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A CHARACTERIZATION FOR FRACTIONAL INTEGRALS ON GENERALIZED MORREY SPACES

Eridani and M. I. Utoyo

(*Airlangga University, Indonesia*)

H. Gunawan

(*Bandung Institute of Technology, Indonesia)*

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Abstract. This paper concerns with the fractional integrals, which are also known as the Riesz potentials. A characterization for the boundedness of the fractional integral operators on generalized Morrey spaces will be presented. Our results can be viewed as a refinement of Nakai's[7].

Key words: *fractional integrals, Morrey spaces*

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1 Introduction

For $0 < \alpha < d$, we define the fractional integral (also known as the Riesz potential) $I_\alpha f$ by

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \mathrm{d}y, \qquad x \in \mathbf{R}^d,$$

for any suitable function f on $\mathbf{R}^d$. Clearly $I_\alpha f$ is well-defined for any locally bounded, compactly supported function f on $\mathbf{R}^d$. It is well-known that I_α is bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$, that is,

$$\|I_\alpha f : L^q\| \le C\, \|f : L^p\|$$

if and only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d},$$

with $1 < p < \dfrac{d}{\alpha}$. This result was proved by Hardy and Littlewood[5,6] and Sobolev[10] around the 1930's. Further development on the subject can be found in [11, 12].

Next, let $\mathbf{R}^+ := (0,\infty)$. For $1 \le p < \infty$ and a suitable function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$, we define the generalized Morrey space $L^{p,\phi} = L^{p,\phi}(\mathbf{R}^d)$ to be the set of all functions $f \in L^p_{\rm loc}(\mathbf{R}^d)$ for which

$$\|f : L^{p,\phi}\| := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p \mathrm{d}y \right)^{1/p} < \infty.$$

Here the supremum are taken over all open balls $B = B(a,r)$ in $\mathbf{R}^d$ and $\phi(B) = \phi(r)$, where $r \in \mathbf{R}^+$. For certain functions ϕ, the spaces $L^{p,\phi}$ reduce to some classical spaces. For instance, if $\phi(r) = r^{(\lambda-d)/p}$, where $0 \le \lambda \le d$, then $L^{p,\phi}$ is the classical Morrey space $L^{p,\lambda}$. For a brief history of the Morrey space and related spaces, see [8]. For more recent results, see [9, 13] and the references therein.

In this short paper, we shall revisit Nakai's theorems on the fractional integrals on the generalized Morrey spaces[7]. In particular, we find that the sufficient condition imposed by Nakai for the boundedness of the operator turns out to be necessary. In other words, we obtain a characterization for which the fractional integral operators are bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

2 Main Results

Let us begin with some assumptions and relevant facts that follow. As customary, the letters C, C_i, C_p and $C_{p,q}$ denote positive constants, which may depend on the parameters such as α, p,q and the dimension d of the ambient space, but not on the function f or the variable x. These constants may vary from line to line.

In the definition of $L^{p,\phi}$, the function ϕ is assumed to satisfy the following conditions:

$$\begin{aligned} \phi \text{ is almost decreasing} \quad &: \quad t \le r \Rightarrow \phi(r) \le C_1\, \phi(t); \\ r^d \phi(r)^p \text{ is almost increasing} \quad &: \quad t \le r \Rightarrow t^d \phi(t)^p \le C_2\, r^d \phi(r)^p, \end{aligned}$$

with $C_1,\ C_2 > 0$ being independent of r and t. These two conditions imply that

$$\phi \text{ satisfies the doubling condition} \quad : \quad 1 \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{C_3} \le \frac{\phi(t)}{\phi(r)} \le C_3,$$

for some $C_3 > 0$ (which is also independent of r and t). Throughout this paper, we shall always assume that ϕ satisfies these conditions.

In [7], Nakai showed that I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$ for

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$$

if ϕ satisfies an additional condition, namely

$$\int_r^\infty t^{\alpha-1}\phi(t)\mathrm{d}t \le C_4 r^\alpha \phi(r), \tag{1}$$

and

$$r^\alpha\phi(r)\le C_5\psi(r), \tag{2}$$

for every $r\in\mathbf{R}^+$. By taking $\phi(r)=r^{(\lambda-d)/p}$ with $0\le\lambda<d-\alpha p$ and $\psi(r)=r^\alpha\phi(r)$ with $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$, Nakai's result contains Spanne's, which states that I_α is bounded form $L^{p,\lambda}$ to $L^{q,\mu}$ for $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$, $0\le\lambda<d-\alpha p$ and $\mu=\frac{q}{p}\lambda$[8]. See also [3] for related results.

In the following, we shall show that the condition (2) is necessary for the fractional integral operator I_α to be bounded from $L^{p,\phi}$ to $L^{q,\psi}$. To do so, we need some lemmas. The first lemma shows particularly that the space $L^{p,\phi}$ is not trivial.

Lemma 2.1. *If $B_0:=B(a_0,r_0)$, then $\chi_{B_0}\in L^{p,\phi}$ where χ_{B_0} is the characteristic function of the ball B_0. Moreover, there exists $C>0$ such that*

$$\frac{1}{\phi(r_0)}\le\|\chi_{B_0}:L^{p,\phi}\|\le\frac{C}{\phi(r_0)}.$$

Proof. Let $B:=B(a,r)$ denote an arbitrary ball in $\mathbf{R}^d$. It is easy to see that

$$\|\chi_{B_0}:L^{p,\phi}\|=\sup_B\frac{1}{\phi(r)}\left(\frac{|B\cap B_0|}{|B|}\right)^{1/p}\ge\frac{1}{\phi(r_0)}\left(\frac{|B_0\cap B_0|}{|B_0|}\right)^{1/p}=\frac{1}{\phi(r_0)}.$$

