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# On a Class of Generalized Sampling Functions

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**Abstract.** In this note, we discuss a class of so-called *generalized sampling functions*. These functions are defined to be the inverse Fourier transform of a family of piecewise constant functions that are either square integrable or Lebegue integrable on the real number line. They are in fact the generalization of the classic *sinc* function. Two approaches of constructing the generalized sampling functions are reviewed. Their properties such as cardinality, orthogonality, and decaying properties are discussed. The interactions of those functions and Hilbert transformer are also discussed.

**Key Words**: Generalized sampling function, sinc function, non-bandlimited signal, sampling theorem, Hilbert transform.

**AMS Subject Classifications**: 41, 42

## 1 Introduction

In signal processing, the classic *sinc* function is fundamentally significant due to the Shannon sampling theorem [1,9,10]. Recall that the classic sinc function is defined at a number $t$ in the set $\mathbb{R}$ of real numbers by the equation

$$\operatorname{sinc}(t) := \frac{\sin t}{t}.$$

The Shannon sampling theorem enables to reconstruct a *bandlimited signal* from translates of sinc functions weighted by the uniformly spaced samples of that signal. It is natural to ask whether similar sampling theorem exists for *non-bandlimited signals*. To that end, recently many efforts have been made to extend the classic sinc to *generalized sampling functions*, for example, in [3–5]. One kind of generalized sampling functions given in [3], denoted by $\operatorname{sinc}_H$, is defined as the *inverse Fourier transform* of a so-called *symmetric cascade filter*, denoted by $H$. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ be the set

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of integers, and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Let $Z$ be a subset of $\mathbb{Z}$, a sequence $\boldsymbol{y} := (y_k : k \in Z)$ is said to be in $l^q(Z)$ if and only if

$$\|\boldsymbol{y}\|_{q,Z} := \Big(\sum_{k\in Z} |y_k|^q\Big)^{1/q} < \infty.$$

The symmetric cascade filter $H$ is a piecewise constant function whose value at $\xi \in \mathbb{R}$ is given by

$$H(\xi) := \sum_{n\in\mathbb{Z}_+} b_n \chi_{I_n}(\xi), \tag{1.1}$$

where the sequence $\boldsymbol{b} = (b_n : n \in \mathbb{Z}_+)$ is in $l^2(\mathbb{Z}_+)$, $\chi_I$ is the *indicator function* of the set $I$, and the interval $I_n$, $n \in \mathbb{Z}_+$, is the union of two symmetric intervals given by the equation

$$I_n := (-(n+1), -n] \cup [n, (n+1)).$$

Let $X$ be a subset of $\mathbb{R}$, and for $q \in \mathbb{N}$, we say a function $f$ is in $L^q(X)$ if and only if

$$\|f\|_{q,X} := \Big(\int_X |f(t)|^q \mathrm{d}t\Big)^{1/q} < \infty.$$

Thus the generalized sampling function $\mathrm{sinc}_H$ is defined by the equation

$$\mathrm{sinc}_H := \sqrt{\frac{\pi}{2}} \mathcal{F}^{-1} H, \tag{1.2}$$

where for any signal $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$ the Fourier transform of $f$ is given by

$$(\mathcal{F}f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\mathrm{i}\xi t} \mathrm{d}t.$$

Of course, we have that $H \in L^2(\mathbb{R})$ because $\boldsymbol{b} = \in l^2(\mathbb{Z}_+)$. The function $H \in L^2(\mathbb{R})$ implies that $\mathrm{sinc}_H \in L^2(\mathbb{R})$ since the Fourier operator is closed in $L^2(\mathbb{R})$.

The primary purpose of this note is to introduce to interested readers the basic concepts, approaches, properties of the generalized sampling functions, and their potential applications. For the remainder of the note, in Section 2, we review two approaches that lead to generating the generalized sampling functions. In Section 3, we discuss the properties such as cardinality, orthogonality, decaying property of the generalized sampling functions. In Section 4, a sampling formula is discussed concerning non-bandlimited functions in the shift-invariant space of the generalized sampling functions. In Section 5, we explore the interaction of the generalized sampling functions and the Hilbert transform operator.

