i}} \frac{1}{i!} \frac{\partial^i H^r}{\partial V}(V)(-dS_{\alpha_1}\psi)..(-dS_{\alpha_i}\psi) J_{\beta*(r)}, \end{aligned} \tag{37}$$

where $V = -dS_0 \cdot \psi(ad_\xi)$. Thus we obtain

$$\begin{aligned} S_{(0)} &= -H^0(V), & S_{(1)} &= -H^1(V), \\ S_{(0,0)} &= \frac{\partial H^0}{\partial V}(V)(dS_{(0)} \cdot \psi), & S_{(1,1)} &= \frac{\partial H^1}{\partial V}(V)(dS_{(1)} \cdot \psi), \\ S_{(0,1)} &= \frac{\partial H^1}{\partial V}(V)(dS_{(0)} \cdot \psi), & S_{(1,0)} &= \frac{\partial H^0}{\partial V}(V)(dS_{(1)} \cdot \psi), \\ &\vdots \end{aligned}$$

Now we truncate the series of $\tilde{S}$ to obtain

$$\bar{S} = S_0 + S_{(1)} J_{(1)} + S_{(0)} J_{(0)} + S_{(1,1)} J_{(1,1)}. \tag{38}$$

Noting that $H_0 = H,\ H_1 = cH$, we get the truncated generating function

$$\bar{S} = S_0 - cH(-R_g^* dS_0)J_{(1)} - H(-R_g^* dS_0)J_{(0)} \tag{39}$$
$$+c^2 \frac{\partial H}{\partial V}(-R_g^* dS_0)(-R_g^* dH(-R_g^* dS_0))J_{(1,1)},$$

which is an approximation of $\tilde{S}$. Then we can use (34) and $\bar{S} \approx \tilde{S}$ to construct a Poisson mapping $Y_0 \to Y$ on $\mathfrak{g}^*$.

4.3. The numerical scheme. The geometric settings for the stochastic rigid body are the same with those for the deterministic rigid body ([2, 26]), stated as follows. The Lie group is $G = SO(3)$, and the Lie algebra is $\mathfrak{g} = so(3)$ which is the space of 3×3 skew-symmetric matrices. Denote by $\hat{y} \in so(3)$ the skew-symmetric matrix related to the vector $y \in \mathbb{R}^3$, defined by $\hat{y} \cdot v = y \times v$ for a vector $v \in \mathbb{R}^3$. The above relation $y \leftrightarrow \hat{y}$ also defines an isomorphism between $\mathbb{R}^3$ and $so(3)$. The Lie bracket on $\mathfrak{g}$ is defined as $[\hat{v}, \hat{w}] = \hat{v}\hat{w} - \hat{w}\hat{v}$, for $\hat{v},\ \hat{w} \in so(3)$, corresponding to the cross product $v \times w$ on $\mathbb{R}^3$.

Using the killing form $\langle \hat{a}, \hat{b} \rangle = \frac{1}{2} trace(\hat{a}^T \hat{b})$ for $a, b \in \mathbb{R}^3$, which corresponds to the inner product $a \cdot b$ on $\mathbb{R}^3$, one can identify $so(3)$ and $so(3)^*$ by defining $\langle \hat{v}, \hat{w} \rangle = v \cdot w$, where the left-hand side of the equation is the duality calculation between $\hat{v} \in so(3)^*$ and $\hat{w} \in so(3)$. Then denoting $\left(\frac{1}{I_1}, \frac{1}{I_2}, \frac{1}{I_3}\right)^T =: I^{-1}$, $\left(\frac{y^1}{I_1}, \frac{y^2}{I_2}, \frac{y^3}{I_3}\right)^T :=$ $I^{-1}(y)$ one can write

$$H(y) = H(\hat{y}) = \frac{1}{2} y \cdot I^{-1}(y) = \frac{1}{2} \langle \hat{y}, \hat{I^{-1}}(\hat{y}) \rangle$$

where $\hat{I^{-1}}(\hat{y}) = \widehat{I^{-1}(y)}$.