Now, if $r\le r_0$, then $\phi(r_0)\le C\phi(r)$ and

$$\frac{1}{\phi(r)}\left(\frac{|B\cap B_0|}{|B|}\right)^{1/p}\le\frac{1}{\phi(r)}\le\frac{C}{\phi(r_0)}.$$

On the other hand, if $r_0\le r$, we have $r_0^d\phi(r_0)^p\le Cr^d\phi(r)^p$ and

$$\frac{1}{\phi(r)}\left(\frac{|B\cap B_0|}{|B|}\right)^{1/p}=\frac{C|B\cap B_0|^{1/p}}{r^{d/p}\phi(r)}\le\frac{C|B_0|^{1/p}}{r^{d/p}\phi(r)}\le\frac{Cr_0^{1/p}}{r_0^{d/p}\phi(r_0)}\le\frac{C}{\phi(r_0)}.$$

This completes the proof.

Lemma 2.2. *If $B_0:=B(a_0,r_0)$, then* $r_0^\alpha\le CI_\alpha\chi_{B_0}(x)$ for every $x\in B_0$.

Proof. If $x, y \in B_0 := B(a_0, r_0)$, then $|x-y| \le |x-a_0| + |a_0-y| < 2r_0$. If we integrate both sides of the following inequality $r_0^{\alpha-d} \le C|x-y|^{\alpha-d}$ over B_0, then we get the desired estimate.

The following theorem gives a characterization of the functions ϕ and ψ for which I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

Theorem 2.3. *Suppose that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d},$$

where $1 < p < \frac{d}{\alpha}$. *Suppose further that* $r^\alpha \phi(r)$ *satisfies the integral condition* (1). *Then,* I_α *is bounded from* $L^{p,\phi}$ *to* $L^{q,\psi}$ *if and only if* $r^\alpha \phi(r) \le C\psi(r)$ *for every* $r \in \mathbf{R}^+$.

Proof. The sufficient part is proved in [7]. We shall now prove the necessary part. Assume that I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$, and let $B_0 := B(a_0, r_0)$. If $x \in B_0$, then $r_0^\alpha \le C I_\alpha \chi_{B_0}(x)$. Integrating over B_0, we get

$$\begin{aligned} r_0^\alpha &\le C\left(\frac{1}{|B_0|}\int_{B_0} |I_\alpha \chi_{B_0}(x)|^q\, dx\right)^{1/q} \le C\psi(r_0)\|I_\alpha \chi_{B_0} : L_\psi^q\| \\ &\le C\psi(r_0)\|\chi_{B_0} : L_\phi^p\| \le C\psi(r_0)\,\phi(r_0)^{-1}. \end{aligned}$$

Note that the first inequality follows from Lemma 2.2, while the last one follows from Lemma 2.1. Since this is true for every $r_0 \in \mathbf{R}^+$, we are done.

3 Additional Results

In [4], there is the following theorem that serves as an extension of Adams and Chiarenza–Frasca's result on the fractional integral operator I_α [1, 2].

Theorem 3.1. (Gunawan-Eridani). *Suppose that* $1 < p < \frac{d}{\alpha}$ *and* ϕ^p *satisfies the integral condition, namely*

$$\int_r^\infty \frac{\phi^p(t)}{t}\, dt \le C_6 \phi^p(r), \tag{3}$$

for every $r \in \mathbf{R}^+$. *If* $\phi(r) \le C r^\beta$ *for* $-\frac{d}{p} \le \beta < -\alpha$, *then, for* $q = \frac{\beta p}{\alpha+\beta}$, *there exists* $C_{p,\beta} > 0$ *such that*

$$\|I_\alpha f : L^{q,\phi^{p/q}}\| \le C_{p,\beta}\, \|f : L^{p,\phi}\|.$$

As in the previous part, we also have the characterization of ϕ for which I_α is bounded from $L^{p,\phi}$ to $L^{q,\phi^{p/q}}$.

Theorem 3.2. *Suppose that* $1 < p < \dfrac{d}{\alpha}$ *and* ϕ^p *satisfies the integral condition* (3). *If* $-\dfrac{d}{p} \le \beta < -\alpha$ *and* $q = \dfrac{\beta p}{\alpha+\beta}$, *then* I_α *is bounded from* L^p_ϕ *to* $L^q_{\phi^{p/q}}$ *if and only if* $\phi(r) \le C r^\beta$ for every $r \in \mathbf{R}^+$.

Proof. The proof of the sufficient part can be found in [4]. As for the necessary part, we have the following observation: if $B_0 := B(a_0, r_0)$, then

$$
\begin{aligned}
r_0^\alpha &\le C\left(\frac{1}{|B_0|}\int_{B_0} |I_\alpha \chi_{B_0}(x)|^q \mathrm{d}x\right)^{1/q} \le C\,\phi(r_0)^{p/q} \|I_\alpha \chi_{B_0} : L^{q,\phi^{p/q}}\| \\
&\le C\,\phi(r_0)^{p/q} \|\chi_{B_0} : L^{p,\phi}\| \le C\,\phi(r_0)^{p/q}\,\phi(r_0)^{-1},
\end{aligned}
$$

which may be rewritten as $\phi(r_0) \le C r_0^\beta$. Since this inequality is valid for every $r_0 \in \mathbf{R}^+$, the theorem is proved.

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Eridani
Department of Mathematics
Airlangga University
Surabaya 60115
Indonesia

E-mail: eridani.dinadewi@gmail.com

M. I. Utoyo
Department of Mathematics
Airlangga University
Surabaya 60115
Indonesia

E-mial: imam_utoyo@unair.ac.id

H. Gunawan
Department of Mathematics
Bandung Institute of Technology
Bandung 40132
Indonesia

E-mail: hgunawan@math.itb.ac.id