## 2 Two approaches to generate the generalized sampling function

Let us first review the approaches to obtain an explicit form of $\mathrm{sinc}_H$. The symmetric cascade filter $H$ can be associated with an analytic function $F$ on the open unit disk

$$\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\},$$

in the complex plane $\mathbb{C}$. The value of $F$ at $z \in \Delta$ is defined by

$$F(z) := \sum_{n \in \mathbb{Z}_+} b_n z^n. \tag{2.1}$$

Definition (2.1) is well-defined for $z \in \Delta$ by the Cauchy-Schwartz inequality as $\boldsymbol{b} \in l^2(\mathbb{Z}_+)$. Recall that the Hardy space $H^2(\Delta)$ consists of all functions $f$ analytic in $\Delta$, with norm given by

$$\|f\|^2_{H^2(\Delta)} = \sup_{r \in (0,1)} \frac{1}{2\pi} \int_{[-\pi,\pi]} |f(re^{\mathrm{i}t})|^2 \mathrm{d}t.$$

Since, by hypothesis, $\boldsymbol{b} \in l^2(\mathbb{Z}_+)$, we have that $F \in H^2(\Delta)$. Consequently its extension to the boundary $\partial\Delta$ of $\Delta$ is in $L^2(\partial\Delta)$.

Thus, from Eqs. (1.2), (1.1) and (2.1) an explicit form of $\mathrm{sinc}_H(t)$, $t \in \mathbb{R}$ can be found as

$$\mathrm{sinc}_H(t) = \mathrm{sinc}\Big(\frac{t}{2}\Big) \mathrm{Re}\Big\{F(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\frac{1}{2}\mathrm{i}t}\Big\}, \text{ a.e.,} \tag{2.2}$$

where $\mathrm{Re}(z)$ is the real part of a complex number $z$.

We observe that if $\boldsymbol{b} \in l^1(\mathbb{Z}_+)$ then $H \in L^1(\mathbb{R})$ and $F$ is continuous on the boundary of $\Delta$, which in turn implies $\mathrm{sinc}_H$ is continuous and bounded.

A very interesting fact, as discovered in the paper [2], is that the function $\mathrm{sinc}_H$ can be generated *alternatively* through a function, denoted by $G$, that is analytic in a neighborhood of the closed unit disc, real on the real axis and normalized so that $G(1) = 1$ and $G'(1) \neq 0$. This requires $F$ be analytic in a neighborhood of the closed unit disc. We point out that if $\boldsymbol{b} \in l^1(\mathbb{Z}_+)$, that is, $H \in L^1(\mathbb{R})$, then $F$ must be analytic in a neighborhood of the closed unit disc. The function $G$ is linked to $F$ by the equation

$$F(z) := \frac{G(z) - 1}{z - 1}. \tag{2.3}$$

Using the function $G$, a real-valued function $\phi_G$ whose value at $t \in \mathbb{R}$ is then defined through the imaginary part of the values of $G$ on the unit circle. Let

$$G(\mathrm{e}^{\mathrm{i}t}) = C(t) + \mathrm{i}S(t). \tag{2.4}$$

The function $\phi_G$ is given by

$$\phi_G(t) := \frac{S(t)}{t}. \tag{2.5}$$

Applying Eq. (2.2) to compute $\mathrm{sinc}_H$ by using Eqs. (2.3) and (2.5) yields for all $t\in\mathbb{R}$,

$$\mathrm{sinc}_H(t)=\phi_G(t). \tag{2.6}$$

Let us look at two important examples that demonstrate the two construction approaches of $\mathrm{sinc}_H$. When $G=z$, i.e., $F=1$, we have $\mathrm{sinc}_H=\phi_G=\mathrm{sinc}$. For the second example, let $G$ be the *Blaschke product* of order $n\in\mathbb{N}$ with parameters $\boldsymbol{a}:=(a_j:j\in\mathbb{Z}_n)\in(-1,1)^n$, that is,

$$G(z)=B_{\boldsymbol{a}}(z):=\prod_{j\in\mathbb{Z}_n}\frac{z-a_j}{1-a_jz}.$$

Then $\mathrm{sinc}_H(t)=\phi_G(t)=\sin\theta_{\boldsymbol{a}}(t)/t$, where $\theta_{\boldsymbol{a}}$ is determined by at $t\in\mathbb{R}$ by the equation

$$\mathrm{e}^{\mathrm{i}\theta_{\boldsymbol{a}}(t)}=B_{\boldsymbol{a}}(\mathrm{e}^{\mathrm{i}t}).$$

# 3 Properties of the generalizes sampling functions

Surprisingly the function $\mathrm{sinc}_H$ has many properties that are similar to the classic sinc, such as cardinal, orthogonal properties and decaying properties. We next list some properties of the function $\mathrm{sinc}_H$.