According to [26], for free rigid body, the function $S_0 = traceA$ generates the identity map, where A is an orthogonal matrix. Then (34) can be written as

$$\hat{y}_0 = -A^T \cdot \nabla \tilde{S}(A), \qquad \hat{y} = -\nabla \tilde{S}(A) \cdot A^T, \tag{40}$$

whereby the result has been skew-symmetrized ([26]) to be kept in $so(3)$.

For the S_0 given above we have $-R_A^* dS_0 = \frac{1}{2}(A - A^T)$. Thus it follows from (39) that

$$\bar{S} = S_0 - cH(\frac{1}{2}(A - A^T))J_{(1)} - H(\frac{1}{2}(A - A^T))J_{(0)} \tag{41}$$
$$+ c^2 \frac{\partial H}{\partial V}(\frac{1}{2}(A - A^T))(-R_A^* dH(\frac{1}{2}(A - A^T))J_{(1,1)},$$

where the calculation between $\frac{\partial H}{\partial V}(\frac{1}{2}(A - A^T))$ and $-R_A^* dH(\frac{1}{2}(A - A^T)$ is the Killing form given above, and $J_{(0)} = h$, $J_{(1)} = \Delta W(h)$, $J_{(1,1)} = \frac{1}{2}\Delta W(h)^2$ with $h = t - t_0$, and $\Delta W(h) = W(t_0 + h) - W(t_0) = \sqrt{h}\xi_0$ where ξ_0 is a normally distributed random variable: $\xi_0 \sim \mathcal{N}(0, 1)$.

Substituting (41) into (40) and replacing the time interval $[t_0, t]$ by $[t_n, t_{n+1}]$, we obtain the following one-step numerical method $\hat{y}_n \to \hat{y}_{n+1}$ $(n \geq 0)$:

$$\hat{y}_n = A_n^S + (A_n^T I(A_n^S))^S (h + c\Delta \hat{W}_n) - \frac{1}{2}(A_n^T d_{A_n} S_{(1,1)}(A_n))^S \Delta \hat{W}_n^2, \tag{42}$$
$$\hat{y}_{n+1} = A_n^S + (I(A_n^S) A_n^T)^S (h + c\Delta \hat{W}_n) - \frac{1}{2}(d_{A_n} S_{(1,1)}(A_n) A_n^T)^S \Delta \hat{W}_n^2,$$

where $h = t_{n+1} - t_n$, $\Delta\hat{W}_n = \sqrt{h}\hat{\xi}_n$ is a truncation of $\Delta W_n := W(t_{n+1}) - W(t_n) = \sqrt{h}\xi_n$ with $\xi_n \sim \mathcal{N}(0,1)$, based on the following truncation of ξ_n to $\hat{\xi}_n$ ([21]):

$$\hat{\xi}_n = \begin{cases} \xi_n, \text{ if } |\xi_n| \le A_h, \\ A_h, \text{ if } \xi_n > A_h, \\ -A_h, \text{ if } \xi_n < -A_h, \end{cases}$$

where $A_h = \sqrt{2k|\ln h|}$ for certain integer $k \ge 1$. This truncation is performed for implicit schemes to avoid the implementation issue caused by the unboundedness of ΔW_n ([21]). The truncation error can be merged into the scheme error by choosing sufficiently large k. In our discussion it is enough to choose $k = 4$.

Further, each A_n $(n \ge 0,\ A_0 := A)$ is first solved from the equation of $\hat{y}_n$, and then substituted into the equation of $\hat{y}_{n+1}$ to calculate $\hat{y}_{n+1}$. From the relation between the matrix $\hat{y}_n$ and the vector y_n mentioned above, we immediately obtain y_n once we get $\hat{y}_n$, for all $n \ge 0$. Therefore we identity the numerical scheme (42) with the final scheme $y_n \to y_{n+1}$ for the stochastic rigid body system (10).

Theorem 4.1. *The numerical scheme* (42) *preserves the Casimir function of the stochastic rigid body system* (10).

