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A Priori Error Estimates of Finite Element Methods for Linear Parabolic Integro-Differential Optimal Control Problems

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Abstract. In this paper, we study the mathematical formulation for an optimal control problem governed by a linear parabolic integro-differential equation and present the optimality conditions. We then set up its weak formulation and the finite element approximation scheme. Based on these we derive the a priori error estimates for its finite element approximation both in H^1 and L^2 norms. Furthermore some numerical tests are presented to verify the theoretical results.

AMS subject classifications: 65N30, 65R20

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1 Introduction

Optimal control problems governed by partial differential equations have been a major research topic in applied mathematics and control theory. Since the milestone work of J. P. Lions [10], a great deal of progress has been made in many aspects like stability, observability and numerical methods, which are too extensive to be mentioned here even very briefly. Among them, finite element approximations of optimal control problems governed by various partial differential equations, either linear or nonlinear, have been

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much studied in the literature. For instance, optimal control problems governed by linear elliptic PDEs have been much studied, and their finite element approximation, and a prior error estimates were established in [3]. Many results in convergence of the standard finite element approximation of optimal control problems governed by linear or nonlinear elliptic and parabolic equations can be found in, for examples, [1, 3, 4, 17–19, 22–24], although it is impossible to give even a very brief review here. Recently, optimal control problems with more complicated state equations have been considered, particularly those with the integro-differential state equations, which are often met in real applications. For example, progress on the finite element method for the optimal control problem governed by elliptic integral equations and integro-differential equations has been made in [8], in which the a priori and a posteriori error estimations were obtained.

Parabolic integro-differential equations and their control are often met in applications such as heat conduction in materials with memory, population dynamics, and viscous-elasticity, cf. e.g., Friedman and Shinbrot [5], Heard [7], and Renardy, Hrusa and Nohel [20]. For equations with nonsmooth kernels, we refer to Grimmer and Pritchard [6], Lunardi and Sinestrari [12], and Lorenzi and Sinestrari [13] and references therein. Furthermore finite element methods for parabolic integro-differential equations problems with a smooth kernel have been discussed in, e.g., Cannon and Lin [2], LeRoux and Thomée [14], Lin, Thomée, and Wahlbin [15], Sloan and Thomée [21], Thomée and Zhang [25], and Yanik and Fairweather [27].

However there exists little research on optimal control problems governed by parabolic integro-differential equations, in spite of the fact that such control problems are widely encountered in practical engineering applications and scientific computations. Furthermore the finite element method of this optimal control problem governed by such equations is not well-studied although there exists much research on the finite element approximation of parabolic integro-differential equations as mentioned above. Those will be studied in this work with numerical verifications.

The content of the paper is as follows. In Section 2, we present the weak formulation and analyze the existence of the solution for the optimal control problem. In Section 3, we give the optimality conditions and the finite element approximation of the optimal control problems. In Section 4, we establish the a priori error estimates for the finite element approximation of the control problem. In the last section, we perform some numerical tests, which illustrate the theoretical results.

Throughout the paper, we adopt the standard notations for Sobolev spaces as in [9–11, 26], such as $W^{m,q}(\Omega)$ on Ω with norm $\|\cdot\|_{m,q,\Omega}$, and semi-norm $|\cdot|_{m,q,\Omega}$ for $1\le q\le\infty$. Set $W_0^{m,q}(\Omega)=\{w\in W^{m,q}(\Omega): w|_{\partial\Omega}=0\}$. Also denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$, with norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$. Denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{1/s}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,q}(\Omega))$ and $C^k(0,T;W^{m,q}(\Omega))$. The details can be found in [11].

2 Model problem, weak formulation and well-posedness

Let Ω and Ω_U be bounded open sets in R^d for $1 \le d \le 3$, and $T > 0$. To fix idea, we shall take the state space $W = L^2(0,T;V)$ with V = $H_0^1(\Omega)$ and the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$. Let the observation space $Y = L^2(0,T;H)$ with $H = L^2(\Omega)$. In addition c or C denotes a general positive constant independent of unknowns and the meshes parameters introduced later.

Introduce the objective functional

$$J(u,y) = \Big\{ \frac{1}{2}\int_0^T \|y - z_d\|^2_{0,\Omega} dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Omega_U} dt \Big\},$$

where α is a positive regularity constant.

We investigate the optimal control problem governed by a parabolic integro-differential equation as follows:

$$\min_{u \in U_{ad}} J(u, y(u)) \tag{2.1}$$

subject to

$$\begin{cases} y_t + \mathbf{A}y + \displaystyle\int_0^t \mathbf{C}(t,\tau)y(\tau)d\tau = f + Bu & \text{in } \ \Omega \times (0,T], \\ y = 0 & \text{on } \ \partial\Omega \times [0,T], \\ y|_{t=0} = y_0 & \text{in } \ \Omega, \end{cases} \tag{2.2}$$

where u is control, y is state, z_d is the observation, U_{ad} is a closed convex subset with respect to the control, f, z_d and y_0 are some suitable functions to be specified later. $\mathbf{A}$ is a linear strongly elliptic self-adjoint partial differential operator of second order with coefficients depending smoothly on spacial variables, and $\mathbf{C}(t,\tau)$ is an arbitrary second-order linear partial differential operator, with coefficients depending smoothly on both of time and spacial variables in the closure of their respective domains, B is a suitable continuous operator. A precise formulation of this problem is given later.

In order to give the weak formulation of problem mentioned-above and study the existence and regularity of the solution, we introduce L^2-inner products:

$$(f_1, f_2) = \int_\Omega f_1 f_2, \quad \forall (f_1, f_2) \in H \times H, \quad (u,v)_U = \int_{\Omega_U} uv, \quad \forall (u,v) \in U \times U,$$

and bilinear forms:

$$a(z,w) = (\mathbf{A}z, w), \quad c(t,\tau; z, w) = (\mathbf{C}(t,\tau)z, w), \quad \forall z, w \in V \times V.$$

In the case that $f_1 \in V$ and $f_2 \in V^*$, the dual pair (f_1, f_2) is understood as $\langle f_1, f_2 \rangle_{V \times V^*}$.