**Proposition 3.1.** Let the generalized function $\mathrm{sinc}_H$ be defined by Eq. (2.2) such that $\boldsymbol{b}\in l^2(\mathbb{Z}_+)$. Then

1. $$\mathcal{F}\mathrm{sinc}_H=\sqrt{\frac{\pi}{2}}H. \tag{3.1}$$
2. $\mathrm{sinc}_H\in L^2(\mathbb{R})$.
3. The set $\{\mathrm{sinc}_H(\cdot-n\pi):n\in\mathbb{Z}\}$ is an orthogonal set, that is
$$\langle\mathrm{sinc}_H,\mathrm{sinc}_H(\cdot-n\pi)\rangle=\pi\|\boldsymbol{b}\|^2_{l^2(\mathbb{Z}_+)}\delta_{n,0}, \tag{3.2}$$
where $\langle u,v\rangle=\int_{\mathbb{R}}u(t)v^*(t)\mathrm{d}t$ denotes the usual inner product of the two functions $u,v\in L^2(\mathbb{R})$, and $v^*$ is the complex conjugate of $v$.
4. If $\boldsymbol{b}\in l^1(\mathbb{Z}_+)$, $\mathrm{sinc}_H(n\pi)=F(1)\delta_{n,0}$, where $\delta_{n,0}=1$ if $n=0$ and $\delta_{n,0}=0$ if $n\in\mathbb{Z}\setminus\{0\}$.
5. If $\boldsymbol{b}\in l^1(\mathbb{Z}_+)$, $\mathrm{sinc}_H$ is even, bounded and continuous, and
$$|\mathrm{sinc}_H(t)|\le\frac{4\|\boldsymbol{b}\|_{l^1(\mathbb{Z}_+)}}{2+|t|},$$
for $t\in\mathbb{R}$.

*Proof.* The first two statements follow immediately from (1.2). The third statement is a special case of Corollary 3.2 of [3]. For the convenience of readers, we provide a direct proof here. By Parseval's theorem and Eq. (1.2) we have

$$\begin{aligned}\int_{\mathbb{R}}\mathrm{sinc}_H(t)\mathrm{sinc}_H(t-n\pi)\mathrm{d}t&=\frac{\pi}{2}\int_{\mathbb{R}}H^2(x)\mathrm{e}^{\mathrm{i}n\pi x}\mathrm{d}x=\frac{\pi}{2}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}_+}b_k^2\chi_{I_k}(x)\mathrm{e}^{\mathrm{i}n\pi x}\mathrm{d}x\\&=\frac{\pi}{2}\sum_{k\in\mathbb{Z}_+}b_k^2\int_{I_k}\mathrm{e}^{\mathrm{i}n\pi x}\mathrm{d}x=\pi\Big(\sum_{k\in\mathbb{Z}_+}b_k^2\Big)\delta_{n,0},\end{aligned}$$

where, in the last equality we have used the orthogonality of the set $\{\mathrm{e}^{-\mathrm{i}n\pi\xi}:n\in\mathbb{Z}\}$ on $I_k$, $k\in\mathbb{Z}_+$. The interchange of the integral operator and the infinite sum is guaranteed by the convergence of the series. The fourth statement follows from Eq. (2.2). The fifth statement follows from Eq. (2.2) and noticing $\mathrm{sinc}(t)\le 2/(1+|t|)$ for any $t\in\mathbb{R}$. $\square$

# 4 Perfect reconstruction sampling formula

With the generalized function $\mathrm{sinc}_H$, a perfect reconstruction sampling theorem was established in [3] for the purpose of reconstructing non-bandlimited signals. This kind of reconstruction sampling theorem may be very useful to study signals with polynomial decaying Fourier spectra that arise in evolution equations and control theories [7, 8]. In [3], a Shannon-type sampling theorem is given concerning functions in the shift-invariant space

$$V_H:=\Big\{\sum_{n\in\mathbb{Z}}c_n\mathrm{sinc}_H(\cdot-n\pi):F(1)=1,\ \boldsymbol{c}=(c_n:n\in\mathbb{Z})\in l^2(\mathbb{Z})\Big\}.$$

The Shannon-type sampling theorem is the direct result of the properties in the previous proposition. We record it here.