Proof. From the relation between $\hat{y}_0$ and $\hat{y}$ in (40) we have

$$\hat{y}_{n+1} = A\hat{y}_n A^T.$$

Since A is an orthogonal matrix, multiplying which will not change the Frobenius norm, we obtain

$$\|\hat{y}_{n+1}\|_F^2 = \|A\hat{y}_n A^T\|_F^2 = \|\hat{y}_n\|_F^2.$$

For the stochastic rigid body system (10), $y = (y^1, y^2, y^3)^T \in \mathbb{R}^3$ whose corresponding $\hat{y} \in so(3)$ is

$$\hat{y} = \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix}.$$

Meanwhile the Casimir function $C(y)$ of (10) satisfies

$$\|\hat{y}_{n+1}\|_F^2 = 4C(y_{n+1}), \quad \|\hat{y}_n\|_F^2 = 4C(y_n).$$

Therefore we have $C(y_{n+1}) = C(y_n)$. $\square$

Combining Theorem 4.1 with the fact that the numerical scheme (42) is a Poisson map by Theorem 3.2, (42) gives a Poisson integrator.

4.4. Numerical tests. In this section we illustrate numerical behavior of our method (42) applied to the stochastic rigid body system (10) from several aspects. Hereby we take $I = [\sqrt{2} + \sqrt{\frac{2}{1.51}}, \sqrt{2} - 0.51\sqrt{\frac{2}{1.51}}, 1]^T$, $c = 0.2$, and $y_0 = [0.7, 0.7, 0]^T$.

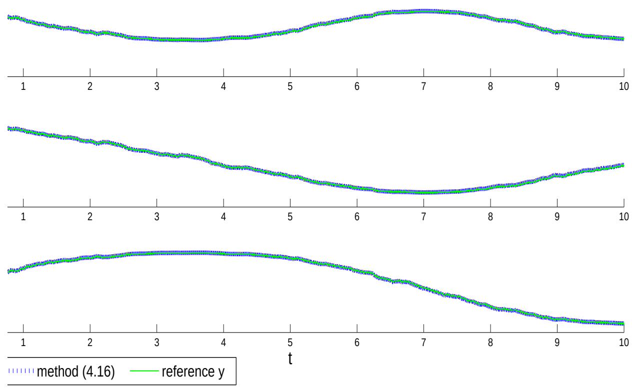
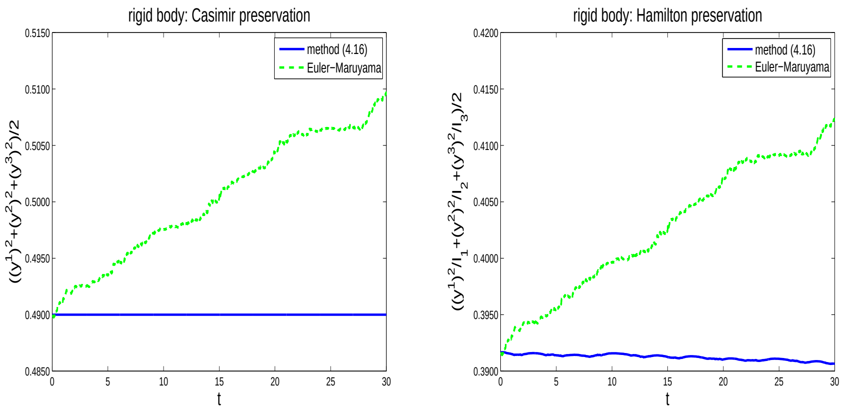


FIGURE 4.1. Sample paths produced by (42).

Figure 4.1 shows one sample path of y^i $(i = 1, 2, 3)$ respectively produced by our method (42) for the stochastic rigid body system (10). Good coincidence between the numerical and the reference (true) paths can be seen. Here we take the time step $h = 10^{-2}$ for our method, and use the midpoint scheme with tiny time step 10^{-5} to simulate the reference (true) paths.