Then a possible weak formulation for the state equation reads:

$$\begin{cases} (y_t, w) + a(y,w) + \int_0^t c(t,\tau; y(\tau), w)d\tau = (f + Bu, w), \quad \forall w \in V, \quad t \in (0,T], \\ y|_{t=0} = y_0. \end{cases} \tag{2.3}$$

From Yanik and Fairweather [27], we know that the above weak formulation has a unique solution in $y \in W(0,T) = \{w \in L^2(0,T;H_0^1(\Omega)),\ w_t \in L^2(0,T;H^{-1}(\Omega))\}$.

Therefore the control problem (2.1)-(2.2) can be restated as (OCP):

$$\min_{u\in U_{ad}} J(u,y(u)), \tag{2.4a}$$

$$\begin{cases} (y_t,w)+a(y,w)+\displaystyle\int_0^t c(t,\tau;y(\tau),w)d\tau=(f+Bu,w), \quad \forall w\in V,\ t\in(0,T], \\ y|_{t=0}=y_0. \end{cases} \tag{2.4b}$$

Assume that $\mathbf{C}_t(t,\tau)$ exists, $c_t(t,\tau;z,w)=(\mathbf{C}_t(t,\tau)\nabla z,\nabla w)$. Then there are constants c and C, such that for all t and τ in $[0,T]$:

$$(a)\ a(z,z)\geq c\|z\|_{1,\Omega}^2, \qquad \forall z\in V, \tag{2.5a}$$
$$(b)\ |a(z,w)|\leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \qquad \forall z,\ w\in V, \tag{2.5b}$$
$$(c)\ |c(t,\tau;z,w)|\leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \qquad \forall z,\ w\in V, \tag{2.5c}$$
$$(d)\ |c_t(t,\tau;z,w)|\leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \qquad \forall z,\ w\in V. \tag{2.5d}$$

In the following, we will analyze the existence, uniqueness of the solution of the system (2.4).

Theorem 2.1. *Assume that the conditions* (2.5a)-(2.5c) *hold. There exists a unique solution (u,y) for the minimization problem* (2.4) *such that $u \in X$ and $y\in L^\infty(0,T;L^2(\Omega))\bigcap L^2(0,T;H_0^1(\Omega))$ and $y_t\in L^2(0,T;H^{-1}(\Omega))$.*

Proof. Let $\{(u^n,y^n)\}_{n=1}^\infty$ be a minimization sequence for the system (2.4), then it is clear that $\{u^n\}_{n=1}^\infty$ are bounded in $L^2(0,T;L^2(\Omega_U))$. Thus there is a subsequence of $\{u^n\}_{n=1}^\infty$ (still denote by $\{u^n\}_{n=1}^\infty$) such that u^n converges to u weakly in $L^2(0,T;L^2(\Omega_U))$.

For the subsequence u^n, we have

$$(y_t^n,w)+a(y^n,w)+\int_0^t c(t,\tau;y^n(\tau),w)d\tau=(f+Bu^n,w), \quad \forall w\in V, \quad t\in(0,T]. \tag{2.6}$$

Taking $w=y^n$ in (2.6), we gives

$$(y_t^n,y^n)+a(y^n,y^n)+\int_0^t c(t,\tau;y^n(\tau),y^n(t))d\tau=(f+Bu^n,y^n), \quad t\in(0,T]. \tag{2.7}$$

Integrating time from 0 to t in (2.7), we have

$$\begin{aligned} &\frac{1}{2}\|y^n(t)\|_{0,\Omega}^2+c\int_0^t\|y^n\|_{1,\Omega}^2d\tau \\ \leq&\frac{1}{2}\|y_0\|_{0,\Omega}^2+\frac{c}{2}\int_0^t\|y^n\|_{1,\Omega}^2d\tau+C\int_0^t(\|f\|_{-1,\Omega}^2+\|u^n\|_{0,\Omega_U}^2)d\tau \\ &+C\int_0^t\int_0^\tau\|y(s)\|_{1,\Omega}^2dsd\tau, \end{aligned} \tag{2.8}$$

such that

$$\int_0^t \|y^n\|_{1,\Omega}^2 d\tau \le C\Big\{\|y_0\|_{0,\Omega}^2 + \int_0^T (\|f\|_{-1,\Omega}^2 + \|u^n\|_{0,\Omega_U}^2)dt + \int_0^t \int_0^\tau \|y^n(s)\|_{1,\Omega}^2 ds d\tau\Big\}. \tag{2.9}$$

Applying Gronwall's inequality to (2.9), we have

$$\int_0^T \|y^n\|_{1,\Omega}^2 dt \le C\Big(\|y_0\|_{0,\Omega}^2 + \int_0^T (\|f\|_{-1,\Omega}^2 + \|u^n\|_{0,\Omega_U}^2)dt\Big)e^{CT} \le C^* < \infty. \tag{2.10}$$

From (2.8) and (2.10), we also obtain

$$\max_{0\le t\le T} \|y^n(t)\|_{0,\Omega}^2 \le C\Big\{\|y_0\|_{0,\Omega}^2 + \int_0^T (\|f\|_{-1,\Omega}^2 + \|u^n\|_{0,\Omega_U}^2)dt\Big\} \le C^* < \infty. \tag{2.11}$$

Then we have $u^n \in L^2(0,T;L^2(\Omega_U))$, $y^n \in L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H_0^1(\Omega))$. Thus

$$\begin{cases} u^n \longrightarrow u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega_U)), \\ y^n \longrightarrow y \quad \text{weakly in} \quad L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H^1(\Omega)), \\ y^n(T) \longrightarrow y(T) \quad \text{weakly in} \quad L^2(\Omega). \end{cases}$$

Integrating time from 0 to T in (2.6), we obtain

$$\begin{aligned} &(y^n(T),w(T)) - (y_0,w(0)) - \int_0^T (y^n,w_t)dt + \int_0^T a(y^n,w)dt \\ &+ \int_0^T \int_0^t c(t,\tau;y^n(\tau),w)d\tau dt = \int_0^T (f+Bu^n,w)dt, \quad \forall w \in W. \end{aligned} \tag{2.12}$$