**Theorem 4.1.** *A signal $f\in V_H$ if and only if*

$$f=\sum_{n\in\mathbb{Z}}f(n\pi)\mathrm{sinc}_H(\cdot-n\pi). \tag{4.1}$$

Eq. (4.1) necessarily implies that the sampling sequence $(f(n\pi):n\in\mathbb{Z})\in l^2(\mathbb{Z})$ by the orthogonality of the set $\{\mathrm{sinc}_H(\cdot-n\pi):n\in\mathbb{Z}\}$. The above equation of course is true in $L^2(\mathbb{R})$ norm. However, if $\boldsymbol{b}\in l^1(\mathbb{Z}_+)$, Eq. (4.1) holds true pointwise, because by Cauchy-Schwartz inequality series on the right side of Eq. (4.1) converges uniformly, hence the limiting function $f$ is continuous.

We remark that, as pointed out in [3], a function $f\in V_H$ can be characterized by its spectrum. Specifically, a function $f\in V_H$ if and only if

$$\mathcal{F}f(\xi)=\sqrt{\frac{\pi}{2}}\Big(\sum_{n\in\mathbb{Z}}f(n\pi)\mathrm{e}^{-\mathrm{i}n\pi\xi}\Big)H(\xi). \tag{4.2}$$

Eq. (4.2) holds true in $L^2(\mathbb{R})$ if $\boldsymbol{b}\in l^2(\mathbb{R})$, and a.e. pointwise if $f\in L^1(\mathbb{R})$ and the sample sequence $(f(n\pi):n\in\mathbb{Z})\in l^1(\mathbb{Z})$.

The following property is true for functions in the space $V_H$ that is similar to Parseval's identity.

**Proposition 4.1.** If $f\in V_H$, then

$$\|f\|^2_{L^2(\mathbb{R})}=\pi\|\boldsymbol{b}\|^2_{l^2(\mathbb{Z}_+)}\sum_{n\in\mathbb{Z}}f^2(n\pi). \tag{4.3}$$

*Proof.* This is a direct result of Eq. (4.1) and Eq. (3.2). □

# 5 Interaction with the Hilbert transformer

In this section we discuss the interactions between the generalized sampling functions and the Hilbert transformer. We first review several basic properties of the Hilbert transform which we will need later. These properties can be found, for example, in the book [6]. First we recall that the Hilbert transform is an *anti-involution*, that is,

$$\mathcal{H}^2=-\mathcal{I}, \tag{5.1}$$

where $\mathcal{I}$ is the identity operator. Second the operator $\mathcal{H}$ is anti-self adjoint, that is,

$$\langle\mathcal{H}u,v\rangle=\langle u,-\mathcal{H}v\rangle. \tag{5.2}$$

Third, for any $f\in L^2(\mathbb{R})$ and $t\in\mathbb{R}$, the composition of the Fourier transform and the Hilbert transform is given by

$$\mathcal{F}(\mathcal{H}f)(t)=-\mathrm{i}\,\mathrm{sgn}(t)\mathcal{F}f(t), \tag{5.3}$$

where $\mathrm{sgn}(\cdot)$ is the *signum function* having values defined by $\mathrm{sgn}(x)=1$ if $x\in\mathbb{R}_+:=\{t\in\mathbb{R}:t>0\}$, $\mathrm{sgn}(x)=-1$ if $x\in\mathbb{R}_-:=\{t\in\mathbb{R}:t<0\}$, and $\mathrm{sgn}(0)=0$.

**Theorem 5.1.** *The system*

$$\Phi:=\{\mathrm{sinc}_H(\cdot-2k\pi),\mathcal{H}\mathrm{sinc}_H(\cdot-2k\pi):k\in\mathbb{Z}\} \tag{5.4}$$

*is an orthogonal system in $L^2(\mathbb{R})$.*

We remark that the translations in Eq. (5.4) is twice as many as that in Eq. (3.2).

*Proof.* By the third statement of Proposition 3.1, we have that

$$\langle\mathrm{sinc}_H,\mathrm{sinc}_H(\cdot-2k\pi)\rangle=\pi\|\boldsymbol{b}\|^2_{l^2(\mathbb{Z}_+)}\delta_{0k}.$$

Invoking Eqs. (5.1) and (5.2) yields

$$\begin{aligned}\langle \mathcal{H}\mathrm{sinc}_H, \mathcal{H}\mathrm{sinc}_H(\cdot-2k\pi)\rangle &= \langle \mathrm{sinc}_H, -\mathcal{H}^2\mathrm{sinc}_H(\cdot-2k\pi)\rangle \\ &= \langle \mathrm{sinc}_H, \mathrm{sinc}_H(\cdot-2k\pi)\rangle = \pi\|\boldsymbol{b}\|^2_{l^2(\mathbb{Z}_+)}\delta_{0k}.\end{aligned}$$