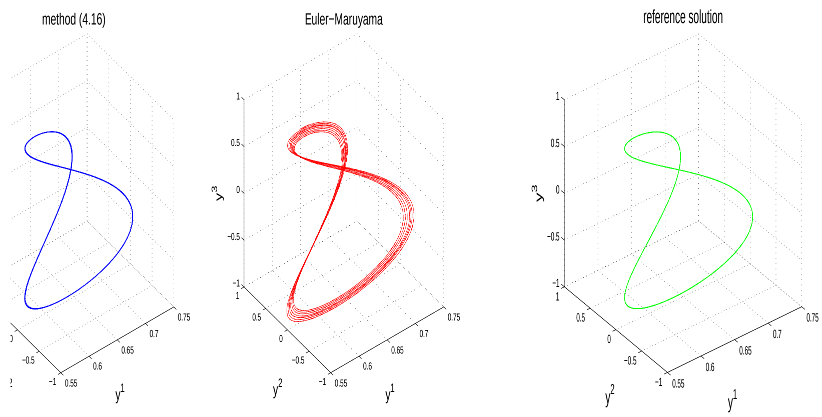


(a) Preservation of Casimir (b) Near preservation of Hamiltonian

FIGURE 4.2. Preservation of Casimir and Hamiltonian by the method (42).

Figure 4.2 (a) illustrates the preservation of the Casimir function by our method (42) for the system (10), whereby the Euler-Maruyama method makes the Casimir function grow along time evolution. In Figure 4.2 (b) we see that our method (42) can nearly preserve the Hamiltonian function of the system (10), while the Euler-Maruyama method fails again to preserve the function.

In Figure 4.3 (a) we compare a phase trajectory created by our method (42) (left) and the Euler-Maruyama method (right), respectively, and Figure 4.3 (b) is the reference (true) phase trajectory simulated by using the midpoint method with



(a) Phase trajectories by (42) and Euler-Maruyama method (b) Reference (true) phase trajectory

FIGURE 4.3. Numerical v.s. true phase trajectories.

time step 10^{-5}. We can see that our method gives a better trajectory simulation than the Euler-Maruyama method.

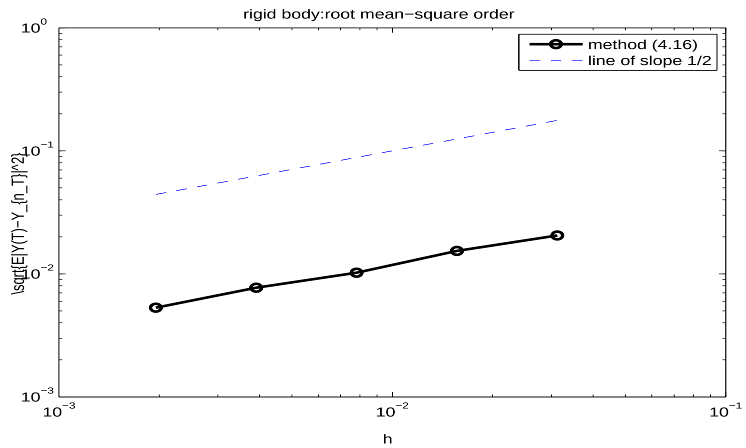


FIGURE 4.4. Root mean-square convergence order of the method (42).

Figure 4.4 shows the root mean-square convergence order of the method (42), which is $\frac{1}{2}$ as seen from the figure. The time steps used are $h = [2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}, 2^{-5}]$, and 500 samples were taken to approximate the expectations.

Remark 4.1. The convergence order results from the truncation of the generating function $\tilde{S}$ to $\bar{S}$ in (38). As was mentioned for deterministic cases ([4, 13, 26]), longer truncation of the series of $\tilde{S}$ may provide higher convergence order also in stochastic situation, as the case by generating functions for SHSs ([6, 23]). This could be further studied in more details.

5. Conclusion

In this paper we propose a Lie-Poisson reduction numerical method for solving a class of stochastic Lie-Poisson systems (SLPSs). We present the reduction procedure for the SLPSs from constrained stochastic Hamiltonian systems (SHSs) on Lie group manifold. Then by reducing stochastic symplectic methods generated by stochastic generating functions for SHSs we construct stochastic Poisson integrators for the SLPSs. The method is applied to a typical model of SLPSs, namely the stochastic rigid body system, for which we give concrete scheme based on the exponential mapping. Numerical tests show effectiveness of the proposed method.

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