Take limits in (2.12) as $n \to \infty$, we have

$$\begin{aligned} &(y(T),w(T)) - (y_0,w(0)) - \int_0^T (y,w_t)dt + \int_0^T a(y,w)dt \\ &+ \int_0^T \int_0^t c(t,\tau;y(\tau),w)d\tau dt = \int_0^T (f+Bu,w)dt, \quad \forall w \in W. \end{aligned}$$

So we have

$$(y_t,w) + a(y,w) + \int_0^t c(t,\tau;y(\tau),w)d\tau = (f+Bu,w), \quad \forall w \in V, \quad t \in (0,T]. \tag{2.13}$$

Further, we have

$$\begin{aligned} \|y_t\|_{L^2(0,T;H^{-1}(\Omega))} &= \sup_{w\in L^2(0,T;H_0^1(\Omega))} \frac{\int_0^T (y_t,w)dt}{\|w\|_{L^2(0,T;H_0^1(\Omega))}} \\ &\le C\Big(\|y_0\|_{0,\Omega}^2 + \int_0^T (\|f\|_{-1,\Omega}^2 + \|u\|_{0,\Omega_U}^2)dt\Big)e^{CT}. \end{aligned}$$

Thus $y_t \in L^2(0,T;H^{-1}(\Omega))$. Since $\int_0^T \|y-z_d\|^2_{0,\Omega}dt$ is a convex function on space $L^2(0,T;L^2(\Omega))$ and $\int_0^T \|u\|^2_{0,\Omega_U}dt$ is a strictly convex function on U, we have

$$\begin{aligned}&\frac{1}{2}\int_0^T \|y-z_d\|^2_{0,\Omega}dt+\frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Omega_U}dt\\ \le&\varliminf_{n\to\infty}\Big\{\frac{1}{2}\int_0^T \|y^n-z_d\|^2_{0,\Omega}dt+\frac{\alpha}{2}\int_0^T \|u^n\|^2_{0,\Omega_U}dt\Big\}.\end{aligned}$$

So (u,y) is one solution of (2.4). Since $J(u,y(u))$ is a strictly convex function on U, hence the solution of the minimization problem (2.4) is unique. This completes the proof. $\square$

The following lemma states the regularity of the solution of (2.4).

Theorem 2.2. *Assume that the condition* (2.5) *holds,* $\mathbf{A}$ *is* H^2*-regular elliptic operator of second order,* $f\in L^2(0,T;L^2(\Omega))$ *and* $y_0\in H_0^1(\Omega)$*. Then the solution of* (2.4) *obeys*

$$y\in L^\infty(0,T;H_0^1(\Omega))\bigcap L^2(0,T;H^2(\Omega)),\quad y_t\in L^2(0,T;L^2(\Omega)). \tag{2.14}$$

Proof. Taking $w=y_t$ in (2.13), we have

$$\begin{aligned}&(y_t,y_t)+\frac{1}{2}\frac{d}{dt}a(y,y)+\frac{d}{dt}\int_0^t c(t,\tau;y(\tau),y)d\tau-c(t,t;y(t),y(t))\\ &-\int_0^t c_t(t,\tau;y(\tau),y)d\tau=(f+Bu,y_t).\end{aligned} \tag{2.15}$$

Integrating time from 0 to t in (2.15), we obtain

$$\begin{aligned}&\int_0^t \|y_t\|^2_{0,\Omega}d\tau+\frac{c}{2}\|y(t)\|^2_{1,\Omega}\\ \le&\frac{1}{2}\Big(\int_0^t \|y_t\|^2_{0,\Omega}d\tau+\frac{c}{2}\|y(t)\|^2_{1,\Omega}\Big)+C\Big\{\|y_0\|^2_{1,\Omega}+(1+T)\int_0^T \|y\|^2_{1,\Omega}dt\\ &+\int_0^t\int_0^\tau \|y(s)\|^2_{1,\Omega}dsd\tau+\int_0^t(\|f\|^2_{0,\Omega}+\|u\|^2_{0,\Omega_U})d\tau\Big\}.\end{aligned} \tag{2.16}$$

Applying Gronwall's inequality to (2.16), we have

$$\int_0^t \|y_t\|^2_{0,\Omega}d\tau+\|y(t)\|^2_{1,\Omega}\le C\Big\{\|y_0\|^2_{1,\Omega}+\int_0^T(\|f\|^2_{0,\Omega}+\|u\|^2_{0,\Omega_U})dt\Big\}. \tag{2.17}$$

Then $y\in L^\infty(0,T;H_0^1(\Omega))$ and $y_t\in L^2(0,T;L^2(\Omega))$. Further we have

$$\begin{aligned}&\|\mathbf{A}y\|_{L^2(0,T;L^2(\Omega))}\\ \le&C\Big\{\|y_t\|_{L^2(0,T;L^2(\Omega))}+\|f\|_{L^2(0,T;L^2(\Omega))}+\|u\|_{L^2(0,T;L^2(\Omega_U))}+\|\mathbf{C}y\|_{L^2(0,T;H^1(\Omega))}\Big\}.\end{aligned}$$

Since $\|y\|_{2,\Omega}\le C\|\mathbf{A}y\|_{0,\Omega}$, we have $y\in L^2(0,T;H^2(\Omega))$. This completes the proof. $\square$

3 Optimality condition and its finite element approximation

By the theory of optimal control problem (see [10]), we can similarly deduce the following optimality conditions of the problem (2.4).

Theorem 3.1. *A pair $(y,u)\in M(0,T)\times X$ is the solution of the optimal control problem* (2.4)*, if and only if there exists a co-state $p\in M(0,T)$ such that the triple (y,p,u) satisfies the following optimality conditions:*

$$\begin{cases} (y_t,w)+a(y,w)+\int_0^t c(t,\tau;y(\tau),w)d\tau=(f+Bu,w), \quad \forall w\in V, \quad t\in(0,T], \\ y|_{t=0}=y_0, \end{cases} \tag{3.1a}$$

$$\begin{cases} -(q,p_t)+a(q,p)+\int_t^T c(\tau,t;q,p(\tau))d\tau=(y-z_d,q), \quad \forall q\in V, \quad t\in[0,T), \\ p|_{t=T}=0, \end{cases} \tag{3.1b}$$

$$\int_0^T (\alpha u+B^*p,v-u)_U dt\geq 0, \quad \forall v\in U_{ad}, \tag{3.1c}$$

where $B:L^2(\Omega_U)\to L^2(\Omega)$ is independent with t. B^* is the adjoint operator of B.