By Parseval's theorem and Eq. (3.1) we have

$$\int_{\mathbb{R}} \mathrm{sinc}_H(t)\mathcal{H}\mathrm{sinc}_H(t-2k\pi)\mathrm{d}t = \frac{\pi \mathrm{i}}{2}\int_{\mathbb{R}} H^2(t)\mathrm{sgn}(t)\mathrm{e}^{\mathrm{i}2k\pi t}\mathrm{d}t. \tag{5.5}$$

Invoking expression (1.1) of $H$, Eq. (5.5) becomes

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{sinc}_H(t)\mathcal{H}\mathrm{sinc}_H(t-2k\pi)\mathrm{d}t \\ =&\frac{\pi \mathrm{i}}{2}\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}_+} b_n^2\chi_{I_n}(t)\mathrm{sgn}(t)\mathrm{e}^{\mathrm{i}2k\pi t}\mathrm{d}t = \frac{\pi \mathrm{i}}{2}\sum_{n\in\mathbb{Z}_+} b_n^2\int_{I_n}\mathrm{sgn}(t)\mathrm{e}^{\mathrm{i}2k\pi t}\mathrm{d}t \\ =&\frac{\pi \mathrm{i}}{2}\sum_{n\in\mathbb{Z}_+} b_n^2\Big[\int_{[n,n+1)}\mathrm{e}^{\mathrm{i}2k\pi t}\mathrm{d}t - \int_{(-(n+1),-n]}\mathrm{e}^{\mathrm{i}2k\pi t}\mathrm{d}t\Big],\end{aligned}$$

where the interchange of the integral and the infinite sum in the second equality is guaranteed by the absolute convergence of the series.

When $k=0$, the difference in the pair of brackets is zero, while when $k\in\mathbb{Z}\setminus\{0\}$, each integral inside the pair of brackets is zero. Therefore we have

$$\langle \mathrm{sinc}_H, (\mathcal{H}\mathrm{sinc}_H)(\cdot-2k\pi)\rangle = 0$$

for any $k\in\mathbb{Z}$. ☐

Next we compute the Hilbert transform of $\mathrm{sinc}_H$. The computation makes use of the function $G$ in Eq. (2.3). Thus in what follows we assume that $\boldsymbol{b}\in l^1(\mathbb{Z}_+)$. Note by Eq. (2.3) we have $G(z)=1+(z-1)F(z)$. Further invoking Eq. (2.1) and recalling (2.4), we obtain

$$C(t) = 1-b_0+\sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\cos kt \tag{5.6}$$

and

$$S(t) = \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\sin kt.$$

Observe that $S(0)=0$ and $C(0)=1$. Define the function at $t\in\mathbb{R}\setminus\{0\}$,

$$\mathrm{cosinc}_H(t) := \frac{1-C(t)}{t}, \tag{5.7}$$

and

$$\mathrm{cosinc}_H(0) = \lim_{t\to 0}\frac{1-C(t)}{t}. \tag{5.8}$$

It will become clear later that this limit exists. We have the following theorem.

**Theorem 5.2.** *The function* $\mathrm{cosinc}_H$ *is continuous and bounded, and*

$$\mathcal{H}\mathrm{sinc}_H = \mathrm{cosinc}_H.$$

*Proof.* Substitute Eq. (5.6) into Eq. (5.7) we have

$$\begin{aligned}\mathrm{cosinc}_H(t) &= \frac{1}{t}\Big[b_0 - \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\cos kt\Big] = \frac{1}{t}\Big[b_0 + \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)(1-\cos kt+1)\Big]\\ &= \frac{1}{t}\Big[b_0 + \sum_{k\in\mathbb{N}}(1-\cos kt) - \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\Big] = \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\frac{1-\cos kt}{t}.\end{aligned}$$

The last equation clear shows that the limit (5.8) exists and the function $\mathrm{cosinc}_H$ is continuous and bounded. Since $\mathrm{sinc}_H(t) = S(t)/t$, we have that the Hilbert transform of $\mathrm{sinc}_H$ is given by

$$\mathcal{H}\mathrm{sinc}_H(t) = \mathcal{H}\frac{S(t)}{t} = \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\mathcal{H}\frac{\sin kt}{t} = \sum_{k\in\mathbb{N}}(b_{k-1}-b_k)\frac{1-\cos kt}{t},$$

in the last equality, we have used the identity $\mathcal{H}\mathrm{sinc}t = (1-\cos t)/t$ and the property that the Hilbert transformer commutes with the positive dilution operator. Thus we have proved the theorem. □

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