In the following, we discuss the finite element approximation of the control problem (2.4). Here we only consider triangular conforming elements.

Let $\Omega^h(\Omega_U^h)$ be a polygonal approximation to $\Omega(\Omega_U)$ with boundary $\partial\Omega^h(\partial\Omega_U^h)$. For simplicity, we assume that $\Omega(\Omega_U)$ are convex polygons so that $\Omega=\Omega^h(\Omega_U=\Omega_U^h)$. Let $T^h(T_U^h)$ be a partitioning of $\Omega^h(\Omega_U^h)$ into disjoint regular n-simplices $\tau(\tau_U)$, so that $\bar\Omega=\bigcup_{\tau\in T^h}\bar\tau(\bar\Omega_U=\bigcup_{\tau_U\in T_U^h}\bar\tau_U)$.

Associated with T^h is a finite-dimensional subspace S^h of $C(\bar\Omega^h)$, such that $\chi|_\tau$ are polynomials of order m $(m\geq 1)$ for all $\chi\in S^h$ and $\tau\in T^h$. Let $V^h=\{v_h\in S^h:\ v_h(P_i)=0\ (i=1,\cdots,J)\}$, $W^h=L^2(0,T;V^h)$. It is easy to see that $V^h\subset V$, $W^h\subset W$.

Associated with T_U^h is another finite-dimensional subspace U^h of $L^2(\Omega_U)$, such that $\chi|_{\tau_U}$ are polynomials of order m $(m\geq 0)$ for all $\chi\in U^h$ and $\tau_U\in T_U^h$. Here there is no requirement for the continuity. Let $X^h=L^2(0,T;U^h)$. It is easy to see that $X^h\subset X$. Let $h_\tau(h_{\tau_U})$ denote the maximum diameter of the element τ (τ_U) in T^h (T_U^h).

Due to the limited regularity of the optimal control u in general, there will be no advantage in considering higher-order finite element spaces than the piecewise constant space for the control. We therefore only consider the piecewise constant finite element space for the approximation of the control, though higher-order finite element spaces will be used to approximate the state and the co-state. Let $P_0(\Omega)$ denote all the 0-order polynomial over Ω. Therefore we always take$X^h=\{u\in X:u(x,t)|_{x\in\tau_U}\in P_0(\tau_U),\ \forall t\in[0,T]\}$. U_{ad}^h is a closed convex set in X^h. For ease of exposition, in this paper we assume that$U_{ad}^h\subset U_{ad}\cap X^h$.

Then the semi-discrete finite element approximation of (OCP) is thus defined by $(\mathrm{OCP})^h$:

$$\min_{u_h\in U_{ad}^h}\Big\{\frac{1}{2}\int_0^T\|y_h-z_d\|_{0,\Omega}^2dt+\frac{\alpha}{2}\int_0^T\|u_h\|_{0,\Omega_U}^2dt\Big\}, \tag{3.2}$$

such that

$$\begin{cases} \Big(\frac{\partial}{\partial t}y_h,w_h\Big)+a(y_h,w_h)+\int_0^t c(t,\tau;y_h(\tau),w_h)d\tau \\ =(f+Bu_h,w_h), \quad \forall w_h\in V^h, \quad t\in(0,T], \\ y_h|_{t=0}=y_0^h, \end{cases} \tag{3.3}$$

where $y_h\in W^h$ and $y_0^h\in V^h$ is a approximation of y_0.

Since (3.3) is linear functional equation, (3.2) is strictly convex and finite dimensional optimal control problem, we can prove that the problem (3.2)-(3.3) has a unique solution $(y_h,u_h)\in W^h\times U_{ad}^h$ in the same way of proving the uniqueness of the solution of (OCP).

It is well known if a pair $(y_h,u_h)\in W^h\times U_{ad}^h$ is a solution of (3.2)-(3.3), if and only there exists a co-state $p_h\in W^h$, such that the triple (y_h,p_h,u_h) satisfies the following optimality conditions:

$$\begin{cases} \Big(\frac{\partial}{\partial t}y_h,w_h\Big)+a(y_h,w_h)+\int_0^t c(t,\tau;y_h(\tau),w_h)d\tau=(f+Bu_h,w_h), \quad \forall w_h\in V^h, \\ y_h|_{t=0}=y_0^h, \end{cases} \tag{3.4a}$$

$$\begin{cases} -\Big(q_h,\frac{\partial}{\partial t}p_h\Big)+a(q_h,p_h)+\int_t^T c(\tau,t;q_h,p_h(\tau))d\tau=(y_h-z_d,q_h), \quad \forall q_h\in V^h, \\ p_h|_{t=T}=0, \end{cases} \tag{3.4b}$$

$$\int_0^T(\alpha u_h+B^*p_h,v_h-u_h)_U dt\geq 0, \quad \forall v_h\in U_{ad}^h. \tag{3.4c}$$

Let π_{h_U} be local averaging operator given by

$$(\pi_{h_U}w)|_{\tau_U}:=\frac{\int_{\tau_U}w}{\int_{\tau_U}1}, \quad \forall \tau_U\in T_U^h. \tag{3.5}$$

It is the obvious fact that $\int_{\Omega_U}w=\int_{\Omega_U}\pi_{h_U}w$ for any $w\in L^2(\Omega_U)$. Then (3.4c) is equivalent to

$$\int_0^T(\alpha u_h+\pi_{h_U}(B^*p_h),v_h-u_h)_U dt\geq 0, \quad \forall v_h\in U_{ad}^h. \tag{3.6}$$

In the following sections, we will establish the a priori error estimates of the approximation solution.

4 A priori error analysis

For simplicity, we consider the following two control constraints:
(1) the zero obstacle problem:

$$U_{ad}=\big\{v\in X;\ v\geq 0,\ \ a.e.\ \text{ in }\Omega_U,\ \ t\in[0,T]\big\}, \tag{4.1}$$

(2) the integral obstacle problem:

$$U_{ad}=\Big\{v\in X;\ \int_{\Omega_U} v\geq 0,\ \ t\in[0,T]\Big\}. \tag{4.2}$$

In the case of (4.1), (3.1c) and (3.4c) yield

$$\alpha u=\max\big\{0,-B^*p\big\},\quad \alpha u_h=\max\big\{0,-\pi_{h_U}(B^*p_h)\big\}. \tag{4.3}$$

In the case of (4.2), (3.1c) and (3.4c) yields

$$\alpha u=-B^*p+\max\Big\{0,\frac{1}{|\Omega_U|}\int_{\Omega_U}B^*p\Big\},\quad \alpha u_h=-\pi_{h_U}(B^*p_h)+\max\Big\{0,\frac{1}{|\Omega_U|}\int_{\Omega_U}B^*p_h\Big\}. \tag{4.4}$$

And following from Theorem 2.2 that:
In the case of (4.1), (3.1c) and (3.4c) yield

$$(y,p,u)\in L^2(0,T;H^2(\Omega))\times L^2(0,T;H^2(\Omega))\times L^2(0,T;H^1(\Omega_U)). \tag{4.5}$$

In the case of (4.2), (3.1c) and (3.4c) yield

$$(y,p,u)\in L^2(0,T;H^2(\Omega))\times L^2(0,T;H^2(\Omega))\times L^2(0,T;H^2(\Omega_U)). \tag{4.6}$$

In the next two subsections, we will give the a priori error estimates in $L^2(0,T;H^1(\Omega))$-norm and $L^2(0,T;L^2(\Omega))$-norm respectively.

4.1 Convergent rate in $L^2(0,T;H^1(\Omega))$-norm.

In order to give the a priori error estimate in $L^2(0,T;H^1(\Omega))$-norm, we need the following lemmas.

Lemma 4.1. *Let U_{ad} be given by* (4.1) *or* (4.2). *Then $\pi_{h_U}w\in U_{ad}^h$, for any $w\in U_{ad}$.*

Lemma 4.2. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c). *Then there holds the estimate:*

$$\begin{aligned}&\alpha\|\pi_{h_U}u-u_h\|^2_{L^2(0,T;L^2(\Omega_U))}\\ \leq&\int_0^T\big[(B^*(p-p_h),u_h-\pi_{h_U}u)_U+(B^*p+\alpha u,\pi_{h_U}u-u)_U\big]dt.\end{aligned} \tag{4.7}$$

Proof. It follows from (3.1c) and (3.4c) that

$$
\begin{aligned}
&\alpha\|\pi_{h_U}u-u_h\|^2_{L^2(0,T;L^2(\Omega_U))}\\
=&\alpha\int_0^T(\pi_{h_U}u-u_h,\pi_{h_U}u-u_h)_U dt\\
=&\alpha\int_0^T[(u,\pi_{h_U}u-u_h)_U-(u_h,\pi_{h_U}u-u_h)_U]dt\\
=&\alpha\int_0^T[(u,u-u_h)_U+(u_h,u_h-\pi_{h_U}u)_U+(u,\pi_{h_U}u-u)_U]dt\\
\le&\int_0^T[(B^*p,u_h-u)_U+(B^*p_h,\pi_{h_U}u-u_h)_U+\alpha(u,\pi_{h_U}u-u)_U]dt\\
=&\int_0^T[(B^*(p-p_h),u_h-\pi_{h_U}u)_U+(B^*p+\alpha u,\pi_{h_U}u-u)_U]dt.
\end{aligned}\tag{4.8}
$$

Thus (4.7) is derived. This completes the proof. ☐

In the following we bounded the terms on the right-hand side of (4.7). Introduce the following auxiliary problem:

$$
\begin{cases}
\Big(\dfrac{\partial}{\partial t}y_h(u),w_h\Big)+a(y_h(u),w_h)+\displaystyle\int_0^t c(t,\tau;y_h(u)(\tau),w_h)d\tau\\
=(f+Bu,w_h),\quad \forall w_h\in V^h,\\
y_h(u)|_{t=0}=y_0^h,
\end{cases}\tag{4.9a}
$$

$$
\begin{cases}
-\Big(q_h,\dfrac{\partial}{\partial t}p_h(u)\Big)+a(q_h,p_h(u))+\displaystyle\int_t^T c(\tau,t;q_h,p_h(u)(\tau))d\tau\\
=(y-z_d,q_h),\quad \forall q_h\in V^h,\\
p_h(u)|_{t=T}=0.
\end{cases}\tag{4.9b}
$$

Lemma 4.3. *Let* $(y_h(u),p_h(u))$ *and* (y_h,p_h,u_h) *be the solutions of the systems* (4.9a)-(4.9b) *and* (3.4a)-(3.4c). *Then there holds the a priori error estimate:*

$$
\begin{aligned}
&\|y_h-y_h(u)\|_{L^2(0,T;H^1(\Omega))}+\|p_h-p_h(u)\|_{L^2(0,T;H^1(\Omega))}\\
\le& C\Big\{\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}+h_U\|u-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\\
&+\|y-y_h(u)\|_{L^2(0,T;L^2(\Omega))}\Big\}.
\end{aligned}\tag{4.10}
$$

Proof. From (4.9a) and (3.4a), we obtain

$$
\begin{cases}
\Big(\dfrac{\partial}{\partial t}(y_h-y_h(u)),w_h\Big)+a(y_h-y_h(u),w_h)+\displaystyle\int_0^t c(t,\tau;(y_h-y_h(u))(\tau),w_h)d\tau\\
=(B(u_h-u),w_h),\quad \forall w_h\in V^h,\\
(y_h-y_h(u))|_{t=0}=0.
\end{cases}\tag{4.11}
$$

Similarly, from (4.9b) and (3.4b), we have

$$\begin{cases} -\Big(q_h,\dfrac{\partial}{\partial t}(p_h-p_h(u))\Big)+a(q_h,p_h-p_h(u))+\displaystyle\int_t^T c(\tau,t;q_h,(p_h-p_h(u))(\tau))d\tau \\ =(y_h-y,q_h), \quad \forall q_h\in V^h, \\ (p_h-p_h(u))|_{t=T}=0. \end{cases} \tag{4.12}$$

Taking $w_h=y_h-y_h(u)$ in (4.11) and $q_h=p_h-p_h(u)$ in (4.12) and noting

$$\begin{aligned} &(B(u_h-u),y_h-y_h(u)) \\ =&(B(u_h-\pi_{h_U}u),y_h-y_h(u))+\Big(\pi_{h_U}u-u,(\mathcal{I}-\pi_{h_U})\big(B^*(y_h-y_h(u))\big)\Big)_U \\ \le& C\varepsilon^{-1}\Big\{\|\pi_{h_U}u-u_h\|^2_{0,\Omega_U}+h_U^2\|\pi_{h_U}u-u\|^2_{0,\Omega_U}\Big\}+\varepsilon\|y_h-y_h(u)\|^2_{1,\Omega}, \quad 0<\varepsilon<1, \end{aligned} \tag{4.13}$$

similar to (2.9), we have

$$\int_0^T\|y_h-y_h(u)\|^2_{1,\Omega}dt\le C\Big\{\int_0^T(\|u_h-\pi_{h_U}u\|^2_{0,\Omega_U}+h_U^2\|u-\pi_{h_U}u\|^2_{0,\Omega_U})dt\Big\}, \tag{4.14}$$

and

$$\int_0^T\|p_h-p_h(u)\|^2_{1,\Omega}dt\le C\int_0^T\|y_h-y\|^2_{0,\Omega}dt. \tag{4.15}$$

Note that

$$\|y_h-y\|_{L^2(0,T;L^2(\Omega))}\le\|y-y_h(u)\|_{L^2(0,T;L^2(\Omega))}+\|y_h(u)-y_h\|_{L^2(0,T;L^2(\Omega))}.$$

Then (4.10) is derived. This completes the proof. □

Since $(y_h(u),p_h(u))$ is the standard finite element of (y,p), from [2, 14, 15, 21, 25, 27], we get the following results.

Lemma 4.4. *Let* $(y_h(u),p_h(u))$ *be the solutions of the systems* (4.9a)-(4.9b). *Then there holds the a priori error estimate:*

$$\|y-y_h(u)\|_{L^2(0,T;H^1(\Omega))}+\|p-p_h(u)\|_{L^2(0,T;H^1(\Omega))}\le Ch, \tag{4.16}$$

and

$$\|y-y_h(u)\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h(u)\|_{L^\infty(0,T;L^2(\Omega))}\le Ch^2. \tag{4.17}$$

Then from Lemma 4.1-4.4, we have the following Theorem.

Theorem 4.1. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of the systems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c). *Then there holds the a priori error estimate:*

$$\|y-y_h\|_{L^2(0,T;H^1(\Omega))}+\|p-p_h\|_{L^2(0,T;H^1(\Omega))}+\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}\le C(h_U+h). \tag{4.18}$$

Proof. Taking $w_h=p_h-p_h(u)$ in (4.11) and $q_h=y_h-y_h(u)$ in (4.12), we get

$$\begin{aligned}
&\int_0^T \Big((B(u_h-u),p_h-p_h(u))-(y_h-y,y_h-y_h(u))\Big)dt\\
=&(y_h-y_h(u),p_h-p_h(u))|_{t=T}-(y_h-y_h(u),p_h-p_h(u))\big|_{t=0}\\
&+\int_0^T\int_0^t c(t,\tau;(y_h-y_h(u))(\tau),(p_h-p_h(u))(t))d\tau dt\\
&-\int_0^T\int_t^T c(\tau,t;(y_h-y_h(u))(t),(p_h-p_h(u))(\tau))d\tau dt\\
=&0.
\end{aligned}$$

Such that

$$\begin{aligned}
&\int_0^T (u_h-\pi_{h_U}u,B^*(p-p_h))_U dt\\
=&\int_0^T (u_h-\pi_{h_U}u,B^*(p-p_h(u)))_U dt+\int_0^T (y_h-y_h(u),y-y_h)dt\\
&+\int_0^T (\pi_{h_U}u-u,B^*(p_h-p_h(u)))_U dt\\
=&\int_0^T (u_h-\pi_{h_U}u,B^*(p-p_h(u)))_U dt+\int_0^T (y_h-y_h(u),y-y_h(u))dt\\
&+\int_0^T (y_h-y_h(u),y_h(u)-y_h)dt+\int_0^T (\pi_{h_U}u-u,B^*(p_h-p_h(u)))_U dt\\
\le&\int_0^T (u_h-\pi_{h_U}u,B^*(p-p_h(u)))_U dt+\int_0^T (y_h-y_h(u),y-y_h(u))dt\\
&+\int_0^T (\pi_{h_U}u-u,B^*(p_h-p_h(u)))_U dt\\
\le&C\Big\{\|y-y_h(u)\|^2_{L^2(0,T;L^2(\Omega))}+\|p-p_h(u)\|^2_{L^2(0,T;L^2(\Omega))}+h_U^2\|u-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}\Big\}\\
&+\varepsilon\Big(\|u_h-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}+\|y_h-y_h(u)\|^2_{L^2(0,T;L^2(\Omega))}+\|p_h-p_h(u)\|^2_{L^2(0,T;H^1(\Omega))}\Big).
\end{aligned} \tag{4.19}$$

On the other hand, we have

$$\begin{aligned}
(B^*p+\alpha u,\pi_{h_U}u-u)_U\le&(B^*p-\pi_{h_U}(B^*p),\pi_{h_U}u-u)_U\\
\le&C\Big(\|B^*p-\pi_{h_U}(B^*p)\|^2_{0,\Omega_U}+\|u-\pi_{h_U}u\|^2_{0,\Omega_U}\Big).
\end{aligned} \tag{4.20}$$

Applying these two estimates and (4.10) to (4.7) yields

$$\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\le C(h_U+h^2). \tag{4.21}$$

Putting (4.21) into (4.10), we have

$$\|y_h-y_h(u)\|_{L^2(0,T;H^1(\Omega))}+\|p_h-p_h(u)\|_{L^2(0,T;H^1(\Omega))}\le C(h_U+h^2). \tag{4.22}$$

Finally, by using triangle inequality and (4.16), we derive (4.18). $\square$

4.2 Convergent rate in L^2-norm

In this subsection, we concern with the a priori error estimate in L^2-norm with the respect to the state. In many cases of engineering applications, L^2 estimates are more useful.

We divide our proofs into two parts for the two control constraints respectively. For first case we only can derive the results when the boundary of the contacting set of the optimal control is made of smooth curves with finite lengths in the 2-D case or smooth surfaces with finite areas in the 3-D case.

Theorem 4.2. *Assume that U_{ad} is given by* (4.1). *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c). *Then there holds the a priori error estimate:*

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C(h_U+h^2). \tag{4.23}$$

Further, let

$$\Omega_h^+(t)=\{\tau\in T_U^h;\ u>0 \text{ in } \tau\},\ \ \Omega_h^0(t)=\{\tau\in T_U^h;\ u=0 \text{ in } \tau\},\ \ \Omega_h^b(t)=\Omega_U\backslash(\Omega_h^+(t)\cup\Omega_h^0(t)),$$

and assume that

$$meas(\Omega_h^b(t))\le Ch_U,\quad \forall t\in[0,T]. \tag{4.24}$$

Then there holds the a priori error estimate:

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C\big(h_U^{\frac{3}{2}}+h^2\big). \tag{4.25}$$

Proof. By tri-inequality and Lemma 4.4, we have for $t\in[0,T]$

$$\begin{aligned}\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}&\le\|y-y_h(u)\|_{L^\infty(0,T;L^2(\Omega))}+\|y_h-y_h(u)\|_{L^\infty(0,T;L^2(\Omega))}\\&\le Ch^2+\|y_h-y_h(u)\|_{L^\infty(0,T;L^2(\Omega))}, \qquad (4.26a)\\ \|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}&\le\|p-p_h(u)\|_{L^\infty(0,T;L^2(\Omega))}+\|p_h-p_h(u)\|_{L^\infty(0,T;L^2(\Omega))}\\&\le Ch^2+\|p_h-p_h(u)\|_{L^\infty(0,T;L^2(\Omega))}. \qquad (4.26b)\end{aligned}$$

Taking $w_h=y_h-y_h(u)$ in (4.11) and $q_h=p_h-p_h(u)$ in (4.12), similar to (2.11), and noting

$$\begin{aligned}&(B(u_h-u),y_h-y_h(u))\\&=(B(u_h-\pi_{h_U}u),y_h-y_h(u))+\Big(\pi_{h_U}u-u,(\mathcal{I}-\pi_{h_U})\big(B^*(y_h-y_h(u))\big)\Big)_U\\&\le C\varepsilon^{-1}\Big\{\|\pi_{h_U}u-u_h\|^2_{0,\Omega_U}+h_U^2\|\pi_{h_U}u-u\|^2_{0,\Omega_U}\Big\}+\varepsilon\|y_h-y_h(u)\|^2_{1,\Omega},\quad 0<\varepsilon<1,\end{aligned}$$

we have

$$\begin{aligned}&\|y_h-y_h(u)\|_{L^\infty(0,T;L^2(\Omega))}\le C(\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}+h_U^2), \qquad (4.27a)\\&\|p_h-p_h(u)\|_{L^\infty(0,T;L^2(\Omega))}\le C\|y_h-y\|_{L^2(0,T;L^2(\Omega))}\\&\le C\|y_h-y_h(u)\|_{L^2(0,T;L^2(\Omega))}+\|y-y_h(u)\|_{L^2(0,T;L^2(\Omega))}\\&\le C\big(\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}+h_U^2+\|y-y_h(u)\|_{L^2(0,T;L^2(\Omega))}\big)\\&\le C(\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}+h^2+h_U^2). \qquad (4.27b)\end{aligned}$$

From (4.26a)-(4.27b), we need estimate $\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}$. Then from Lemma 4.2, we have

$$
\begin{aligned}
&\alpha\|\pi_{h_U}u-u_h\|^2_{L^2(0,T;L^2(\Omega_U))}\\
\le&\int_0^T\big[(B^*(p-p_h),u_h-\pi_{h_U}u)_U+(B^*p+\alpha u,\pi_{h_U}u-u)_U\big]dt.
\end{aligned}\tag{4.28}
$$

Note that:

(1) In $\Omega_h^+(t)$, since $\alpha>0$, $u>0$, then $\alpha u>0$. From (4.3), we have

$$\alpha u=-B^*p,\quad B^*p+\alpha u=0;$$

(2) In $\Omega_h^0(t)$, since $u=0$, then

$$(B^*p+\alpha u,\pi_{h_U}u-u)=0.$$

So we have

$$
\begin{aligned}
(B^*p+\alpha u,\pi_{h_U}u-u)_U=&\int_{\Omega_h^b}(B^*p+\alpha u)(\pi_{h_U}u-u)\\
\le&C\Big(\|B^*p-\pi_{h_U}(B^*p)\|^2_{0,\Omega_h^b}+\|u-\pi_{h_U}u\|^2_{0,\Omega_h^b}\Big)\\
\le&Ch_U^2meas(\Omega_h^b).
\end{aligned}\tag{4.29}
$$

On the other hand

$$\int_0^T(B^*(p-p_h),u_h-\pi_{h_U}u)_Udt\le C\|p-p_h\|^2_{L^2(0,T;L^2(\Omega))}+\varepsilon\|u_h-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}. \tag{4.30}$$

Note that

$$\|p-p_h\|^2_{L^2(0,T;L^2(\Omega))}\le\|p-p_h(u)\|^2_{L^2(0,T;L^2(\Omega))}+\|p_h(u)-p_h\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.31}$$

Letting ε be small enough, then from (4.28)-(4.31) and Lemma4.4, we obtain

$$\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\le C\Big(h_U\sqrt{meas(\Omega_h^b)}+h^2+\|p_h(u)-p_h\|^2_{L^2(0,T;L^2(\Omega))}\Big). \tag{4.32}$$

Then we need to estimate $\|p_h(u)-p_h\|^2_{L^2(0,T;L^2(\Omega))}$.

Let ψ be the solution of the equation

$$a(\omega,\psi)=(p_h(u)-p_h,\omega),\quad \forall\omega\in W. \tag{4.33}$$

Note that Ω is convex, we have

$$\|\psi\|_{2,\Omega}\le C\|p_h(u)-p_h\|_{0,\Omega},\quad \forall t\in[0,T]. \tag{4.34}$$

Let $I_h\psi \in W^h$ be the Lagrange interpolator of ψ, then

$$\begin{aligned}\|p_h(u)-p_h\|_{0,\Omega}^2 &= a(p_h(u)-p_h,\psi)=a(p_h(u)-p_h,\psi-I_h\psi)\\ &\le C\|p_h(u)-p_h\|_{1,\Omega}\|\psi-I_h\psi\|_{1,\Omega}\\ &\le C\|p_h(u)-I_h(p_h(u))\|_{1,\Omega}\|\psi-I_h\psi\|_{1,\Omega}\\ &\le Ch^2\|p_h(u)-p_h\|_{0,\Omega}. \end{aligned} \tag{4.35}$$

We obtain

$$\|p_h(u)-p_h\|_{0,\Omega}\le Ch^2, \quad \forall t\in[0,T]. \tag{4.36}$$

Then from (4.32) and (4.36), we have

$$\|u_h-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}\le C\Big(h_U\sqrt{meas(\Omega_h^b)}+h^2\Big). \tag{4.37}$$

Combing (4.27a)-(4.27b), (4.37), we obtain

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C(h_U+h^2).$$

Further, assume that $meas(\Omega_h^b(t))\le Ch_U$, $\forall t\in[0,T]$, we get

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C\Big(h_U\sqrt{meas(\Omega_h^b)}+h^2\Big)\le C\big(h_U^{\frac{3}{2}}+h^2\big).$$

This completes the proof. □

Theorem 4.3. *Assume that U_{ad} is given by* (4.2). *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c). *Then there holds the a priori error estimate:*

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C(h_U^2+h^2). \tag{4.38}$$

Proof. In this case, from (4.2), we know that $\alpha u+B^*p$ is a constant, then we have

$$(B^*p+\alpha u,\pi_{h_U}u-u)_U=0.$$

Then from Lemma 4.2 and (4.36), we have

$$\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}\le C(h^2+h_U^2). \tag{4.39}$$

Combing (4.27a)-(4.27b), (4.39), we have

$$\|y-y_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}\le C(h_U^2+h^2).$$

This completes the proof. □

5 Numerical experiment

In this section, we carry out numerical experiment to check if the a priori error estimates derived in Section 4 are sharp. The numerical tests were done by using AFEpack software package (see [16]).

In the numerical examples, $\Omega=\Omega_U=[0,1]^2$. We use linear finite element spaces to approximate the state and co-state, and the piecewise constant finite element spaces to approximate the control. For time variable, a Euler backward-difference procedure is used to solve discrete system. Here time step size is controlled to demonstrate the relation between the error function and spacial sizes.

The numerical example is the following control problem

$$\min_{u\geq 0}\frac{1}{2}\int_0^1\Big\{\int_\Omega (y-z_d)^2+\int_\Omega u^2\Big\}dt, \tag{5.1a}$$

$$\begin{cases} y_t-\Delta y-\displaystyle\int_0^t (t-\tau)\Delta y d\tau = f+u, & x\in\Omega, \quad 0<t<1, \\ y|_{\partial\Omega}=0. \end{cases} \tag{5.1b}$$

The solutions of (5.1a)-(5.1b) are:

$$\begin{cases} p=-(T-t)\sin\pi x_1\sin\pi x_2, \quad T=1, \\ u=\max\{-p,0\}, \\ y=tx_1(1-x_1)x_2(1-x_2), \\ z_d=y+p_t+\Delta p+\displaystyle\int_t^T (t-\tau)\Delta p d\tau, \\ f=y_t-\Delta y-\displaystyle\int_0^t (t-\tau)\Delta y d\tau - u. \end{cases} \tag{5.2}$$

The numerical results are put into the following Table 1. In the Table, the errors in $L^2(0,T;H^1(\Omega))$-norm and $L^\infty(0,T;L^2(\Omega))$-norm are listed.

From the Table, we see that the L^2-norm convergent rate of the control variable $u-u_h$ is $\mathcal{O}(h)$, i.e., the first order accuracy with the respect to the spacial size; that the H^1-norm convergent rate of the state and costate variables $y-y_h$ and $p-p_h$ also are $\mathcal{O}(h)$; however that the L^2-norm convergent rate of the state and costate approximation errors $y-y_h$ and $p-p_h$ are $\mathcal{O}(h^{1.8})$, which is less than 2, but consistent with our theoretical analysis.

Table 1: Numerical result: for adaptive time steps 50.

			L^2-L^2	L^2-H^1		$L^\infty-L^2$		
# nodes	# sides	# elements	$u-u_h$	$y-y_h$	$p-p_h$	$u-u_h$	$y-y_h$	$p-p_h$
3468	8823	5406	2.3e-01	1.1e-01	1.3e+00	6.7e-02	1.8e-03	1.5e-02
12291	33864	21624	1.1e-01	5.5e-02	6.9e-01	3.3e-02	5.2e-04	4.0e-03
46155	132600	86496	5.8e-02	2.6e-02	3.4e-01	1.6e-02	1.3e-04	1.0e-03
178755	524688	345984	2.9e-02	1.3e-02	1.7e-01	8.3e-03	3.6e-05	2.8e-04

6 Conclusions

In this paper, we study the semi-discrete finite element method for optimal control problem governed by a linear parabolic integro-differential equation. We extend the existing methods in studying finite element approximation of optimal control governed by a parabolic equation to the control governed by a parabolic integro-differential equation. The weak formulation is given, the existence and regularity of the solution for the optimal control problem are analyzed. Further, the a priori error estimates are derived and we carry out some numerical experiments to verify the numerical algorithm is effective and the a priori error estimates derived in Section 4 is reliable and accurate. The work will pave a way to derive the a posteriori error estimates of full discrete finite element approximations of this optimal control problem.

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